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Categories of Locally Diagonal Partial Algebras

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Abstract. We define and study the property of local diagonality for partial algebras and also a pair of related properties. We show that each of the three properties, together with idempotency, gives a cartesian closed initially structured subcategory of the category of all partial algebras of a given type.

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It is well known that cartesian closed categories have a plenty of applications to many branches of mathematics and, in particular, to computer science. An especially important role is played by the cartesian closed categories that are initially structured because such categories have well-behaved function spaces for all pairs of objects. In this note, we define and study three properties of partial algebras: local diagonality, local antidiagonality and weak local antidiagonality. We show that each of the three properties, together with idempotency, gives a cartesian closed initially structured subcategory of the category of all partial algebras of a given arity (with the usual homomorphisms as morphisms)

For the categorical terminology used, see e.g. [4]. Throughout the paper, all categories are considered to be constructs (i.e., concrete categories of structured sets and structure-compatible maps). If A is an object of a category, then the underlying set of A is denoted by $|A|$. A category $\mathcal{K}$ is called *initially structured* [5] if the following three conditions are satisfied:

(1) $\mathcal{K}$ is fibre-small,
(2) All constant maps between objects of $\mathcal{K}$ are morphisms in $\mathcal{K}$,

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(3) $\mathcal{K}$ has uniquely determined initial structures for arbitrary mono-sources (i.e., sources $f_i : X \to A_i$, $i \in I$, such that for any pair of maps $r, s : Y \to X$ from $f_i \circ r = f_i \circ s$ for all $i \in I$ it follows that $r = s$).

If A, B are objects of a given category $\mathcal{K}$, we denote by $\mathrm{Mor}_{\mathcal{K}}(A, B)$ the set of all morphisms from A to B in $\mathcal{K}$.

Definition 1. ([8]) *Let $\mathcal{K}$ be a category with finite products and $\mathcal{S}, \mathcal{T}$ be full and isomorphism closed subcategories of $\mathcal{K}$. Let $\mathcal{T}$ be finitely productive (i.e., closed under finite products) in $\mathcal{K}$. We say that $\mathcal{T}$ is exponential for $\mathcal{S}$ in $\mathcal{K}$ provided that the following condition is satisfied: For any objects $A \in \mathcal{S}$ and $B \in \mathcal{T}$ there exists an object $A^B \in \mathcal{S} \cap \mathcal{T}$ with $|A^B| = \mathrm{Mor}_{\mathcal{K}}(B, A)$ such that the pair (A^B, e), where $e : B \times A^B \to A$ is the evaluation map (given by $e(x, f) = f(x)$), is a co-universal map for A with respect to the functor $B \times - : \mathcal{T} \to \mathcal{K}$.*

If a category $\mathcal{T}$ is exponential for $\mathcal{K}$ in $\mathcal{K}$, then $\mathcal{T}$ is called an *exponential subcategory* of $\mathcal{K}$ (cf. [6]).

The objects A^B from the definition are called *function spaces* and they are uniquely determined up to the isomorphisms carried by identity maps (and hence uniquely determined whenever $\mathcal{K}$ is transportable, i.e., whenever for any object $A \in \mathcal{K}$ and any bijection $f : |A| \to Y$ there exists a unique object $B \in \mathcal{K}$ with $|B| = Y$ such that $f : A \to B$ is an isomorphism in $\mathcal{K}$. Function spaces fulfill a very important condition – the so-called first exponential law $(A^B)^C \backsimeq A^{B \times C}$. If a concrete category $\mathcal{K}$ with finite products is exponential for $\mathcal{K}$ in $\mathcal{K}$, then $\mathcal{K}$ is cartesian closed [3], i.e., the functor $B \times - : \mathcal{K} \to \mathcal{K}$ has a right adjoint (and vice versa whenever all constant maps between objects of $\mathcal{K}$ are morphisms in $\mathcal{K}$). Especially, if $\mathcal{T}$ is exponential for $\mathcal{S}$ in $\mathcal{K}$ and if also $\mathcal{S}$ is finitely productive in $\mathcal{K}$, then $\mathcal{S} \cap \mathcal{T}$ is cartesian closed.

Let K be a set. By a *K-ary partial algebra* we understand a pair (X, p) where X is a set and p is a K-ary partial operation on X, i.e., a partial map $p : X^K \to X$. As usual, total maps (and thus also total operations) are considered to be special cases of partial ones. We denote by D_p the domain of the partial operation p, i.e., the subset of X^K having the property that $p(x_k; k \in K)$ is defined if and only if $(x_k; k \in K) \in D_p$. Thus, $(x_k; k \in K) \in D_p$ is implicitly assumed whenever we write $p(x_k; k \in K) = x$ ($x \in X$ an element). Given a pair of K-ary partial algebras $G = (X, p)$ and $H = (Y, q)$, by a *homomorphism* of G into H we understand any (total) map $f : X \to Y$ such that $p(x_k; k \in K) = x \Rightarrow q(f(x_k); k \in K) = f(x)$. The set of all homomorphisms of G into H will be denoted by $\mathrm{Hom}(G, H)$. We denote by Pal_K the category of K-ary partial algebras with homomorphisms as morphisms. Clearly, Pal_K is transportable and has products (given by direct products, cf. [1]).

Definition 2. ([8]) *Let $G = (X, p)$, $H = (Y, q)$ be K-ary partial algebras. The power of G and H is the K-ary partial algebra $[H, G] = (Hom(G, H), r)$ where r is given by $r(f_k; k \in K) = f$ if and only if $f \in Hom(H, G)$ is a unique homomorphism with the property that $q(y_k; k \in K) = y \Rightarrow p(f_k(y_k); k \in K) = f(y)$.*

A K-ary partial algebra (X,p) is said to be *idempotent* if for any $x \in X$ we have $p(x_k; k \in K) = x$ whenever $x_k = x$ for all $k \in K$. The full subcategory of Pal_K whose objects are precisely the idempotent K-ary partial algebras will be denoted by $IPal_K$. Clearly, $IPal_K$ is productive (i.e., closed under products) in Pal_K and initially structured. (The mono-sources in $IPal_K$ are precisely the sources separating points; given a mono-source $f_i : X \to (X_i, p_i), i \in I$, in $IPal_K$, the initial K-ary partial operation p on X for this mono-source is given by $p(x_k; k \in K) = x$ if and only if $p_i(f_i(x_k); k \in K) = f_i(x)$ for each $i \in I$.)

Definition 3. *A K-ary partial algebra (X,p) is called locally diagonal if, whenever $(x_{kl}; k,l \in K) \in X^{K\times K}$ is an element such that there exists $x \in X$ with $x_{kk} = x$ for all $k \in K$, from $(x_{kl}; l \in K) \in D_p$ for all $k \in K$ it follows that $p(p(x_{kl}; l \in K); k \in K) = x$.*

We denote by $LdPal_K$ the full subcategory of Pal_K given by the K-ary partial algebras which are locally diagonal. Further, we put $LdIPal_K = IPal_K \cap LdPal_K$. Clearly, $LdPal_K$ is transportable and has products (it is isomorphism closed and productive in Pal_K), and $LdIPal_K$ is productive in $LdPal_K$ and initially structured (it is closed under formation of initial structures for monosources in $IPal_K$).

Remark 1. By Definition 3, a partial groupoid, i.e., a partial binary algebra, X is locally diagonal if and only if, for any $x, y, z \in X$, we have $(xy)(zx) = x$ whenever xy and zx are defined (we use multiplicative denotation for partial binary operations).

Examples.

1. Let X be a set and for any $l \in K$ let $p_l : X^K \to X$ be the l-th projection, i.e., the K-ary (total) operation on X given by $p_l(x_k; k \in K) = x_l$. Then (X, p_l) is idempotent and locally diagonal for any $l \in K$.

2. Each total n-ary algebra (n a positive integer) which is diagonal in the sense of [7] is idempotent and locally diagonal. In particular, each rectangular band (see [2]) is idempotent and locally diagonal.

3. A K-ary partial algebra (X,p) is said to be diagonal (cf. [9], Remark 2) provided that, whenever $(x_{kl}; k,l \in K) \in X^{K\times K}$ is an element such that $(x_{kl}; l \in K) \in D_p$ for each $k \in K$, we have $p(p(x_{kl}; l \in K); k \in K) = x$ if and only if $p(x_{kk}; k \in K) = x$. It is obvious that each idempotent and diagonal partial algebra is locally diagonal.

4. Let X be a set and let ρ be a binary relation on X (i.e., $\rho \subseteq X^2$). Define a binary partial operation Δ_ρ on X by $x\Delta_\rho y = x \Leftrightarrow x\rho y$. Then the partial groupoid (X, Δ_ρ) has the following properties:

(1) ρ is reflexive if and only if (X, Δ_ρ) is idempotent,
(2) If ρ is symmetric, then (X, Δ_ρ) is locally diagonal (but not vice versa in general),
(3) (X, ρ) is a tolerance space (i.e., ρ is both reflexive and symmetric) if and only if (X, Δ_ρ) is both idempotent and locally diagonal.

Proposition 1. *Let (X,p) be an idempotent K-ary partial algebra with card $K < \aleph_0$. Let, for any $(x_k; k \in K) \in D_p$, any $k_0 \in K$, and any $(z_k; k \in K) \in D_p$, from $z_{k_0} = x_{k_0}$ it* *follows that the element* $(y_k; k \in K) \in X^K$ *given by* $y_{k_0} = p(z_k; k \in K)$ *and* $y_k = x_k$ *for all* $k \in K - \{k_0\}$ *fulfills* $p(x_k; k \in K) = p(y_k; k \in K)$*. Then* (X, p) *is locally diagonal.*

Proof. Let the assumptions of the statement be fulfilled. Without loss of generality we can suppose $K = \{1, \dots, n\}$, n a positive integer. Let $(x_{kl}; k, l \in K) \in X^{K \times K}$ be an element such that there exists $x \in X$ with $x = x_{kk}$ for all $k \in K$ and let $(x_{kl}; l \in K) \in D_p$ for each $k \in K$. Then we have $x = p(x_{11}, \dots, x_{nn}) = p(p(x_{11}, \dots, x_{1n}), x_{22}, \dots, x_{nn}) = p(p(x_{11}, \dots, x_{1n}), p(x_{21}, \dots, x_{2n}), x_{33}, \dots, x_{nn}) = \cdots = p(p(x_{11}, \dots, x_{1n}), p(x_{21}, \dots, x_{2n}), \dots, p(x_{n1}, \dots, x_{nn}))$. Hence (X, p) is locally diagonal. □

Remark 2. Let X be an idempotent partial groupoid. If, for any elements $x, y, z, t \in X$ for which xy, xz and ty are defined, we have $xy = (xz)y = x(ty)$, then, by Proposition 1, X is locally diagonal.

Proposition 2. *Let* (X, p) *be an idempotent locally diagonal* K*-ary partial algebra. Let* $(x_k; k \in K) \in D_p$, $p(x_k; k \in K) = y$*, let* $k_0 \in K$ *and put* $y_{k_0} = y$ *and* $y_k = x_{k_0}$ *for all* $k \in K - \{k_0\}$*. Then* $p(y_k; k \in K) = x_{k_0}$*.*

Proof. Put $x_{kl} = x_{k_0}$ for all $k, l \in K$, $k \neq k_0$, and $x_{k_0 l} = x_l$ for all $l \in K$. Then $x_{kk} = x_{k_0}$ for each $k \in K$. We have $p(x_{k_0 l}; l \in K) = y$ and, because (X, p) is idempotent, $p(x_{kl}; l \in K) = x_{k_0}$ for each $k \in K - \{k_0\}$. Since (X, p) is locally diagonal, we get $p(p(x_{kl}; l \in K); k \in K) = x_{k_0}$. As $p(x_{kl}; l \in K) = y_k$ for each $k \in K$, the proof is complete. □

Remark 3. Let X be an idempotent locally diagonal partial groupoid and let $x, y \in X$. Then, by Proposition 2, $(xy)x = x$ and $y(xy) = y$ whenever xy is defined.

We will need the following result proved in [8]:

Lemma 1. $IPal_K$ *is an exponential subcategory of* Pal_K *and the corresponding function spaces are given by powers.*

Theorem 1. $LdIPal_K$ *is an exponential subcategory of* $LdPal_K$ *and the corresponding function spaces are given by powers.*

Proof. Let $G = (X, p) \in LdPal_K$, $H = (Y, q) \in LdIPal_K$ and let $[H, G] = (\mathrm{Hom}(H, G), r)$. Let $(f_{kl}; k, l \in K) \in (\mathrm{Hom}(H, G))^{K \times K}$ be an element such that there exists $f \in \mathrm{Hom}(H, G)$ with $f = f_{kk}$ for every $k \in K$. For each $k \in K$, let $(f_{kl}; l \in K) \in D_r$, $r(f_{kl}; l \in K) = f_k$. Let $(y_k; k \in K) \in D_q$, $q(y_k; k \in K) = y$. For each $k, l \in K$ put $y_{kl} = y_k$ whenever $k \neq l$ and $y_{kk} = y$. By Proposition 2, we have $q(y_{kl}; l \in K) = y_k$ for every $k \in K$. Put $x_{kl} = f_{kl}(y_{kl})$ for each $k, l \in K$. Then $p(x_{kl}; l \in K) = p(f_{kl}(y_{kl}); l \in K) = f_k(y_k)$ and $x_{kk} = f_{kk}(y_{kk}) = f(y)$ for every $k \in K$. As (X, p) is locally diagonal, we have $p(f_k(y_k); k \in K) = f(y)$. Since (Y, q) is idempotent, f is uniquely determined and hence $r(f_k; k \in K) = f$. Therefore, $[H, G] \in LdPal_K$ and the statement follows from Lemma 1. □

Corollary 1. *$LdIPal_K$ is an initially structured cartesian closed category.*

Remark 4. Denote by $DIPal_K$ the full subcategory of $IPal_K$ given by the idempotent K-ary partial algebras which are diagonal – see Example 3. Then we get a chain of three initially structured cartesian closed categories $DIPal_K \subseteq LdIPal_K \subseteq IPal_K$ where $\subseteq$ denotes the full categorical inclusion (the cartesian closedness of $DIPal_K$ is proved in [9]).

With respect to Definition 3, it is natural to define:

Definition 4. *A K-ary partial algebra (X,p) is called locally antidiagonal if, whenever $(x_{kl}; k,l \in K) \in X^{K\times K}$ is an element such that there exists $x \in X$ with $x_{kk} = x$ for all $k \in K$, from $(x_{kl}; l \in K) \in D_p$ for all $k \in K$ and $p(p(x_{kl}; l \in K); k \in K) = x$ it follows that $x_{kl} = x$ for all $k,l \in K$.*

We denote by $LaPal_K$ the full subcategory of Pal_K whose objects are precisely the locally antidiagonal K-ary partial algebras. Further, we put $LaIPal_K = IPal_K \cap LaPal_K$ Clearly, $LaPal_K$ is productive in Pal_K and $LaIPal_K$ is an initially structured category.

Example 5. Let X be a set and let $*$ be the partial operation on the set $\mathcal{P}X$ of all subsets of X given as follows: $A * B$ is defined if and only if either $A = B$ or A and B are nonempty and disjoint, and then $A * B = A \cap B$. Then $(\mathcal{P}X, *)$ is a locally antidiagonal (and idempotent) partial groupoid.

Theorem 2. *$IPal_K$ is exponential for $LaPal_K$ in Pal_K and the corresponding function spaces are given by powers.*

Proof. Let $G = (X,p) \in LaPal_K$, $H = (Y,q) \in IPal_K$ and let $[H,G] = (\mathrm{Hom}(H,G), r)$. Let $(f_{kl}; k,l \in K) \in (\mathrm{Hom}(H,G))^{K\times K}$ be an element such that there exists $f \in \mathrm{Hom}(H,G)$ with $f_{kk} = f$ for all $k \in K$, let $(f_{kl}; l \in K) \in D_r$ for each $k \in K$ and let $r(r(f_{kl}; l \in K); k \in K) = f$. Let $y \in Y$ be an arbitrary element. Then $(f_{kl}(y); k,l \in K) \in X^{K\times K}$, $f_{kk}(y) = f(y)$ for each $k \in K$ and, since H is idempotent, $(f_{kl}(y); l \in K) \in D_p$ for each $k \in K$ and $p(p(f_{kl}(y); l \in K); k \in K) = f(y)$. As G is locally antidiagonal, we have $f_{kl}(y) = f(y)$ for all $k,l \in K$. Hence $f_{kl} = f$ for all $k,l \in K$. It follows that $[H,G]$ is locally antidiagonal. Now the statement follows from Lemma 1. □

Corollary 2. *$LaIPal_K$ is a cartesian closed initially structured category.*

Clearly, a K-ary partial algebra (X,p) is locally diagonal if and only if, whenever $(x_{kl}; k,l \in K) \in X^{K\times K}$ is an element such that there exists $x \in X$ with $x_{kk} = x$ for all $k \in K$, from $(x_{kl}; l \in K) \in D_p$ for all $k \in K$ it follows that $p(p(x_{kl}; l \in K); k \in K) = x$ and from $(x_{kl}; k \in K) \in D_p$ for all $l \in K$ it follows that $p(p(x_{kl}; k \in K); l \in K) = x$. With respect to this fact, the notion of local antidiagonality, which seems to be rather strong, can be weakened as follows:

Definition 5. *A K-ary partial algebra (X,p) is called weakly locally antidiagonal if, whenever $(x_{kl}; k,l \in K) \in X^{K\times K}$ is an element such that there exists $x \in X$ with $x_{kk} = x$*

for all $k \in K$*, from* $(x_{kl}; l \in K) \in D_p$ *for all* $k \in K$*,* $(x_{kl}; k \in K) \in D_p$ *for all* $l \in K$*, and* $p(p(x_{kl}; l \in K); k \in K) = p(p(x_{kl}; k \in K); l \in K) = x$ *it follows that* $x_{kl} = x$ *for all* $k, l \in K$*.*

We denote by $WlaPal_K$ the full subcategory of Pal_K whose objects are precisely the weakly locally antidiagonal K-ary partial algebras, and we put $WlaIPal_K = IPal_K \cap WlaPal_K$. Of course, $WlaPal_K$ is productive in Pal_K and $WlaIPal_K$ is initially structured.

Example 6. Let X be a set, ρ a binary relation on X, and let (X, Δ_ρ) be the partial groupoid given in Example 4. Then (X, Δ_ρ) is weakly locally antidiagonal if and only if ρ is antisymmetric.

Theorem 3. *$IPal_K$ is exponential for $WlaPal_K$ in Pal_K and the corresponding function spaces are given by powers.*

Proof. The proof of the statement is analogous to the proof of Theorem 2. □

Corollary 3. *$WlaIPal_K$ is a cartesian closed initially structured category.*

Remark 5. Again (see Remark 4), we have a chain of three cartesian closed initially structured categories $LaIPal_K \subseteq WlaIPal_K \subseteq IPal_K$.

Example 7. It is well known that the category *Rel* of sets endowed with a reflexive binary relation, with relational homomorphisms as morphisms, is an initially structured cartesian closed category (see e.g. [9]). Given a pair $G = (X, \rho)$ and $H = (Y, \sigma)$ of objects of *Rel*, their function space in *Rel* is defined by $G^H = (\mathrm{Hom}(H, G), \tau)$ where $\tau \subseteq (\mathrm{Hom}(H, G))^2$ is given by $f\tau g$ if and only if $x\rho y \Rightarrow f(x)\sigma g(y)$. Putting $F(X, \rho) = (X, \Delta_\rho)$ for each object $(X, \rho) \in Rel$ – see Example 4 – we get a full concrete embedding of *Rel* into the category $IPal_2$ of idempotent partial groupoids. Thus, *Rel* can be considered to be a full subcategory of $IPal_2$ and then it can easily be seen that the function spaces in *Rel* are inherited from $IPal_2$. In other words, for any pair G, H of objects of *Rel* we have $F(G^H) = [F(H), F(G)]$. We denote by Tol and Pos the full subcategories of *Rel* whose objects are precisely the tolerance spaces and the pseudo-ordered sets (i.e., sets endowed with a reflexive and antisymmetric binary relation) respectively. Then F (restricted to Tol and Pos) is a full concrete embedding of Tol into $LdIPal_2$ and of Pos into $WlaIPal_2$, i.e., Tol and Pos can be considered to be full subcategories of $LdIPal_2$ and $WlaIPal_2$, respectively. By [10], Tol and Pos are cartesian closed initially structured categories and their function spaces are inherited from *Rel*. By Corollaries 1 and 2, also $LdIPal_2$ and $WlaIPal_2$ are cartesian closed initially structured categories and their function spaces are inherited from $IPal_2$. Thus, the function spaces in Tol and Pos are inherited from $IPal_2$, hence from $LdIPal_2$ and $WlaIPal_2$, respectively.

Remark 6. The definitions 2–5 can be extended in a natural way to universal partial algebras of an arbitrary type $(K_i; i \in I)$, i.e., pairs $(X, (p_i; i \in I))$ where (X, p_i) is a K_i-ary partial algebra for each $i \in I$. All results proved for K-ary partial algebras will remain valid also for universal partial algebras of a given type.

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