Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 2, 581-594.

# Radii of Simplices and some Applications to Geometric Inequalities

*Dedicated to the memory of Bernulf Weißbach*

**René Brandenberg  Thorsten Theobald**

*Zentrum Mathematik, Technische Universität München*
*Boltzmannstr. 3, D–85747 Garching bei München*
*e-mail: brandenb@ma.tum.de  theobald@ma.tum.de*

**Abstract.** We review some recent results on inner and outer $j$-radii of simplices and general convex bodies. In particular, we discuss two lines of research whose present-day developments were strongly influenced by work of Bernulf Weißbach: radii of regular simplices and geometric inequalities among the radii.

MSC 2000: 52A20, 52A40, 52B12
Keywords: Radii, geometric inequalities, simplices, convex bodies, enclosing cylinders, orthogonal projections

## 1. Introduction

Let $\mathcal{L}_{j,n}$ be the set of all $j$-dimensional linear subspaces (hereafter *$j$-spaces*) in $n$-dimensional Euclidean space $\mathbb{E}^n$. The *inner $j$-radius* $r_j(C)$ of a convex body $C \subset \mathbb{E}^n$ is the radius of a largest *$j$-ball* ($j$-dimensional ball) contained in $C$, and the *outer $j$-radius* $R_j(C)$ is the radius of the smallest enclosing *$j$-ball* in an optimal orthogonal projection of $C$ onto a $j$-space $J \in \mathcal{L}_{j,n}$, where the optimization is performed over $\mathcal{L}_{j,n}$.

Studying radii of polytopes is a fundamental topic in convex geometry (see [1, 2, 9, 11, 12, 15]. Applications in functional analysis, statistics, computer vision, robotics, and medical diagnosis (see [13] and the references therein) have initiated additional interest from the computational point of view. In the investigation of the inner and outer radii, the following two questions immediately arise.

a) What are the radii of special classes of convex bodies, such as (regular) simplices?

b) Is there a general order on the radii?

The main purpose of this paper is to review the developments on these questions and to stress the influences given by Weißbach, which then allows to provide some further generalizations and refinements.

The modern developments on the first question were initiated by Pukhov [17] and Weißbach [19] who computed the $j$-radii of the regular $n$-simplex for most pairs $(j, n)$. However, for the remaining pairs $(j, n)$ these values were not known until recently (see [7]). In section 3 we present a unifying proof consisting of the major geometric ideas within the various parts of the meanwhile complete characterization.

Concerning relations among the radii, it is easy to see that for any convex body $C \subset \mathbb{E}^n$ we have $r_n(C) \leq \cdots \leq r_1(C)$, $R_1(C) \leq \cdots \leq R_n(C)$, and $R_1(C) \leq r_1(C)$. Moreover, in [14] it was shown $r_j(C) < R_{n+1-j}(C)$. The stated geometric inequalities are also displayed in Figure 1. It is not difficult to see that for important classes of bodies (e.g., symmetric

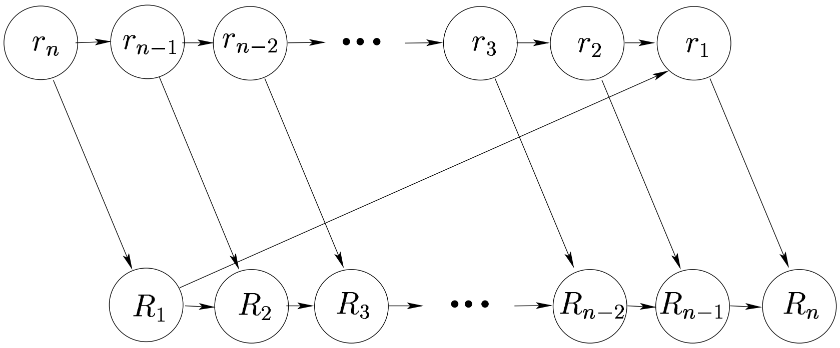


Figure 1: An arc between two radii represents a less than or equal relationship (from the origin to the sink), which holds for all $n$-dimensional bodies.

bodies) $R_{n-1}(C) \leq r_1(C)$ holds. Eggleston [10] first showed that for every $n \geq 3$ there are also bodies with $R_{n-1}(C) > r_1(C)$. A much simpler proof was provided by Weißbach [19], using the regular simplices for $n \geq 4$ and a special construction for $n = 3$, which we will refer as the Weißbach polytope.

In Section 4 this result is strengthened twofold. On the one hand for the first time the relevant radii of the Weißbach polytope are explicitly computed, which can then be used to give an explicit quotient for the ratio of $R_2$ and $r_1$. On the other hand we review a generalization, which was developed recently [8], showing that even for $n \geq 4$ there exist bodies $C$ such that $R_j > r_1$ for all $j \geq 2$. From these results it follows that Figure 1 is complete in the sense that for any two radii which are not connected by a directed path there exist bodies $C_1, C_2$ such that the relationship between the two radii is 'less than' for $C_1$ and 'greater than' for $C_2$.

Finally, we give an application of these geometric inequalities. In [12] it was shown that $R_n(C) = \sup_{\substack{S \subset C \\ S \text{ Simplex}}} R_n(S)$ for every convex body $C$. Using the concept of totally non-spherical bodies, we show that this result is not true for general outer $j$-radii.

## 2. Preliminaries

Throughout the paper we work in Euclidean space $\mathbb{E}^n$, i.e., $\mathbb{R}^n$ with the usual scalar product $x \cdot y = \sum_{i=1}^n x_i y_i$ and norm $\|x\| = (x \cdot x)^{1/2}$. $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ denote the (closed) unit ball and unit sphere, respectively. For a set $A \subset \mathbb{E}^n$, the linear hull of $A$ is denoted by $\operatorname{lin} A$ and the convex hull of $A$ is denoted by $\operatorname{conv} A$.

A set $C \subset \mathbb{E}^n$ is called a *(convex) body* if it is bounded, closed, convex and contains an interior point. Let $1 \leq j \leq n$. A *j-flat* is an affine subspace of dimension $j$, and a *j-cylinder* is a set of the form $J + \rho\mathbb{B}^n$ with an $(n-j)$-flat $J$ and $\rho > 0$. For a body $C \subset \mathbb{E}^n$, the outer $j$-radius $R_j(C)$ of $C$ (as defined in the introduction) is the radius $\rho$ of a smallest enclosing $j$-cylinder of $C$. It follows from a standard compactness argument that this minimal radius is attained (see, e.g., [12]). Let $1 \leq j \leq k < n$. If $C' \subset \mathbb{E}^n$ is a compact, convex set whose affine hull $F$ is a $k$-flat then $R_j(C')$ denotes the radius of a smallest enclosing $j$-cylinder $\mathcal{C}'$ relative to $F$, i.e., $\mathcal{C}' = J' + R_j(C')(\mathbb{B}^n \cap E)$ with a $(k-j)$-flat $J' \subset F$ and $E$ the linear $k$-space parallel to $F$.

For a simplex $S = \operatorname{conv}\{v^{(1)}, \ldots, v^{(n+1)}\}$, let $S^{(i)}$ denote the facet of $S$ which does not contain the vertex $v^{(i)}$, $i = 1, \ldots, n+1$. Whenever a statement is invariant under orthogonal transformations and translations we denote by $T^n$ the regular simplex in $\mathbb{E}^n$ with edge length $\sqrt{2}$. The reason for the choice of $\sqrt{2}$ stems from the convenient embedding of $T^n$ into $\mathbb{E}^{n+1}$. Let $\mathcal{H}_\alpha^n = \{x \in \mathbb{E}^{n+1} : \sum_{i=1}^{n+1} x_i = \alpha\}$. Then the *standard embedding* $\mathbf{T}^n$ of $T^n$ is defined by

$$\mathbf{T}^n \; := \; \operatorname{conv}\left\{e^{(i)} \in \mathbb{E}^{n+1} \, : \, 1 \leq i \leq n+1\right\} \subset \mathcal{H}_1^n \, ,$$

where $e^{(i)}$ denotes the $i$-th unit vector in $\mathbb{E}^{n+1}$. By $\mathcal{S}^{n-1} := \mathbb{S}^n \cap \mathcal{H}_0^n$ we denote the set of unit vectors parallel to the $n$-flat in which $\mathbf{T}^n$ is embedded.

A $j$-cylinder $\mathcal{C}$ containing some simplex $S$ is called a *circumscribing* cylinder of $S$ if all the vertices of $S$ are contained in the boundary of $\mathcal{C}$.

## 3. Outer radii of regular simplices

One access to understand the geometry of functionals such as radii and volumes is to investigate their behavior on special subclasses of convex bodies. Here we mainly consider the class of regular simplices. In particular, for the analysis of geometric inequalities, they often attain the extreme values. In [19, Theorem 1] Weißbach showed that

$$R_{n-1}(T^n) \geq \sqrt{\frac{n-1}{n+1}} \; , \text{ with equality if and only if } n \text{ is odd.} \tag{3.1}$$

While working on generalizations of (3.1) we recently discovered a paper of Pukhov [17] in which the following result is obtained.

**Theorem 1.** *Let $1 \leq j \leq n$ such that $n$ is odd or $j \notin \{1, n-1\}$. Then $R_j(T^n) = \sqrt{\frac{j}{n+1}}$.*

We describe an alternative proof discovered in [5]. Before doing so, some auxiliary statements are needed.

**Lemma 2.** *For $1 \le j \le n$ it holds $R_j(T^n) \ge \sqrt{\frac{j}{n+1}}$, and in case of equality every minimal enclosing $j$-cylinder of $T^n$ is a circumscribing $j$-cylinder of $T^n$.*

*Proof.* Let $s^{(1)}, \dots, s^{(n)}$ be an orthonormal basis of $\mathcal{H}_0^n$, $p \in \mathcal{H}_1^n$ and $\rho > 0$ such that $\mathcal{C} = J + \rho(\mathbb{B}^{n+1} \cap \mathcal{H}_0^n)$ is an enclosing $j$-cylinder of $\mathbf{T}^n$ with $J = p + \mathrm{lin}\{s^{(j+1)}, \dots, s^{(n)}\}$. Further let $x \mapsto Px$ be the orthogonal projection onto $\mathrm{lin}\{s^{(1)}, \dots, s^{(j)}\}$, where $P = \sum_{k=1}^{j} s^{(k)}(s^{(k)})^T \in \mathbb{E}^{(n+1)\times(n+1)}$. Then

$$\|Pe^{(i)}\|^2 = \sum_{k=1}^{j} (s_i^{(k)})^2 \ . \tag{3.2}$$

Assume there exists a point $x \in \mathcal{H}_0^n$ such that $\|x - Pe^{(i)}\| < \sqrt{j/(n+1)}$ for all $i = 1, \dots, n+1$. Since $\sum_{i=1}^{n+1} s_i^{(k)} = 0$ and $\sum_{i=1}^{n+1} (s_i^{(k)})^2 = 1$, we obtain from summing over all $i$

$$\begin{aligned} j > \sum_{i=1}^{n+1} \|x - Pe^{(i)}\|^2 &= (n+1)\|x\|^2 - 2\sum_{i=1}^{n+1}\sum_{k=1}^{j} s_i^{(k)} x^T s^{(k)} + \sum_{i=1}^{n+1}\sum_{k=1}^{j} (s_i^{(k)})^2 \\ &= (n+1)\|x\|^2 + j \ge j \end{aligned} \tag{3.3}$$

which is a contradiction. This proves the first part of the theorem. For the second part, it is easy to see that if $R_j(T^n) = \sqrt{j/(n+1)}$ then (3.3) becomes an equality chain if and only if $x = 0$ and $\|x - Pe^{(i)}\|^2 = j/(n+1)$ for all $1 \le i \le n+1$. □

If a sequence of orthogonal vectors $s^{(1)}, \dots, s^{(j)} \in \mathcal{S}^{n-1}$ satisfies $\sum_{k=1}^{j} (s_i^{(k)})^2 = j/(n+1)$ for all $i = 1, \dots, n+1$, we call it a *good subspace basis* (shortly, *gsb*). The proof of Lemma 2 implies that $R_j(T^n) = \sqrt{j/(n+1)}$ if and only if there exists a gsb for the pair $(j, n)$. It is not hard to see that

$$\text{any orthonormal basis } s^{(1)}, \dots, s^{(n)} \text{ of } \mathcal{H}_0^n \text{ is a gsb.} \tag{3.4}$$

It directly follows $R_n(T^n) = \sqrt{n/(n+1)}$, which is a long known fact, not only since the famous work of Jung [16].

As an easy consequence of (3.4), the basis completion theorem implies

$$R_j(T^n) = \sqrt{\frac{j}{n+1}} \ \text{ if and only if } R_{n-j}(T^n) = \sqrt{\frac{n-j}{n+1}} \ . \tag{3.5}$$

Property (3.5) immediately shows that Weißbach's 'if and only if $n$ is odd' statement on equality in (3.1) corresponds directly to the old result, shown by Steinhagen [18], about the (half) width of regular simplices.

$$R_1(T^n) = \begin{cases} \sqrt{\frac{1}{n+1}}, & \text{if } n \text{ is odd,} \\ \sqrt{\frac{n+1}{n(n+2)}}, & \text{if } n \text{ is even.} \end{cases} \tag{3.6}$$

In fact, Steinhagen first showed that all minimal enclosing 1-cylinders of a regular simplex are circumscribing. It was shown in [12] that minimal enclosing 1-cylinders are always circumscribing even for general simplices, and a similar statement for general $j$ will be given in Proposition 6.

Weißbach showed that the axis direction $s^{(n)} \in \mathcal{S}^{n-1}$ of a minimal enclosing $(n-1)$-cylinder of $\mathbf{T}^n$ for odd $n$ must be of the form $\sqrt{1/(n+1)}(1,\dots,1,-1,\dots,-1)^T$ where both the number of 1's and $(-1)$'s are $(n+1)/2$. However, for the analysis in which cases the lower bound $\sqrt{j/(n+1)}$ holds for general $j$, it seems to be more convenient to describe the space where $\mathbf{T}^n$ is projected on.

**Definition 3.** *Let $1 \le j < m$. A sequence $v^{(1)},\dots,v^{(m)} \in \mathbb{S}^{j-1}$ is called $(j,m)$-isotropic if $\sum_{i=1}^m v^{(i)} = 0$ and $\sum_{i=1}^m v^{(i)}(v^{(i)})^T = m/j\ I$, where $I$ denotes the $j$-dimensional identity matrix. A polytope $P \subset \mathbb{B}^j$ is called $(j,m)$-isotropic if there exists a $(j,m)$-isotropic sequence $v^{(1)},\dots,v^{(m)}$ whose convex hull is $P$.*

The next two propositions are taken from [5]. Their proofs are quite simple but purely technical.

**Proposition 4.** *Let $1 \le j \le n$. There exists a gsb $s^{(1)},\dots,s^{(j)} \in \mathcal{S}^{n-1}$ if and only if there exists a $(j,n+1)$-isotropic polytope $P \subset \mathbb{B}^j$. Moreover, for any gsb $s^{(1)},\dots,s^{(j)} \in \mathcal{S}^{n-1}$ we can choose $P$ such that it is the projection of $\mathbf{T}^n$ on $\mathrm{lin}\{s^{(1)},\dots,s^{(j)}\}$ up to a linear transformation.*

Obviously, for every odd $n \ge 1$ the unique $(1,n+1)$-isotropic polytope is $[-1,1]$, and the underlying sequence consists of 1's and $(-1)$'s, both $(n+1)/2$ times.

Also it is easy to see that every regular $j$-dimensional polytope with $n+1$ vertices, all on $\mathbb{S}^{n-1}$, is $(j,n+1)$-isotropic. Hence, at least for $j=2$ there exist $(2,n+1)$-isotropic polytopes for all $n \ge 2$. Moreover, for odd $n$ we can choose a 3-dimensional prism with a regular $((n+1)/2)$-gon as the base. By appropriately choosing the height of the prism, it becomes $(3,n+1)$-isotropic. Proposition 5 shows how to combine lower dimensional isotropic polytopes to obtain higher dimensional ones.

**Proposition 5.** *Let $0 \le j_i < m_i$, $i=1,2$ such that $m_2 j_1 > m_1 j_2$. Let $j = j_1 + j_2$, $m = m_1 + m_2$, $\alpha = \sqrt{(m_2 j_1 - m_1 j_2)/m_2 j}$, and $\beta = \sqrt{m j_2 / m_2 j}$, and suppose there exists a $(j_1,m_1)$-isotropic polytope $K_1 = \mathrm{conv}\{u^{(1)},\dots,u^{(m_1)}\}$, a $(j_1,m_2)$-isotropic polytope $K_2 = \mathrm{conv}\{v^{(1)},\dots,v^{(m_2)}\}$, and a $(j_2,m_2)$-isotropic polytope $K_3 = \mathrm{conv}\{w^{(1)},\dots,w^{(m_2)}\}$, such that $K' = \mathrm{conv}\left\{\sqrt{\tfrac12}\begin{pmatrix} v^{(1)} \\ w^{(1)}\end{pmatrix},\dots,\sqrt{\tfrac12}\begin{pmatrix} v^{(m_2)} \\ w^{(m_2)}\end{pmatrix}\right\}$ is a $(j,m_2)$-isotropic polytope. Then there exists a $(j,m)$-isotropic polytope*

$$K = \mathrm{conv}\left\{\begin{pmatrix} u^{(1)} \\ 0\end{pmatrix},\dots,\begin{pmatrix} u^{(m_1)} \\ 0\end{pmatrix},\begin{pmatrix} \alpha v^{(1)} \\ \beta w^{(1)}\end{pmatrix},\dots,\begin{pmatrix} \alpha v^{(m_2)} \\ \beta w^{(m_2)}\end{pmatrix}\right\}.$$

The reader may convince himself that neither it is possible to construct a $(1,n+1)$-isotropic polytope by the additive rule if $n$ is even, nor it is possible to construct an $(n-1,n+1)$-isotropic polytope by the additive rule at all.

Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* By Proposition 4, it suffices to show the existence of an $(j, n+1)$-isotropic polytope for every $(j, n)$ with $n$ odd or $j \notin \{1, n-1\}$. We do an inductive proof over $j$ and $n$ using Proposition 5 to construct the higher-dimensional isotropic polytopes from lower dimensional ones.

From (3.6) and since every regular $(n+1)$-gon with vertices on $\mathbb{S}^1$ is $(2, n+1)$-isotropic we see that the claim is true for pairs $(j, n)$ with $j \leq 2$. Moreover, by (3.5) the claim is true for $j \geq n-2$.

Now assume that the claim is true for every pair $(j', n')$ with $j' < j$, $n' \leq n$ or $j' \leq j$, $n' < n$. Due to the initial statements we can assume $j \geq 3$ and because of (3.5) that $j < (n+1)/2$. We distinguish three cases:

Case 1, $(j, n+1) = (3, 9)$:
In this case we choose $j_1 = 2, j_2 = 1, m_1 = 3, m_2 = 6$. For sure $K_1 = K_2 = \sqrt{3/2}\, T^2$ are $(2,3)$-isotropic and also $(2,6)$-isotropic by duplicating every vertex. Now, $K_3 = \sqrt{1/2}\, T^1 = [-1, 1]$ is $(1,6)$-isotropic (triplicating the two vertices) and

$$K' = \operatorname{conv}\left\{ \sqrt{\frac{1}{2}} \begin{pmatrix} v^{(1)} \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v^{(2)} \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v^{(3)} \\ 1 \end{pmatrix}, \right.$$
$$\left. \sqrt{\frac{1}{2}} \begin{pmatrix} v^{(1)} \\ -1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v^{(2)} \\ -1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v^{(3)} \\ -1 \end{pmatrix} \right\}$$

is $(3,6)$-isotropic. Hence $K_1, K_2$ and $K_3$ satisfy the conditions of the additive rule and therefore there exists a $(3,9)$-isotropic polytope.

Case 2, $j \geq 5$ odd and $n + 1 = 2j + 3$:
Let $m = n+1$ and choose $j_1 = j-2, j_2 = 2, m_1 = m-j-1, m_2 = j+1$. Since $j < m/2$ it holds $j_1 < m_1$ and since $j_1 = j-2 \neq j = m-j-3 = m_1 - 2$ there exists a $(j_1, m_1)$-isotropic polytope $K_1$. Completing the conditions of the additive rule we choose an $m_2$-gon for $K_3$ and the projection of $\sqrt{j/(j+1)}\, T^j$ onto $(\operatorname{lin} K_3)^{\perp}$ as $K_2$ (thus $K' = \sqrt{j/(j+1)}\, T^j$). Note that $m_2 j_1 = j^2 + j > 2m = m_1 j_2$ since $j \geq 5$.

Case 3, $j$ even or $n + 1 \neq 2j + 3$:
Then we set $j_1 = j, j_2 = 0, m_1 = j+1$ and $m_2 = m-j-1$. Since $j < m/2$ it holds $m_2 > j$ and if $j+2$ is odd $m_2 \neq j+2$ since $m \neq 2j+3$. Hence there exists a $(j, m_2)$-isotropic polytope $K_2$ by the induction hypothesis and $K_1 = \sqrt{(j+1)/(j+2)}\, T^{j+1}$ is a $(j, m_1)$-isotropic polytope, which obviously satisfies the conditions of the additive rule. $\square$

From the two cases which are excluded in Theorem 1 one is the even case in (3.6). So it remains only $R_{n-1}(T^n)$ with even $n$. A formula for this case was claimed by Pukhov [17] and first shown by Weißbach [20], but that proof contained an error (see [6]). At the end of 2002, Weißbach suggested to us to work jointly on a new proof, but unfortunately he died in June 2003. In a letter, he stressed his opinion that a major step would be to show that every minimal enclosing $(n-1)$-cylinder of the regular simplex is always circumscribing. Recently, in [7] we were able to show the following statement:

**Proposition 6.** *Let $1 \leq j \leq n$ and $S$ be a simplex in $\mathbb{E}^n$ with facets $S^{(1)}, \ldots, S^{(n+1)}$. Then one of the following is true.*

a) *Every minimal enclosing $j$-cylinder of $S$ is a circumscribing $j$-cylinder of $S$.*
b) $R_j(S) = R_{j-1}(S^{(i)})$ *for some* $i \in \{1, \dots, n+1\}$ *and* $j \geq 2$.
c) $R_j(S) = R_k(F)$, *for some* $k \in \{1, \dots, j-1\}$, *where $F$ is a $k$-face of $S$.*
*If $j = 1$ or if $S = T^n$ then always case* a) *holds.*

Rather than presenting the whole proof of Proposition 6 we will concentrate on the case $j = n-1$ and $S = T^n$, as major ideas are already contained in this case.

**Lemma 7.** *The sequence* $(R_{n-1}(T^n))_{n \geq 2}$ *is strictly increasing.*

*Proof.* Let $T^{n+1}$ be embedded in $\mathbb{E}^{n+1}$ such that $S^{(n+1)}$ (which is $T^n$) lies within $H := \{x \in \mathbb{E}^{n+1} : x_{n+1} = 0\}$, and let $\mathcal{C} = \ell + R_{n-1}(T^n)(\mathbb{B}^{n+1} \cap H)$ be any minimal enclosing cylinder of $S^{(n+1)}$ (taken in $n$-dimensional space) with $\ell$ a line in $H$. But, since

$$\operatorname{dist}(v^{(n+1)}, \ell) \geq \operatorname{dist}(v^{(n+1)}, H) = \sqrt{\frac{n+1}{n}} > 1 > R_n(T^{n+1})$$

(where $\operatorname{dist}(\cdot,\cdot)$ denotes the Euclidean distance), $\ell$ cannot be the axis of a minimal enclosing cylinder of $T^{n+1}$. Hence $R_{n-1}(T^n) < R_n(T^{n+1})$. $\square$

**Theorem 8.** *Let $n \geq 2$ and $\ell \subset \mathbb{E}^n$ be a line, such that $\mathcal{C} = \ell + R_{n-1}(T^n)\mathbb{B}^n$ is a minimal enclosing cylinder of $T^n$. Then $\mathcal{C}$ is also a circumscribing cylinder of $T^n$.*

*Proof.* The proof is split into three parts. In the first part we will exclude the special cases that $\ell$ is parallel or perpendicular to one of the facets $S^{(1)}, \dots, S^{(n+1)}$ of $T^n$. The other two parts deal with the previously excluded cases. We can assume that $T^n$ is embedded in $\mathbb{E}^n$ such that $S^{(n+1)} \subset H := \{x \in \mathbb{E}^n : x_n = 0\}$.
Part 1: Suppose $\ell$ is neither perpendicular nor parallel to any of the facets of $T^n$. Now assume $v^{(n+1)} \not\in \operatorname{bd}(\mathcal{C})$. Let $p, s \in \mathbb{E}^n$ such that $\ell = p + \operatorname{lin}\{s\}$. Since, by assumption, $\ell$ is not parallel to $H$, we can assume $p = 0 \in \ell \cap H$, and $s_n > 0$. For every $s'_n \in (0, s_n)$ and $s' := (s_1, \dots, s_{n-1}, s'_n) \in \mathbb{E}^n$ let $\ell' = p + \operatorname{lin}\{s'\}$. Geometrically, $\ell'$ results from $\ell$ by rotating $\ell$ into the direction of the hyperplane $H$ in such a way that the orthogonal projection of $\ell$ onto $H$ remains invariant (see Figure 2). Since $\ell$ and $H$ are not perpendicular we obtain $\ell \neq \ell'$. Further, since $v^{(1)}, \dots, v^{(n)} \in H$, we have

$$\operatorname{dist}(v^{(i)}, \ell') \leq \operatorname{dist}(v^{(i)}, \ell), \quad 1 \leq i \leq n, \tag{3.7}$$

where "$<$" holds whenever $v^{(i)} \not\in K := \ell^{\perp} \cap H$. Obviously, $\dim(K) = n-2$. If none of the $v^{(i)}$ lies in $K$, then, by choosing $s'_n$ sufficiently close to $s_n$, all vertices of $T^n$ lie in the interior of $\mathcal{C}' = \ell' + R_j(T^n)\mathbb{B}^n$, a contradiction to the minimality of $R_j(T^n)$. Hence, there must be some vertex of $S$ in $K \cap \operatorname{bd}(\mathcal{C})$.

Let $k+1$ be the number of vertices in $K \cap \operatorname{bd}(\mathcal{C})$. By renumbering the vertices we can assume $v^{(1)}, \dots, v^{(k+1)} \in K \cap \operatorname{bd}(\mathcal{C})$, $0 \leq k \leq n-2$. Then $F := \operatorname{conv}\{v^{(1)}, \dots, v^{(k+1)}\}$ is a $k$-face of $T^n$. Since $F$ and $\ell$ are perpendicular, $F$ is congruent to $T^k$ and $p = 0$ is the unique center of the circumball of $F$. Hence,

$$R_{n-1}(T^n) = R_k(F) = R_k(T^k) = \sqrt{\frac{k}{k+1}} = \sqrt{\frac{2k}{2k+2}}\,.$$

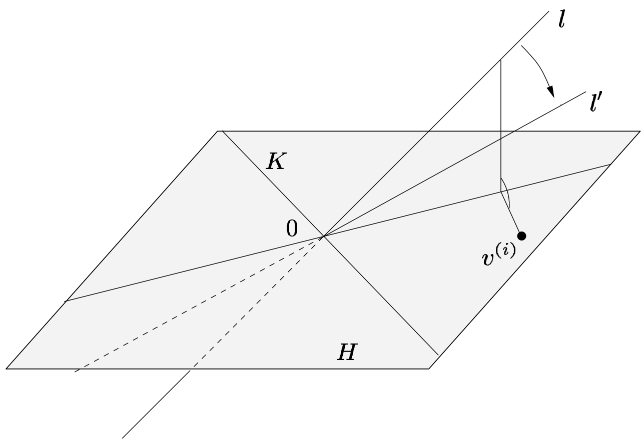


Figure 2: For $n = 3$ the figure shows how the underlying line $\ell$ of the cylinder $\mathcal{C}$ is rotated towards its orthogonal projection onto the plane $H$. The distances between the vertices $v^{(i)}$, $1 \le i \le n$, and the $j$-cylinder axis are not increased, and decreased if $v^{(i)} \notin K$.

Now Theorem 1 and Lemma 7 imply that $n = 2k + 1$. But in this case it follows already from Lemma 2 that $\mathcal{C}$ circumscribes $T^n$, contradicting $v^{(n+1)} \notin \mathrm{bd}(\mathcal{C})$.

Part 2: Now consider the case that $\ell$ is perpendicular to $H$ and assume again $v^{(n+1)} \notin \mathrm{bd}(\mathcal{C})$. In this case any small perturbation of $\ell$ around $p := \ell \cap H$ keeps $v^{(n+1)}$ within the cylinder not increasing the distances of all the other vertices to the new axis. So the same argumentation as in the non-perpendicular case shows a contradiction to the assumptions.

Part 3: Finally, consider the case that $\ell$ is parallel to one of the facets of $T^n$. By Lemma 2 and Theorem 1, we only have to consider the case $n$ even. Suppose $\ell$ is parallel to $S^{(n+1)}$ and that $v_n^{(n+1)} > 0$. Since $R_{n-1}(T^{n-1}) = \sqrt{(n-1)/n}$ we have $v_n^{(n+1)} = \sqrt{(n+1)/n}$. Let $p \in \ell$. Since $v_n^{(n+1)} > 0$ it holds $p_n \ge 0$ and obviously

$$R_{n-1}(T^n) \ge \sqrt{\frac{n+1}{n}} - p_n \; . \tag{3.8}$$

On the other hand, since $\ell$ is parallel to $S^{(n+1)}$, and since the cylinder radius of $T^{n-1}$ is $\sqrt{(n-2)/n}$ (recall that $n-1$ is odd)

$$R_{n-1}(T^n)^2 = \frac{n-2}{n} + p_n^2. \tag{3.9}$$

Let

$$p_n^* = \frac{3}{2\sqrt{n(n+1)}} > 0$$

be the unique minimal solution for $p_n$ to (3.8) and (3.9). As $\mathcal{C}$ is an optimal cylinder it must hold $p_n = p_n^*$. But because of Lemma 2 and the equalities in (3.8) and (3.9) for $p_n^*$ this means that all vertices of $T^n$ have the same distance to $\ell$, which shows that $\mathcal{C}$ is circumscribing. $\square$

As shown in [7], the optimal choice of $\ell$ for even $n$ is the one parallel to a facet. The proof includes study of the relevant polynomial equations, which we do not review here in detail.

However, a crucial step towards a solution was the transformation of the original problem into an optimization problem over symmetric polynomials, an idea which also Weißbach had in mind for solving the problem. In [7] this transformation is presented for general outer $j$-radii of the regular simplex, but here we will keep our focus on the case $j = n-1$.

Let $\ell = p + \mathrm{lin}\{s\}$, where $s \in \mathcal{S}^{n-1}$ and $p \in \mathrm{lin}\{s\}$. Suppose $\mathcal{C} = \ell + R_{n-1}(T^n)(\mathbb{B}^{n+1} \cap \mathcal{H}_0^n)$ is a minimal enclosing (and circumscribing) cylinder of the regular simplex $\mathbf{T}^n$ in standard embedding. The orthogonal projection of a vector $x \in \mathcal{H}_1^n$ onto the orthogonal complement of $\mathrm{lin}\{s\}$ (relative to $\mathcal{H}_1^n$) can be written as $x \mapsto Px$ with $P = I - ss^T \in \mathbb{E}^{(n+1)\times(n+1)}$. Hence, for a general polytope with vertices $v^{(1)}, \ldots, v^{(m)}$ (embedded in $\mathcal{H}_1^n$) the computation of the square of $R_{n-1}$ can be expressed by the following optimization problem. Here, we use the convention $x^2 := x \cdot x$.

$$\begin{array}{llrcll} & & \min \rho^2 & & & \\ \text{(i)} & \text{s.t.} & (p - Pv^{(i)})^2 & \leq & \rho^2, & i = 1, \ldots, m, \\ \text{(ii)} & & p \cdot s & = & 0, & \\ \text{(iii)} & & s & \in & \mathcal{S}^{n-1}, & \\ \text{(iv)} & & p & \in & \mathcal{H}_1^n . & \end{array} \tag{3.10}$$

In the case of $\mathbf{T}^n$, (i) can be replaced by

$$\text{(i')} \qquad (p - e^{(i)} + s_i s)^2 \; = \; \rho^2, \quad i = 1, \ldots, n+1, \tag{3.11}$$

where the equality sign comes from the fact that $\mathcal{C}$ is circumscribing. By (ii) and $s \in \mathcal{S}^{n-1}$, (i') can be simplified to

$$\text{(i")} \qquad p^2 - \rho^2 \; = \; s_i^2 + 2p_i - 1, \quad i = 1, \ldots, n+1 .$$

Summing over all $i$ gives $(n+1)(p^2 - \rho^2) = 1 + 2 - (n+1)$, i.e., $p^2 - \rho^2 = \frac{2-n}{n+1}$. We substitute this value into (i") and obtain $p_i = \frac{1}{2}\left(\frac{3}{n+1} - s_i^2\right)$. Hence, the $p_i$ can be replaced in terms of the $s_i$,

$$\begin{aligned} \rho^2 &= \frac{4n-5}{4(n+1)} + \frac{1}{4}\sum_{i=1}^{n+1} s_i^4 , \\ p \cdot s &= -\frac{1}{2}\sum_{i=1}^{n+1} s_i^3 . \end{aligned}$$

We arrive at the following characterization of the minimal enclosing cylinders:

**Theorem 9.** *Let $n \geq 2$. A vector $s \in \mathcal{S}^{n-1}$ represents the axis direction of a minimal enclosing cylinder of $\mathbf{T}^n \subset \mathcal{H}_1^n$ if and only if it is an optimal solution of the problem*

$$\begin{array}{rlcl} & \min \sum_{i=1}^{n+1} s_i^4 & & \\ s.t. & \sum_{i=1}^{n+1} s_i^3 & = & 0\,, \\ & \sum_{i=1}^{n+1} s_i^2 & = & 1\,, \\ & \sum_{i=1}^{n+1} s_i & = & 0\,. \end{array} \tag{3.12}$$

Solving this system for even $n$ in connection with Weißbach's results for odd $n$ yields the following proposition.

**Proposition 10.** *Let $n \in \mathbb{N}$ and $T^n$ be a regular simplex in $\mathbb{E}^n$ with edge length $\sqrt{2}$. Then*

$$R_{n-1}(T^n) = \begin{cases} \sqrt{\frac{n-1}{n+1}} & \text{if } n \text{ is odd,} \\ \frac{2n-1}{2\sqrt{n(n+1)}} & \text{if } n \text{ is even.} \end{cases}$$

## 4. Geometric inequalities

One reason why Weißbach considered the outer $(n-1)$-radius of the regular simplex was an older result of Eggleston [10], showing that for $n \geq 3$ there exists a body $C \subset \mathbb{E}^n$ with $R_{n-1}(C) > r_1(C)$. In [19] Weißbach gives a much simpler proof. By Lemma 2,

$$R_{n-1}(T^n) \geq \sqrt{\frac{n-1}{n+1}} > \sqrt{\frac{1}{2}} = r_1(T^n) \quad \text{for } n \geq 4\,. \tag{4.1}$$

However, for $n = 3$ it holds $R_{n-1}(T^n) = r_1(T^n)$. By an elegant construction, Weißbach also provides a proof of the remaining case.

**Proposition 11.** *The Weißbach polytope*

$$W = \operatorname{conv}\left(\left\{ \begin{pmatrix} 0 \\ -\sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{1}{2}} \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\sqrt{\frac{1}{2}} \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1-\sqrt{6}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}-2\sqrt{3}}{4} \\ \frac{-\sqrt{2}+2\sqrt{3}}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{-\sqrt{2}+2\sqrt{3}}{4} \\ \frac{-\sqrt{2}+2\sqrt{3}}{4} \\ 0 \end{pmatrix} \right\}\right)$$

*fulfills $R_2(W) > r_1(W)$.*

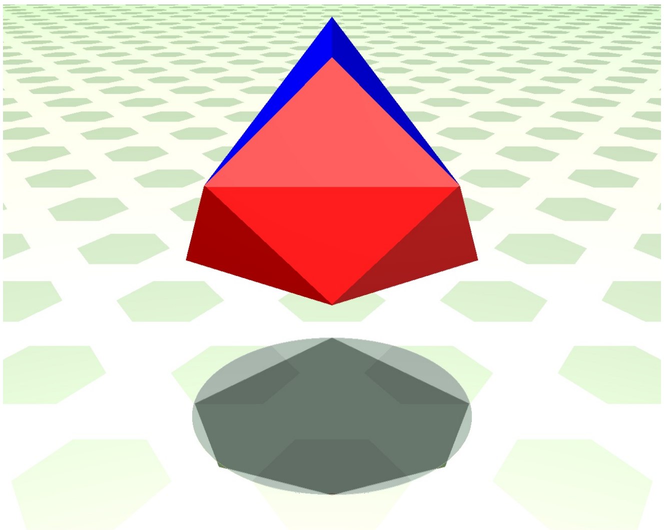

Figure 3: In the picture, all seven vertices of the Weißbach polytope $W$ are visible. The upper two and the left most and right most belong to the original regular simplex $T^3$. In all three optimal projection directions of $T^3$ two of the additional vertices of $W$ are projected outside the circumball of the projection of $T^3$. The picture shows one of these three projections as well as the circumball of the projection of $T^3$.

Note that the first four vertices of $W$ are the vertices of a regular simplex with edge length $\sqrt{2}$. Hence, it suffices to show that $r_1(W) = r_1(T^3)$ and in each of the three optimal projection directions of $T^3$ the projection of $W$ does not fit into the same circumball. This situation is visualized in Figure 3.[1]

It turns out that the boundary of a minimal enclosing cylinder of $W$ contains five vertices of $W$, and that this condition can be used to compute the numerical value of $R_2(W)$. E.g., one of the minimal enclosing cylinders of $W$ contains the vertices 1, 3, 4, 6, and 7. For any five points in general position, there are six (complex) cylinders whose surface passes through the given points (see [3]). Thus, solving the corresponding systems of polynomial equations yields six cylinders (which are all real for our configuration), among which the one with radius $\rho \approx \sqrt{2} \cdot 0.50095$ is the smallest one containing $W$. Hence, $R_2(W)/r_1(W) \approx 1.0019$.

In fact, $R_2 > r_1$ would already be obtained by adding only two of the three new vertices to the regular simplex.

Since $R_{n-1}(C) < r_1(C)$ for many symmetric bodies, (4.1) and Proposition 11 show that $R_{n-1}$ and $r_1$ are *incomparable*, i.e., no arc between the nodes for $R_{n-1}$ and $r_1$ can be added in Figure 1. Now the question arises which other pairs of radii (which are not connected by a directed path in that figure) are provably incomparable. For certain pairs, already the regular simplex in addition with the class of ellipsoids shows incomparability, but for example it holds $R_2(T^n) \leq r_1(T^n)$ for $n \geq 3$ and $R_1(T^n) \leq r_2(T^n)$ for $n \geq 5$ like for all symmetric bodies. On the other hand it follows from the monotonicity of the outer and inner radii

[1] A colored, dynamic 3D model of the Weißbach polytope is available from the homepages of the authors (www-m9.ma.tum.de/ brandenb/; www-m9.ma.tum.de/ theobald/).

chains, that bodies $C, C'$ with $R_2(C) > r_1(C)$ and $R_1(C') > r_2(C')$ imply that the diagram in Figure 1 is already complete (since in both cases the other direction of the inequalities is fulfilled by ellipsoids).

Bodies $C$ which satisfy the inequalities $R_2(C) > r_1(C)$ and $R_1(C) > r_2(C)$ simultaneously were considered in [8]. There, a body of constant breadth $C$ with $R_2(C) > r_1(C)$ is called a *totally non-spherical body* as every projection of $C$ onto arbitrary subspaces of dimension at least 2 is different from the ball.

**Proposition 12.** *For all $n \geq 3$ there exists a totally non-spherical body $C \subset \mathbb{E}^n$, i.e.,*

$$r_n(C) \leq \cdots \leq r_2(C) < r_1(C) = R_1(C) < R_2(C) \leq \cdots \leq R_n(C) .$$

Finally, we want to describe a small application of the existence of bodies (such as the Weißbach polytope) which satisfy radii relations in the unusual direction (say, $R_2(C) > r_1(C)$).

The following Proposition was shown in [12, (1.11)] by use of Helly's theorem.

**Proposition 13.** *If $C$ is a body in $\mathbb{E}^n$ then*

$$R_n(C) = \sup_{\substack{S \subset C \\ S \text{ Simplex}}} R_n(S) .$$

Proposition 13 provides an algorithmic reduction of the problem to compute $R_n(P)$ of a polytope $P$ to the computation of the outer $n$-radius of simplices defined by vertices of $P$. Using the concept of non-spherical bodies, the following results imply that a similar result does not hold for general outer $j$-radii, not even for the outer 2-radius in $\mathbb{E}^3$ (the radius of the smallest enclosing cylinder). We make use of the following result, shown in [4, Theorem 3.17].

**Proposition 14.** *Let $1 \leq j \leq n$, such that $n - j + 1$ divides $n + 1$, and let $S$ be an $n$-dimensional simplex. Then*

$$\frac{R_j(S)}{r_1(S)} \leq \sqrt{\frac{2j}{n+1}} ,$$

*with equality if $S = T^n$.*

Proposition 11 and the case $(j, n) = (2, 3)$ in Proposition 14 imply that the Weißbach polytope $W$ satisfies

$$R_2(W) > r_1(W) \geq r_1(S) \geq R_2(S) \quad \text{for all simplices } S \subset W.$$

More generally, we can state the following theorem.

**Theorem 15.** *If $1 \leq j \leq \frac{n+1}{2}$ then every totally non-spherical body $C \subset \mathbb{E}^n$ satisfies*

$$R_j(C) > \sup_{\substack{S \subset C \\ S \text{ Simplex}}} R_j(S) .$$

*Proof.* In case $j = 1$, note that the constant breadth of $C$ implies $r_1(C) = R_1(C)$. However, since no simplex can be of constant breadth, $R_1(S) < r_1(S) \le r_1(C)$ for any simplex $S$ contained in $C$.

Now suppose $j \ge 2$ and let $S \subset C$ be a simplex. Since $r_1(S) \le r_1(C) < R_j(C)$, it remains to show $R_j(S) \le r_1(S)$. If $n+1$ is even, this follows from the monotonicity of the outer radii and Proposition 14. If $n$ is even, let $\bar{S}$ be an $(n+1)$-simplex, such that $S$ is congruent to one of the facets of $\bar{S}$ and $r_1(S) = r_1(\bar{S})$. Since $j+1 \le \frac{n+2}{2}$, we have $R_j(S) \le R_{j+1}(\bar{S}) \le r_1(\bar{S}) = r_1(S)$. □

## References


[1] Ball, K.: *Ellipsoids of maximal volume in convex bodies.* Geom. Dedicata **41** (1992), 241–250. Zbl 0747.52007

[2] Bonnesen, T.; Fenchel, W.: *Theorie der konvexen Körper.* Springer-Verlag, Berlin 1934. English translation: BCS Associates, Moscow, Idaho 1987. Zbl 0008.07708

[3] Bottema, O.; Veldkamp, G. R.: *On the lines in space with equal distances to n given points.* Geom. Dedicata **6** (1977), 121–129. Zbl 0363.50017

[4] Brandenberg, R.: *Radii of convex bodies.* Ph.D. thesis, Dept. of Mathematics, Technische Universität München 2002.

[5] Brandenberg, R.: *Radii of regular polytopes.* Submitted.

[6] Brandenberg, R.; Theobald, T.: *Algebraic methods for computing smallest enclosing and circumscribing cylinders of simplices.* Appl. Algebra Eng. Commun. Comput. **14** (2004), 439–460.

[7] Brandenberg, R.; Theobald, T.: *Radii minimal projections of simplices and constrained optimization of symmetric polynomials.* Preprint, `arXiv:math.MG/0311017`.

[8] Brandenberg, R.; Larman, D.: *Dark clouds on spheres and totally non-spherical bodies of constant breadth.* Beitr. Algebra Geom. **44** (2003), 531–538. Zbl pre01973846

[9] Chakerian, G. D.; Groemer, H.: *Convex bodies of constant width.* In: P. Gruber and J. Wills (eds.), *Convexity and its applications*, 49–96, Birkhäuser 1983. Zbl 0518.52002

[10] Eggleston, H. G.: *Minimal universal covers in $\mathbb{E}^n$.* Israel J. Math. **1** (1963), 149–155. Zbl 0117.39502

[11] Everett, H.; Stojmenovíc, I.; Valtr, P.; Whitesides, S.: *The largest $k$-ball in a $d$-dimensional box.* Comput. Geom. **11** (1998), 59–67. Zbl 0911.68198

[12] Gritzmann, P.; Klee, V.: *Inner and outer $j$-radii of convex bodies in finite-dimensional normed spaces.* Discrete Comput. Geom. **7** (1992), 255–280. Zbl 0747.52003

[13] Gritzmann, P.; Klee, V.: *Computational complexity of inner and outer $j$-radii of polytopes in finite-dimensional normed spaces.* Math. Program. Ser. A **59** (1993), 163–213. Zbl 0784.90076

[14] Henk, M.: *Ungleichungen für sukzessive Minima und verallgemeinerte In- und Umkugelradien.* Ph.D. thesis, Dept. of Mathematics, Universität-GH Siegen 1991. Zbl 0734.52008

[15] Henk, M.: *A generalization of Jung's theorem.* Geom. Dedicata **42** (1992), 235–240. Zbl 0759.52008

[16] Jung, H. W. E.: *Über die kleinste Kugel, die eine räumliche Figur einschliesst.* J. Reine Angew. Math. **123** (1901), 241–257. JFM 32.0296.05

[17] Pukhov, S. V.: *Kolmogorov diameters of a regular simplex.* Mosc. Univ. Math. Bull. **35** (1980), 38–41. Zbl 0479.41019

[18] Steinhagen, P.: *Über die grösste Kugel in einer konvexen Punktmenge.* Abh. Math. Sem. Univ. Hamburg **1** (1921), 15–26. JFM 48.0837.03

[19] Weißbach, B.: *Über die senkrechten Projektionen regulärer Simplexe.* Beitr. Algebra Geom. **15** (1983), 35–41. Zbl 0528.52008

[20] Weißbach, B.: *Über Umkugeln von Projektionen regulärer Simplexe.* Beitr. Algebra Geom. **16** (1983), 127–137. Zbl 0534.51012

Received December 1, 2003