Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 1, 13-20.

On Buffon's Problem for a Lattice and its Deformations

Giuseppe Caristi **Massimiliano Ferrara**

Department of Economic and business branches of knowledge
Faculty of Economics, University of Messina, Via dei Verdi, 75 - 98121 Messina, Italy
e-mail: gcaristi@dipmat.unime.it massiferrara@tiscalinet.it

Abstract. We consider the Buffon's problem for the lattice $R_{\alpha,a}$ which has the fundamental cell composed by the union of octagon, with all sides of lengths a and the angles $(\pi - \alpha)$ and $(\frac{\pi}{2} + \alpha)$ with $\alpha \in \,]0, \frac{\pi}{2}[$, and of the square with side of length a (see Fig. 1). We determine the probability of intersection of a body test needle of length l, $l < a$. For $\alpha = \frac{\pi}{4}$ we also give the estimate of this probability for the cases, when the segment is non-small with respect to $R_{\frac{\pi}{4},a}$ (see [1], [2]).

MSC 2000: 60D05, 52A22
Keywords: geometric probability, stochastic geometry, random sets, random convex sets and integral geometry

Consider a lattice $R_{\alpha,a}$ in euclidean space E_2 with the fundamental cell composed by the union of octagon with all sides of length a and the angles $(\pi - \alpha)$ and $(\frac{\pi}{2} + \alpha)$, and of the square with side of length a (see Fig. 1).

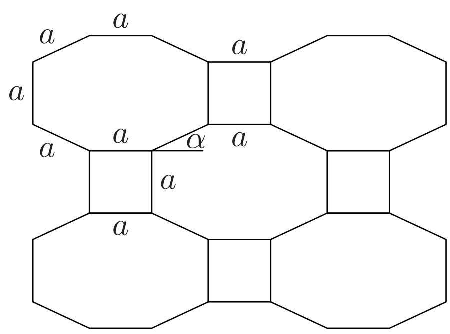

Figure 1

We want to solve the Buffon's problem for a test body segment s of length l which has a random uniformly distributed in a bounded domain of the euclidean plane.

Denoting by M the family of segments s, of length l, whose middle point inside a fixed tile C_0 of $R_{\alpha,a}$ and by N the set of segments s of length l, that are completely contained in C_0, we have [4, p. 53]

$$p_l = 1 - \frac{\mu(N)}{\mu(M)} \tag{1}$$

for the probability p that a random segment intersects $R_{\alpha,a}$. The measures $\mu(M)$ and $\mu(N)$ can be computed by means of the elementary kinematic meausure in the euclidean plane E_2 [3, p. 126], i.e.

$$dK = dx \wedge dy \wedge d\varphi,$$

where x and y are the coordinates of the middle point of the segment s and φ an angle between a fixed side of C_0 and s.

1. Consider the case $l \leq a$, i.e. s is small with respect to $R_{\alpha,a}$ and we prove

Theorem 1. *The probability that a segment s, of length $l \leq a$, intersects a side of one of the cells of the lattice $R_{\alpha,a}$ is*

$$p_l = \frac{6}{\pi(1+\sin\alpha)\cdot(1+\cos\alpha)} \cdot \frac{l}{a} - \frac{3 - \alpha\cot\alpha - (\frac{\pi}{2} - \alpha)\tan\alpha}{2\pi(1+\sin\alpha)\cdot(1+\cos\alpha)} \cdot \left(\frac{l}{a}\right)^2. \tag{2}$$

Proof. Taking into account the symmetries of the set C_0 with respect to straight line in Figure 1, it suffices to consider the values of φ in the interval $\left[0, \frac{\pi}{2}\right]$. We denote by $C_0(\varphi)$ the set with vertices in the middle points of the "boundary" positions of the segment s entirely contained in the set C_0. The set $C_0(\varphi)$ is composed by an octagon $C_o(\varphi)$ and by a rectangle $C_q(\varphi)$ as you can see in the following figure

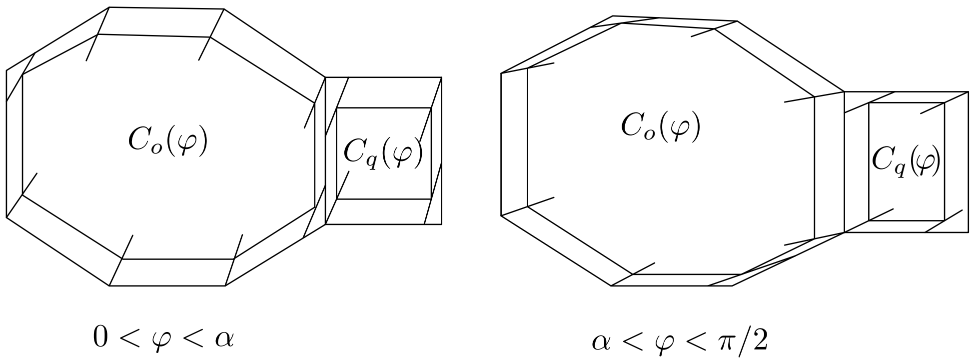

Figure 2

We have that:

$$\text{area}C_0 = 2(1+\sin\alpha)\cdot(1+\cos\alpha)\cdot a^2,$$
$$\text{area}C_q(\varphi) = a^2 + (\sin\varphi + \cos\varphi)\cdot al + (\sin\varphi\cos\varphi)\cdot l^2,$$

and if $0 < \varphi < \alpha$

$$\text{area}C_o(\varphi) = (1 + 2\cos\alpha + 2\sin\alpha + 2\sin\alpha\cos\alpha)\cdot a^2 -$$
$$(\sin\varphi + \cos\varphi + 2\sin\alpha\cos\alpha)\cdot al + \left(\sin\varphi\cos\varphi - \frac{\sin\alpha}{\cos\alpha}\cos^2\varphi\right)\cdot l^2,$$

if $\alpha < \varphi < \frac{\pi}{2}$

$$\text{area}C_o(\varphi) = (1 + 2\cos\alpha + 2\sin\alpha + 2\sin\alpha\cos\alpha)\cdot a^2 -$$
$$(\sin\varphi + \cos\varphi + 2\sin\alpha\cos\alpha)\cdot al + \left(\sin\varphi\cos\varphi - \frac{\sin\alpha}{\cos\alpha}\cos^2\varphi\right)\cdot l^2.$$

Then

$$\mu(M) = \int_0^{\frac{\pi}{2}} \text{area}C_0 d\varphi = \pi(1 + sin\alpha)\cdot(1+\cos\alpha)\cdot a^2,$$

$$\mu(N) = \int_0^{\alpha} \text{area}C_o(\varphi)d\varphi + \int_{\alpha}^{\frac{\pi}{2}} \text{area}C_o(\varphi)d\varphi + \int_0^{\frac{\pi}{2}} \text{area}C_q(\varphi)d\varphi =$$
$$\alpha(1 + 2\sin\alpha + 2\cos\alpha + 2\sin\alpha\cos\alpha)\cdot a^2 - (\sin\alpha + 1 - \cos\alpha + 2\sin^2\alpha)\cdot al +$$
$$\left(1 - \alpha\frac{\cos\alpha}{\sin\alpha}\right)\cdot\frac{l^2}{2} + \left(\frac{\pi}{2} - \alpha\right)\cdot(1 + 2\sin\alpha + 2\cos\alpha + 2\sin\alpha\cos\alpha)\cdot a^2 -$$
$$(\cos\alpha + 1 - \sin\alpha + 2\cos^2\alpha)\cdot al - \left(1 - \left(\frac{\pi}{2} - \alpha\right)\cdot\frac{\sin\alpha}{\cos\alpha}\right)\cdot\frac{l^2}{2} + \frac{\pi}{2}a^2 - 2al + \frac{l^2}{2} =$$
$$\pi[(1+\sin\alpha)\cdot(1+\cos\alpha)]\cdot a^2 - 6al + \left(3 - \alpha\frac{\cos\alpha}{\sin\alpha} - \left(\frac{\pi}{2} - \alpha\right)\cdot\frac{\sin\alpha}{\cos\alpha}\right)\cdot\frac{l^2}{2}.$$

From relation (1) we get the probability (2). □

Remark 1. If $\alpha = 0$ or $\alpha = \frac{\pi}{2}$ we obtain the same lattice of the form

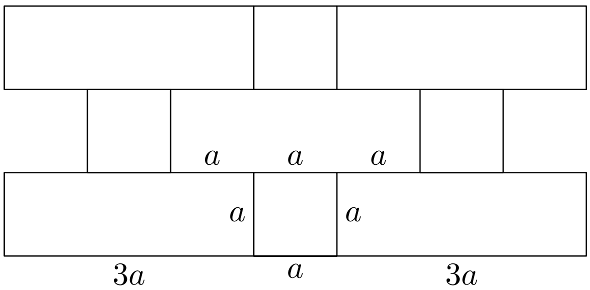

Figure 3

and the probability is

$$p = \frac{3}{\pi}\cdot\frac{l}{a} - \frac{1}{2\pi}\cdot\left(\frac{l}{a}\right)^2. \tag{3}$$

Remark 2. If $\alpha = \frac{\pi}{4}$ the fundamental cell is composed by the regular octagon with the side a and by the square with side a, the probability for this special lattice $R_{\frac{\pi}{4},a}$ is:

$$p = \frac{12}{\pi(3+2\sqrt{2})}\cdot\frac{l}{a} - \frac{3 - \frac{\pi}{2}}{\pi(3+2\sqrt{2})}\cdot\left(\frac{l}{a}\right)^2. \tag{4}$$

2. We also consider, now for the lattice $R := R_{\frac{\pi}{4},a}$, the possibility that $l \geq a$, i.e. the case that s is non-small with respect to the lattice R. The diagonal of the square and the segment between two vertices non-near of the octagon have the lengths $a\sqrt{2}$, $a\sqrt{2+\sqrt{2}}$, $a(\sqrt{2}+1)$, $a\sqrt{4+2\sqrt{2}}$. For the relation between l and these four length we must consider four cases (since the geometric situations in these cases are different):

(i) $a \leq l \leq a\sqrt{2}$,

(ii) $a\sqrt{2} \leq l \leq a\sqrt{2+\sqrt{2}}$,

(iii) $a\sqrt{2+\sqrt{2}} \leq l \leq a(\sqrt{2}+1)$,

(iv) $a(\sqrt{2}+1) \leq l \leq a\sqrt{4+2\sqrt{2}}$.

For all these cases we have a symmetry which permits to consider φ only in the interval $\left[0, \frac{\pi}{4}\right]$.

Case (i): $a \leq l \leq a\sqrt{2}$.
We denote by φ_1 and φ_2 the angles between 0 and $\frac{\pi}{4}$ with the properties $\cos\varphi_1 = \frac{a}{l}$ resp. $\sin\left(\frac{\pi}{4} - \varphi_2\right) = \frac{a}{l\sqrt{2}}$, i.e. $\varphi_2 = \frac{\pi}{4} - \arcsin\frac{a}{l\sqrt{2}}$. We have that $0 \leq \varphi_2 \leq \varphi_1 \leq \frac{\pi}{4}$. Using the same relations of the case with l small with respect to R, we obtain $C_q(\varphi) = \emptyset$ for $0 \leq \varphi < \varphi_1$ and

$$\text{area}C_q(\varphi) = a^2 - (\sin\varphi + \cos\varphi) \cdot al + (\sin\varphi\cos\varphi) \cdot l^2$$

if $\varphi_1 \leq \varphi \leq \frac{\pi}{4}$. Then

$$\int_0^{\frac{\pi}{4}} \text{area}C_q(\varphi)d\varphi = \int_{\varphi_1}^{\frac{\pi}{4}} \text{area}C_q(\varphi)d\varphi = \left(\frac{\pi}{4} - \varphi_1\right) \cdot a^2 - (\cos\varphi_1 - \sin\varphi_1) \cdot al + \left(\frac{1}{4} - \frac{\sin^2\varphi_1}{2}\right) \cdot l^2.$$

For $\varphi \in]0, \varphi_2[$ the set $C_o(\varphi)$ is a hexagon with the sides of length $a + \frac{a}{\sqrt{2}} - l\sin\left(\frac{\pi}{4} - \varphi\right)$ and $a + \frac{a}{\sqrt{2}} - l\sin\left(\frac{\pi}{4} + \varphi\right)$ and the angles $\frac{3\pi}{4}$, $\frac{\pi}{4}$, and $\frac{3\pi}{4}$ (see Fig. 4).

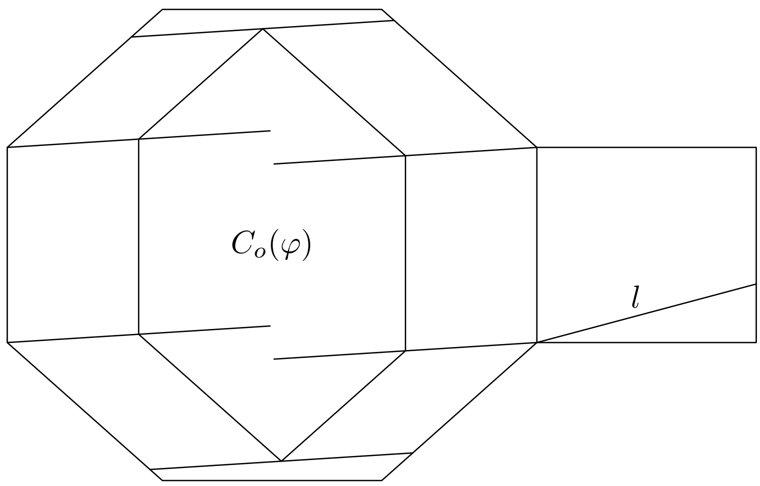

Figure 4

The area of the hexagon is

$$\text{area}C_o(\varphi) = \left(\frac{5}{2} + 2\sqrt{2}\right) \cdot a^2 - (2 - \sqrt{2}) \cdot al\cos\varphi + \frac{l^2}{2}\cos 2\varphi.$$

For $\varphi \in]0, \varphi_2[$ we have that $C_o(\varphi) = C_o\left(\frac{\pi}{4} - \varphi\right)$ and then

$$\int_0^{\varphi_2} \text{area}C_o(\varphi)d\varphi = \int_{\frac{\pi}{4}-\varphi_2}^{\frac{\pi}{4}} \text{area}C_o(\varphi)d\varphi.$$

We have that

$$\int_0^{\varphi_2} \text{area}C_o(\varphi)d\varphi = \varphi_2\left(\frac{5}{2} + 2\sqrt{2}\right) \cdot a^2 - (2 + \sqrt{2}) \cdot al\sin\varphi_2 + \frac{l^2}{4}\sin 2\varphi_2.$$

For $\varphi \in]\varphi_2, \frac{\pi}{4} - \varphi_2[$ we use Figure 2 on the left. The area of $C_o(\varphi)$ is:

$$\text{area}C_o(\varphi) = (2 + \sqrt{2}) \cdot a^2 - [\sin\varphi + (1 + \sqrt{2})\cos\varphi] \cdot al + (\sin\varphi\cos\varphi - \sin^2\varphi) \cdot l^2.$$

Then

$$\int_{\varphi_2}^{\frac{\pi}{4}-\varphi_2} \text{area}C_o(\varphi)d\varphi = \left(\frac{\pi}{2} - 4\varphi_2\right)(1 + \sqrt{2}) \cdot a^2 - 2[\cos\varphi_2 - (1 + \sqrt{2})\sin\varphi_2] \cdot al +$$
$$\left(\frac{\cos 2\varphi_2}{2} - \frac{\sin 2\varphi_2}{2} - \frac{\pi}{8} + \varphi_2\right) \cdot l^2,$$

$$\int_0^{\frac{\pi}{4}} \text{area}C_o(\varphi)d\varphi = \left[\frac{\pi}{2}(1 + \sqrt{2}) + \varphi_2\right] \cdot a^2 - [2\sin\varphi_2 + 2\cos\varphi_2] \cdot al +$$
$$\left(\frac{\cos 2\varphi_2}{2} - \frac{\pi}{8} + \varphi_2\right) \cdot l^2,$$

$$\int_0^{\frac{\pi}{4}} \text{area}C_o(\varphi)d\varphi + \int_0^{\frac{\pi}{4}} \text{area}C_q(\varphi)d\varphi = \left[\frac{\pi}{4}(3 + 2\sqrt{2}) - \varphi_1 + \varphi_2\right] \cdot a^2 -$$
$$[2\sin\varphi_2 + 2\cos\varphi_2 + \cos\varphi_1 - \sin\varphi_1] \cdot al + \left(\frac{\cos 2\varphi_2}{2} - \frac{\sin^2\varphi_1}{2} + \frac{1}{4} - \frac{\pi}{8} + \varphi_2\right) \cdot l^2.$$

From relation (1) and $\int_{\varphi_2}^{\frac{\pi}{4}} \text{area}C_o(\varphi)d\varphi = \frac{\pi}{4}(3 + 2\sqrt{2}) \cdot a^2$ we have that

Theorem 2. *The probability that a random segment s of length l, $a \leq l \leq a\sqrt{2}$, intersects a side of the lattice R is*

$$p_l = \frac{4(\varphi_1 - \varphi_2)}{\pi(3+2\sqrt{2})} + \frac{4}{\pi} \cdot \frac{2\sin\varphi_2 + 2\cos\varphi_2 + \cos\varphi_1 - \sin\varphi_1}{(3+2\sqrt{2})} \cdot \frac{l}{a} - \frac{2\cos 2\varphi_2 - 2\sin^2\varphi_1 + 1 - \frac{\pi}{2} + 4\varphi_2}{\pi(3+2\sqrt{2})} \cdot \left(\frac{l}{a}\right)^2. \tag{5}$$

Remark 3. For $l = a$ we have an extreme case of the Theorem 1 and Theorem 2 and we have that:

$$p = \frac{\frac{9}{\pi} + \frac{1}{2}}{3 + 2\sqrt{2}} \approx 0,577306519. \tag{6}$$

Case (ii): $a\sqrt{2} \le l \le a\sqrt{2+\sqrt{2}}$. If $l > a\sqrt{2}$, then $C_q(\varphi)$ is empty. The computing of $\int_0^{\frac{\pi}{4}} \text{area}C_o(\varphi)d\varphi$ is the same as in case (i) and then:

Theorem 3. *The probability that a random segment s of length l, $a\sqrt{2} \le l \le a\sqrt{2+\sqrt{2}}$, intersects a side of the lattice R is*

$$p = \frac{1 - \frac{4\varphi_2}{\pi}}{3+2\sqrt{2}} + \frac{8(\sin\varphi_2 + \cos\varphi_2)}{\pi(3+2\sqrt{2})}\frac{l}{a} - \frac{2\cos\varphi_2 - \frac{\pi}{2} + 4\varphi_2}{\pi(3+2\sqrt{2})}\left(\frac{l}{a}\right)^2. \tag{7}$$

Case (iii): $a\sqrt{2+\sqrt{2}} \le l \le a(\sqrt{2}+1)$. Let $\varphi_3 \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right]$ be defined by $\cos\varphi_3 = \frac{a}{l}\left(1 + \frac{1}{\sqrt{2}}\right)$. If $\varphi \in \left(0, \frac{\pi}{4} - \varphi_3\right]$ then $C_o(\varphi)$ and $C_o\left(\frac{\pi}{4} - \varphi\right)$ have the same area, and this area is computed in the same way used for $C_o(\varphi)$ in the case (i), Figure 4, i.e.,

$$\text{area}C_o(\varphi) = a^2\left(\frac{5}{2} + 2\sqrt{2}\right) - (2+\sqrt{2}) \cdot al\cos\varphi + \frac{l^2}{2}\cos 2\varphi,$$

then

$$\int_0^{\frac{\pi}{4}-\varphi_3} \text{area}C_o(\varphi)d\varphi = \left(\frac{\pi}{4} - \varphi_3\right) \cdot \left(\frac{5}{2} + 2\sqrt{2}\right) \cdot a^2 - (\sqrt{2}+1) \cdot (\cos\varphi_3 - \sin\varphi_3) \cdot al + (\cos 2\varphi_3) \cdot \frac{l^2}{4}.$$

If $\varphi \in \left[\frac{\pi}{4} - \varphi_3, \varphi_3\right]$, then $C_o(\varphi)$ is a parallelogram with the sides of length $(2+\sqrt{2}) \cdot a - \sqrt{2}l\cos\varphi$ and $(2+\sqrt{2}) \cdot a - l(\sin\varphi + \cos\varphi)$ and the angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. The area of $C_o(\varphi)$ is

$$\text{area}C_o(\varphi) = a^2(4+3\sqrt{2}) - al \cdot [(1+\sqrt{2})\sin\varphi + (3+2\sqrt{2})\cos\varphi] + l^2(\sin\varphi\cos\varphi + \cos^2\varphi). \tag{8}$$

Since we have that:

$$\int_{\frac{\pi}{4}-\varphi_3}^{\varphi_3} \text{area}C_o(\varphi)d\varphi = (4+3\sqrt{2}) \cdot \left(2\varphi_3 - \frac{\pi}{4}\right) \cdot a^2 - [(6+4\sqrt{2}) \cdot \sin\varphi_3 - (2+2\sqrt{2}) \cdot \cos\varphi_3] \cdot al + \left(\varphi_3 - \frac{\pi}{8} + \frac{1}{2}\sin 2\varphi_3 - \frac{1}{2}\cos 2\varphi_3\right) \cdot l^2$$

and $\text{area}C_o(\varphi) = \text{area}C_o\left(\frac{\pi}{4} - \varphi\right)$ for any $\varphi \in \left[0, \frac{\pi}{4} - \varphi_3\right]$ and we obtain that:

$$\int_0^{\frac{\pi}{4}} \text{area}C_o(\varphi)d\varphi = 2\int_0^{\frac{\pi}{4}-\varphi_3} \text{area}C_o(\varphi)d\varphi + \int_{\frac{\pi}{4}-\varphi_3}^{\varphi_3} \text{area}C_o(\varphi)d\varphi =$$

$$\left[(1+\sqrt{2})\frac{\pi}{4} + (3+2\sqrt{2})\cdot\varphi_3\right]\cdot a^2 - [(4+2\sqrt{2})\sin\varphi_3]\cdot al + \left(\varphi_3 - \frac{\pi}{8} + \frac{1}{2}\sin 2\varphi_3\right)\cdot l^2,$$

then we prove the following

Theorem 4. *The probability that a segment s of length l, $a\sqrt{2+\sqrt{2}} \le l \le a(\sqrt{2}+1)$, intersects a side of the lattice R is*

$$p = \frac{2+\sqrt{2}}{3+2\sqrt{2}} - \frac{4\varphi_3}{\pi} + \frac{8}{\pi}\cdot\frac{(2+2\sqrt{2})\sin\varphi_3}{3+2\sqrt{2}}\cdot\frac{l}{a} - \frac{4\varphi_3 - \frac{\pi}{2} + 2\sin 2\varphi_3}{\pi(3+2\sqrt{2})}\cdot\left(\frac{l}{a}\right)^2. \tag{9}$$

Remark 4. If $l = a\sqrt{2+\sqrt{2}}$, then $\varphi_2 = \varphi_3 = \frac{\pi}{8}$ and from relation (7) and (9) we have that

$$p = \frac{1}{2(3+2\sqrt{2})} + \frac{6(1+\sqrt{2})}{\pi(3+2\sqrt{2})} \approx 0,706555366.$$

Case (iv): $a(\sqrt{2}+1) \le l \le a\sqrt{2+2\sqrt{2}}$. In this case, let $\varphi_4 \in \left[0, \frac{\pi}{8}\right]$ defined univocally from equality $\cos\varphi_4 = \frac{a(1+\sqrt{2})}{l}$. If $\varphi \in [0, \varphi_4[\cup\,]\frac{\pi}{4} - \varphi_4, \frac{\pi}{4}\,]$ then we have $C_o(\varphi) = \emptyset$ and if $\varphi \in \left[\varphi_4, \frac{\pi}{4} - \varphi_4\right]$ the set $C_o(\varphi)$ is a parallelogram (see in Fig. 5), then the area is computed with the formula (8).

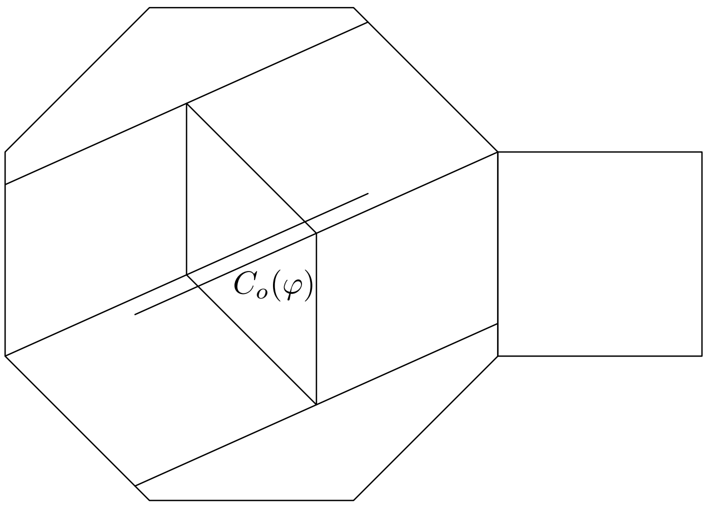

Figure 5

We have that:

$$\int_0^{\frac{\pi}{4}} \text{area} C_o(\varphi) d\varphi = \int_{\varphi_4}^{\frac{\pi}{4}-\varphi_4} \text{area} C_o(\varphi) d\varphi = (4+3\sqrt{2}) \cdot \left(\frac{\pi}{4} - 2\varphi_4\right) \cdot a^2 -$$
$$[(2\sqrt{2}+2)\cdot\cos\varphi_4 - (6+4\sqrt{2})\cdot\sin\varphi_4]\cdot al + \left(\frac{\pi}{8} - \varphi_4 + \frac{1}{2}\cos 2\varphi_4 - \frac{1}{2}\sin 2\varphi_4\right)\cdot l^2,$$

then we obtain the following result:

Theorem 5. *The probability that a segment s of length l,* $a(\sqrt{2}+1) \leq l \leq a\sqrt{4+2\sqrt{2}}$, *intersects a side of the lattice R is*

$$p = \frac{8(4+3\sqrt{2})\varphi_4}{\pi(3+2\sqrt{2})} - \frac{1+\sqrt{2}}{3+2\sqrt{2}} + \frac{8}{\pi} \cdot \frac{(\sqrt{2}+1)\cdot\cos\varphi_4 - (3+3\sqrt{2})\cdot\sin\varphi_4}{(3+2\sqrt{2})} \cdot \frac{l}{a} - \frac{\frac{\pi}{2} - 4\varphi_4 + 2(\cos 2\varphi_4 - \sin 2\varphi_4)}{\pi(3+2\sqrt{2})} \cdot \left(\frac{l}{a}\right)^2 \tag{10}$$

Remark 5. If $l = a(\sqrt{2}+1)$, then we have that $\varphi_4 = 0$ and $\varphi_3 = \frac{\pi}{4}$. From relations (9) and (10) we obtain (the same) probability

$$p = \frac{6}{\pi} - \frac{1}{2} - \frac{1+\sqrt{2}}{3+2\sqrt{2}} \approx 0,9956.$$

References

[1] Duma, A.: *Problems of Buffon type for "non-small" needles.* Rend. Circ. Mat. Palermo, Serie II, Tomo XLVIII (1999), 23–40. Zbl 0944.60024

[2] Duma, A.: *Problems of Buffon type for "non-small" needles (II).* Rev. Roum. Math. Pures Appl., Tome XLIII, 1–2 (1998), 121–135. Zbl 0941.60022

[3] Poincaré, H.: *Calcul des probabilités.* Ed. 2, Carré, Paris 1912.

[4] Rizzo, S.: *Probabilità geometriche di tipo Buffon per reticoli con cellula fondamentale non convessa.* Rend. Semin. Mat. Brescia, vol. 9 (1988), 1–17. Zbl 0672.60024

[5] Stoka, M.: *Probabilités geometriques de type "Buffon" dans le plan euclidien.* Atti Acc. Sci. Torino, vol. 110 (1975–1976), 53–59. Zbl 0351.52005

Received September 2, 2001