Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 42 (2001), No. 2, 509-516.

Inflection Points on Real Plane Curves Having Many Pseudo-Lines

Johannes Huisman

Institut Mathématique de Rennes, Université de Rennes 1
Campus de Beaulieu, 35042 Rennes Cedex, France
e-mail: huisman@univ-rennes1.fr

Abstract. A pseudo-line of a real plane curve C is a global real branch of $C(\mathbb{R})$ that is not homologically trivial in $\mathbb{P}^2(\mathbb{R})$. A geometrically integral real plane curve C of degree d has at most $d-2$ pseudo-lines, provided that C is not a real projective line. Let C be a real plane curve of degree d having exactly $d-2$ pseudo-lines. Suppose that the genus of the normalization of C is equal to $d-2$. We show that each pseudo-line of C contains exactly 3 inflection points. This generalizes the fact that a nonsingular real cubic has exactly 3 real inflection points.

MSC 2000: 14H45, 14P99
Keywords: real plane curve, pseudo-line, inflection point

1. Introduction

Let $C \subseteq \mathbb{P}^2$ be a real algebraic plane curve. The set $C(\mathbb{R})$ of real points is a real analytic subset of $\mathbb{P}^2(\mathbb{R})$ and has a finite number of global branches [1]. Let B be such a branch. Since B is a real analytic subset of $\mathbb{P}^2(\mathbb{R})$, it has a fundamental class $[B]$ in the homology group $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ of $\mathbb{P}^2(\mathbb{R})$ [2]. We say that B is a *pseudo-line* of the curve C if $[B] \neq 0$ (see Figure 1). A convenient and equivalent way to express that B is a pseudo-line is that there is a projective real line $L \subseteq \mathbb{P}^2(\mathbb{R})$ intersecting B in an odd number of points (counted with multiplicity). In fact, if B is a pseudo-line then any projective real line $L \subseteq \mathbb{P}^2(\mathbb{R})$ intersects B in an odd number of points (provided, of course, that $L \neq B$). These statements all follow from the fact that $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, that the intersection product on $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is nondegenerate, and that the fundamental class $[L]$ of a projective real line $L \subseteq \mathbb{P}^2(\mathbb{R})$ is nonzero in $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ [8].

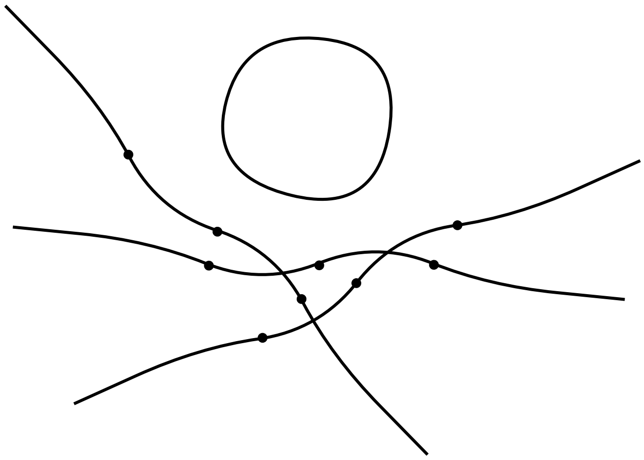

Figure 1. A real plane curve having 4 real branches, exactly 3 of them are pseudo-lines. The marked points are the real inflection points of the curve.

Real plane curves with many pseudo-lines have rarely been studied. The reason may be that such curves are necessarily singular. Indeed, if $C \subseteq \mathbb{P}^2$ is a real algebraic curve having many pseudo-lines then two distinct pseudo-lines B and B' of C intersect each other in a singular point of C (since $[B] \cdot [B'] \neq 0$). Curves with many pseudo-lines seem interesting to study because they turn out to have a more uniform behavior than, for example, nonsingular plane curves. In fact, one may think of the class of curves having many pseudo-lines and the class of nonsingular plane curves as lying at opposite ends on the spectrum of all plane curves of given degree. A study of the passage from curves having many pseudo-lines to curves having less pseudo-lines might shed a different light on the geometry of nonsingular real plane curves. In this paper we study the geometry of curves with many pseudo-lines.

Let us make precise what we mean by real plane curves having many pseudo-lines. Let C be a geometrically integral real plane curve, i.e., its complexification $C \times_{\mathbb{R}} \mathbb{C}$ is reduced and irreducible [5]. Let d be the degree of C. We say that C has *many pseudo-lines* if C has exactly $d-2$ pseudo-lines and if the genus of the normalization of C is equal to $d-2$ (Figure 1 is a picture of a curve of degree 5 having many pseudo-lines). In Section 2 we give a motivation for the present definition of a real plane curve having many pseudo-lines.

There are many examples of real plane curves having many pseudo-lines: all nonsingular real conics and all nonsingular real cubics have many pseudo-lines. In fact, there are curves having many pseudo-lines of arbitrary degree $d \geq 2$. Indeed, choose a nonsingular real conic X in $\mathbb{P}^2$ and choose $d-2$ real projective lines $L_1, \dots, L_{d-2}$ in $\mathbb{P}^2$ in general position such that $L_i(\mathbb{R})$ does not intersect $X(\mathbb{R})$ for $i = 1, \dots, d-2$. Let C' be the union of X and $L_1, \dots, L_{d-2}$. Then, by a well known result of Brusotti one can “deform away” all the nonreal singularities of the real plane curve C'. What one gets is a real plane curve C of degree d having exactly $d-2$ pseudo-lines such that the genus of the normalization of C is equal to $d-2$, i.e., C is a real plane curve of degree d having many pseudo-lines (see [6] for another proof of the existence of such curves).

The paper is devoted to the study of real inflection points on real plane curves having many pseudo-lines. The main result is the following:

Theorem 1. *Let C be a real plane curve having many pseudo-lines. Then, each pseudo-line of C contains exactly 3 inflection points.*

Theorem 1 generalizes that what is known for nonsingular real cubics [9]: A nonsingular real cubic has a unique pseudo-line which admits the structure of a real Lie group isomorphic to the circle group. The set of inflection points on the pseudo-line coincide with the 3-torsion subgroup. The latter subgroup is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Therefore, the pseudo-line contains exactly 3 inflection points.

Using Bezout's Theorem and the fact that a real projective line necessarily intersects a pseudo-line, it can easily be seen that a real branch of C that is not a pseudo-line cannot contain inflection points. Hence, Theorem 1 implies that C has exactly $3(d-2)$ real inflection points. This total of $3(d-2)$ real inflection points can also be obtained from the generalized Klein Equation [10, 11]. However, that equation does not imply anything concerning the distribution of the real inflection points over the different real branches.

In order to prove Theorem 1, one may show that a smooth pseudo-line of any real plane curve contains at least 3 inflection points. Then, applying the generalized Klein Equation, one deduces that each pseudo-line of C contains exactly 3 inflection points. However, we present here a proof which does not use the generalized Klein Equation. The proof is inspired on the case of a nonsingular real cubic, in spite of the absence of a natural structure of a Lie group on a pseudo-line of C.

Theorem 1 has already been proved in [6] in the case where the curve C has, besides the $d-2$ pseudo-lines, yet another real branch. The proof in [6] does not seem to generalize to the case of curves having many pseudo-lines.

Acknowledgement. I thank J.-J. Risler and the referee for their remarks on an earlier version of the paper.

2. The number of pseudo-lines of a real plane curve

In this section we briefly justify the present definition of a real curve having many pseudo-lines.

Proposition 2. *Let $C \subseteq \mathbb{P}^2$ be a geometrically integral real plane curve of degree d. If C has at least $d-1$ pseudo-lines then C is a real projective line in $\mathbb{P}^2$.*

Proof. Let $\tilde{C}$ be the normalization of C and let $\tilde{g}$ be its genus. By the genus formula,

$$\tilde{g} = \tfrac{1}{2}(d-1)(d-2) - \mu,$$

where μ is the multiplicity of the singular locus of C [3]. By hypothesis, C has at least $d-1$ pseudo-lines. Since any two distinct pseudo-lines of C intersect each other, μ is greater than or equal to $\frac{1}{2}(d-1)(d-2)$. Hence, $\tilde{g} = 0$ and C is a rational curve. Then, C can have at most 1 pseudo-line. It follows that $d-1 \leq 1$, i.e., $d = 1$ or $d = 2$. Then, C is a real projective line or a real conic. But a geometrical integral real conic has no pseudo-lines. Since C is supposed to have at least $d-1$ pseudo-lines, C is not a conic. Therefore, C is a real projective line in $\mathbb{P}^2$. □

By the above proposition, real plane curves of degree d having at least $d-1$ pseudo-lines do not constitute an interesting class of real curves as far as geometry is concerned. Therefore, we concentrate on real plane curves having exactly $d-2$ pseudo-lines:

Proposition 3. *Let $C \subseteq \mathbb{P}^2$ be a geometrically integral real plane curve of degree d having exactly $d-2$ pseudo-lines. Then, the genus $\tilde{g}$ of the normalization of C is equal to $d-3$ or $d-2$. Moreover, one has $\tilde{g} = d-2$ if and only if any two distinct pseudo-lines of C intersect in one point only—the intersection being transverse—and C has no other singularities.*

Proof. Let $\tilde{C}$ be the normalization of C. By the genus formula [3], the genus $\tilde{g}$ of $\tilde{C}$ satisfies

$$\tilde{g} \leq \tfrac{1}{2}(d-1)(d-2) - \tfrac{1}{2}(d-2)(d-3) = d-2.$$

Equality holds if and only if any two distinct pseudo-lines of C intersect in one point only—the intersection being transverse—and C has no other singularities. Note that, by definition, the intersection of two distinct real branches of C is transverse in a point P if and only if both branches are smooth at P and both tangent lines are distinct.

Harnack's Inequality [4] states that the number s of real branches of C satisfies the inequality

$$s \leq \tilde{g} + 1.$$

Since C has at least $d-2$ real branches, one has $d-2 \leq s \leq \tilde{g}+1$, i.e., $d-3 \leq \tilde{g}$. Therefore, $\tilde{g}$ is equal to $d-3$ or $d-2$. □

In view of the two preceding propositions, the definition of a real plane curve having many pseudo-lines now seems reasonable.

3. Inflection points on pseudo-lines

We fix, throughout this section, an integer $d \geq 2$ and a geometrically integral curve $C \subseteq \mathbb{P}^2$ of degree d having many pseudo-lines. In particular, C has exactly $d-2$ pseudo-lines. Let $\nu\colon \tilde{C} \to C$ be the normalization of C. By hypothesis, the genus $\tilde{g}$ of $\tilde{C}$ is equal to $d-2$.

Let us collect in the following statement some immediate properties satisfied by C, some of which have already been mentioned above:

Proposition 4.

1. *The curve C has either $d-2$ or $d-1$ real branches.*
2. *Two distinct pseudo-lines of C intersect in one point only. These singularities are the only singularities of C. They are all real ordinary multiple points. In particular, each real branch of C is a smooth real analytic curve in $\mathbb{P}^2(\mathbb{R})$.*
3. *The curve C has only ordinary real inflection points and no real multitangent lines.*

Proof. 1. By Harnack's Inequality [4] the number s of real branches of C satisfies $s \leq \tilde{g}+1 = d-1$. Since C has many pseudo-lines, $s \geq d-2$. Therefore, $s = d-2$ or $d-1$.

2. These properties have already been shown above (cf. Proposition 3).

3. Suppose that there is a real branch B of C containing a nonordinary real inflection point P. Let $L \subseteq \mathbb{P}^2$ be a real projective line such that $L(\mathbb{R})$ is tangent to B at P. By hypothesis, the

order of contact of L at P is at least 4. Since $L(\mathbb{R})$ intersects at least $d-3$ pseudo-lines of C different from B, the degree of the intersection product $L \cdot C$ is at least $4+(d-3)=d+1$. Contradiction by Bezout's Theorem since C is of degree d. Therefore, C does not have nonordinary real inflection points.

Let us make precise what we mean by real multitangent lines. Let $L \subseteq \mathbb{P}^2$ be a real line. We say that L is a multitangent line of C if either L is tangent to C at a nonreal closed point, or L is tangent to C at two distinct real points. This is in accordance with the fact that a nonreal closed point is a point of degree 2.

Suppose that L is a multitangent line of C that is tangent to C at a nonreal point P. Since $L(\mathbb{R})$ intersects each of the $d-2$ pseudo-lines of C, the degree of $L \cdot C$ is at least $(d-2)+2\deg(P)=d+2$. This contradicts the fact that $L \cdot C$ is of degree d. Hence, there are no multitangent lines of C that are tangent at a nonreal point.

Suppose that L is a multitangent line of C that is tangent to C at two distinct real points P and P'. Let B (resp. B') be the real branch of C to which $L(\mathbb{R})$ is tangent at P (resp. P'). There are 3 cases to consider:

1. B and B' are distinct pseudo-lines of C,
2. B and B' are one and the same pseudo-line of C, and
3. B or B' is not a pseudo-line of C.

We show that each of these cases leads to a contradiction. In each of the first two cases, the main observation is that $L(\mathbb{R})$ intersects each pseudo-line of C in an odd number of points.

In case 1, the degrees of $L(\mathbb{R}) \cdot B$ and $L(\mathbb{R}) \cdot B'$ are at least 3. Since $L(\mathbb{R})$ intersects each of the remaining $d-4$ pseudo-lines, the degree of $L \cdot C$ is at least $3+3+(d-4)=d+2$. Contradiction.

In case 2, the degree of $L(\mathbb{R}) \cdot B$ is at least 5. Since $L(\mathbb{R})$ intersects each of the remaining $d-3$ pseudo-lines, the degree of $L \cdot C$ is at least $5+(d-3)=d+2$. Contradiction.

In case 3, $L(\mathbb{R})$ is tangent to a real branch O of C that is not a pseudo-line, i.e., the degree of $L(\mathbb{R}) \cdot O$ is at least 2. Since $L(\mathbb{R})$ intersects each of the $d-2$ pseudo-lines of C and since the degree of $L \cdot C$ is equal to $d=2+(d-2)$, the real line $L(\mathbb{R})$ is not tangent to a pseudo-line of C and is tangent to O at only one point. Contradiction, since L was supposed to be tangent to C at two distinct real points. □

Let us, for completeness, include a proof of the following statement:

Lemma 5. *Any pseudo-line of C contains at least* 1 *inflection point.*

Proof. We show first the more general statement that, for any geometrically integral real plane curve D, the dual curve $B^\curlyvee$ of a smooth real branch B of D is homologically trivial in $\mathbb{P}^2(\mathbb{R})^\curlyvee$. Indeed, choose a general point $P \in \mathbb{P}^2(\mathbb{R})$ such that $P \notin B$. Let $p\colon B \to \mathbb{P}^1(\mathbb{R})$ be the restriction of the linear projection from $\mathbb{P}^2(\mathbb{R}) \setminus \{P\}$ onto $\mathbb{P}^1(\mathbb{R})$ with center P. Then, p is a nonconstant real analytic map from the real analytic curve B into the real analytic curve $\mathbb{P}^1(\mathbb{R})$. Such a map is necessarily ramified at an even number of points of B. Therefore, the number of lines passing through P and tangent to B is even. Dually, this means that the line dual to P intersects $B^\curlyvee$ in an even number of points. Therefore, $B^\curlyvee$ is homologically trivial in $\mathbb{P}^2(\mathbb{R})^\curlyvee$.

Now, let B be a pseudo-line of C. It is clear that $B^\curlyvee$ is a real branch of $C^\curlyvee$. By the preceding paragraph, $B^\curlyvee$ is not a smooth real branch of $C^\curlyvee$ since otherwise $B = (B^\curlyvee)^\curlyvee$ would be homologically trivial. Therefore, $B^\curlyvee$ contains singularities. By Proposition 4 (3), $B^\curlyvee$ can only contain ordinary cusps as singularities. Hence, $B^\curlyvee$ contains at least 1 ordinary cusp. It follows that B contains at least 1 inflection point. □

Proof of Theorem 1. Let B be a pseudo-line of C. By Proposition 4 (2), B is a smooth real analytic curve in $\mathbb{P}^2(\mathbb{R})$. Let Q be a point on B and let $L \subseteq \mathbb{P}^2(\mathbb{R})$ be the tangent line at Q to B. Since B is a pseudo-line, L has to intersect B in yet another point P. Using Bezout's Theorem, one sees that the point $P \in B$ is uniquely determined by Q. Let $\alpha\colon B \to B$ be the map defined by $\alpha(Q) = P$. It is clear that α is continuous.

An inflection point of B is a fixed point of α and conversely. Therefore, we have to show that α has exactly 3 fixed points. The idea is to show that α is a topological covering of B of degree -2. It will then follow from Lemma 7 below that α has exactly 3 fixed points.

For $P \in B$, let $\pi_P\colon \mathbb{P}^2 \setminus \{P\} \to \mathbb{P}^1_P$ be the linear projection with center P. Here, $\mathbb{P}^1_P$ denotes the real algebraic curve of projective lines in $\mathbb{P}^2$ passing through P. Of course, $\mathbb{P}^1_P$ is isomorphic to $\mathbb{P}^1$, however, not canonically. That will be crucial below. By definition, the π_P-image of a real point Q of $\mathbb{P}^2 \setminus \{P\}$ is the real projective line passing through P and Q.

Let $f_P\colon \tilde{C} \to \mathbb{P}^1_P$ be the unique morphism such that $f_P = \pi_P \circ \nu$ on $\tilde{C} \setminus \nu^{-1}(P)$. Then, f_P is a morphism of degree $d - \mu$, where μ is the multiplicity of C at P or, equivalently, μ is the number of real branches of C passing through P. By Proposition 4 (2), one may identify a real branch of C with the corresponding real branch of $\tilde{C}$. The morphism f_P is ramified at a point $Q \in B$ if and only if the real line $L \subseteq \mathbb{P}^2(\mathbb{R})$ through P and Q is tangent to B at Q. It follows that the fiber $\alpha^{-1}(P)$ is equal to the set of ramification points of the restriction of f_P to B.

Let B' be a pseudo-line of C not passing through P. Since B' is contained in $\mathbb{P}^2(\mathbb{R}) \setminus \{P\}$, the restriction of f_P to B' is a continuous map from B' into $\mathbb{P}^1(\mathbb{R})$ of odd topological degree (see [7] for the notion of topological degree mod 2). In particular, the restriction of f_P to B' is surjective. Since there are $(d-2)-\mu$ pseudo-lines of C not passing through P and since the degree of f_P is equal to $d-\mu$, each fiber of the restriction of f_P to B has cardinality at most 2. In fact, more precisely, for all points $R \in \mathbb{P}^1_P(\mathbb{R})$, the degree of the divisor $(f_P|_B)^\star(R)$ on B is at most 2. Since the topological degree of the restriction of f_P to B is even, there are either 0 or 2 points of B at which f_P is ramified. In the former case $f_P|_B$ is not null-homotopic, in the latter case $f_P|_B$ is null-homotopic.

Let $\mathcal{T} = \mathcal{T}_{B/\mathbb{P}^2(\mathbb{R})}$ be the restriction to B of the tangent bundle of $\mathbb{P}^2(\mathbb{R})$. Since B is a pseudo-line, the real analytic vector bundle $\mathcal{T}$ is isomorphic to the direct sum of a trivial line bundle and a nontrivial, i.e. a Möbius line bundle on B. Denote by $\mathbb{P}(\mathcal{T})$ the projectivization of $\mathcal{T}$. The total space of $\mathbb{P}(\mathcal{T})$ is a Klein bottle. The fiber $\mathbb{P}(\mathcal{T})_P$ of $\mathbb{P}(\mathcal{T})$ over a point P of B is canonically isomorphic to $\mathbb{P}^1_P(\mathbb{R})$, i.e., we have made the collection of all real projective lines $\{\mathbb{P}^1_P(\mathbb{R})\}_{P\in B}$ into a locally trivial real analytic fiber bundle over B. Define

$$F\colon B \times B \longrightarrow \mathbb{P}(\mathcal{T})$$

by $F(P,Q) = f_P(Q)$ for all $(P,Q) \in B \times B$. The map F is real analytic, and, when we consider $B \times B$ to be fibered over B through the projection on the first factor, F is a map of locally trivial real analytic fiber bundles over B. The fiber F_P of F over $P \in B$ is the map $f_P|_B$, i.e., we have made the collection of all maps $f_P|_B$ into a real analytic family of maps over B.

Above, we have seen that $f_P|_B$ is ramified at exactly 0 or 2 points of B. Moreover, $f_P|_B$ is not null homotopic in the former case and is null homotopic in the latter case. Since the maps $f_P|_B$ vary continuously in a connected family, either all maps $f_P|_B$ are null homotopic, or all maps $f_P|_B$ are not null homotopic. Hence, either all maps $f_P|_B$ are unramified, or all maps $f_P|_B$ are ramified at exactly 2 points. Now, there is, of course, a point $P_0 \in B$ such that $\alpha^{-1}(P_0)$ is nonempty. But then, as we have seen above, $f_{P_0}|_B$ is ramified. Hence, all maps $f_P|_B$ are ramified at exactly 2 points of B. Moreover, since, for all $R \in \mathbb{P}^1_P(\mathbb{R})$, the degree of the divisor $(f_P|_B)^\star(R)$ is at most 2, the image $(f_P|_B)(B)$ is an interval I_P in $\mathbb{P}^1_P(\mathbb{R})$. The union I of all I_P is an interval subbundle of $\mathbb{P}(\mathcal{T})$. Since the latter fiber bundle is not globally trivial, the interval bundle I is a Möbius bundle over B. This implies that the ramification locus of F, i.e., the union of the ramification loci of the maps $f_P|_B$, is a nontrivial topological covering of the base B of degree ± 2. Now, this ramification locus is the transpose of the graph of α. Hence, α is of degree ± 2.

In order to show that α is of degree -2, recall that B has at least 1 inflection point by Lemma 5 and that such an inflection point is necessarily ordinary by Proposition 4 (3). A local study of α at an ordinary inflection point of B reveals that α is orientation-reversing. Hence, the topological degree of α is equal to -2. It follows from Lemma 7 below that the number of fixed points of α is equal to 3. Therefore, B contains exactly 3 inflection points. $\square$

Before proving Lemma 7 one needs the following preliminary statement:

Lemma 6. *Let $\alpha, \beta\colon S^1 \to S^1$ be topological coverings either both orientation-preserving or both orientation-reversing. Then, the product $\alpha\beta\colon S^1 \to S^1$, defined by $(\alpha\beta)(z) = \alpha(z) \cdot \beta(z)$ for any $z \in S^1$, is also a topological covering.*

Proof. Let $p\colon \mathbb{R} \to S^1$ be the universal covering defined by $p(t) = \exp(2\pi i t)$ for $t \in \mathbb{R}$. Let $\tilde\alpha, \tilde\beta\colon \mathbb{R} \to \mathbb{R}$ be liftings of α and β, respectively, i.e., $p \circ \tilde\alpha = \alpha \circ p$ and $p \circ \tilde\beta = \beta \circ p$. Then, $\tilde\alpha + \tilde\beta$ is a lifting of $\alpha\beta$. Since α and β are topological coverings, $\tilde\alpha$ and $\tilde\beta$ are homeomorphisms of $\mathbb{R}$ onto itself. Since α and β are either both orientation-preserving or both orientation-reversing, $\tilde\alpha$ and $\tilde\beta$ are either both strictly ascending or both strictly descending real functions. It follows that $\tilde\alpha + \tilde\beta$ is strictly ascending or strictly descending. In particular, $\tilde\alpha + \tilde\beta$ is a homeomorphism and, therefore, $\alpha\beta$ is a topological covering. $\square$

Lemma 7. *Let $\alpha\colon S^1 \to S^1$ be a topological covering of degree $-e$, for some $e > 0$. Then, α has exactly $e + 1$ fixed points.*

Proof. Let $\beta\colon S^1 \to S^1$ be the standard topological covering of degree $-e$, i.e., $\beta(z) = \bar z^e$ for $z \in S^1$. Then, α and β are isotopic coverings, i.e., there is a homotopy $F\colon S^1 \times [0, 1] \to S^1$ such that $F_0 = \alpha$, $F_1 = \beta$ and F_t is a topological covering for all $t \in [0, 1]$. Define $F'\colon S^1 \times [0, 1] \to S^1$ by $F'(z, t) = F(z, t) \cdot \bar z$ for $(z, t) \in S^1 \times [0, 1]$. By Lemma 6, F' is an isotopy of topological coverings of S^1. One has $F'_1(z) = \bar z^{e+1}$. Hence, the fiber $F_1'^{-1}(1)$ consists of $e + 1$ points. Then, the fiber $F_0'^{-1}(1)$ consists of $e + 1$ points too, i.e., there are exactly $e+1$ points $z \in S^1$ such that $\alpha(z)\bar{z} = 1$. Therefore, α has exactly $e+1$ fixed points. $\square$

References

[1] Bochnak, J., Coste, M., Roy, M.-F.: *Géométrie algébrique réelle.* Ergebn. Math. 3.Folge, Bd. 12, Springer Verlag 1987.

[2] Borel, A., Haefliger, A.: *La classe d'homologie fondamentale d'un espace analytique.* Bull. Soc. Math. France **89** (1961), 461–513.

[3] Brieskorn, E., Knörrer, H.: *Plane algebraic curves.* Birkhäuser Verlag 1986.

[4] Harnack, A.: *Über die Vieltheiligkeit der ebenen algebraischen Curven.* Math. Ann. **10** (1876), 189–198.

[5] Hartshorne, R.: *Algebraic geometry.* Grad. Texts in Math. **52**, Springer Verlag 1977.

[6] Huisman, J.: *Nonspecial divisors on real algebraic curves and embeddings into real projective spaces.* (Submitted.)

[7] Milnor, J. W.: *Topology from the differentiable viewpoint.* Univ. Press Virginia 1965.

[8] Munkres, J. R.: *Elements of algebraic topology.* Addison-Wesley 1984.

[9] Silverman, J. H.: *The arithmetic of elliptic curves.* Grad. Texts in Math. **106**, Springer Verlag 1986.

[10] Viro, O. Ya.: *Some integral calculus based on Euler characteristic.* Topology and geometry—Rohlin Seminar, Lect. Notes in Math. **1346**, Springer Verlag 1988, 127–138.

[11] Wall, C. T. C.: *Duality of real projective plane curves: Klein's equation.* Topology **35** (1996), 355–362.

Received March 1, 2000