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A Classification and some Constructions of p-harmonic Morphisms

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1. Introduction

We give a classification of p-harmonic morphisms between Riemannian manifolds of equal dimensions (Theorem 3.1), which generalizes the corresponding results of B. Fuglede on harmonic morphisms in [13] on one hand, and on p-harmonic morphisms between Euclidean domains in [25] on the other. We also prove that pre- or post-composition by a horizontally homothetic harmonic map makes a weakly conformal map a p-harmonic morphism, for $p =$ dim M or dim N. This can be used to construct infinitely many p-harmonic morphisms which are not harmonic morphisms. Our examples include the well-known Hopf fibration $\varphi : S^{2n-1} \longrightarrow S^n$ ($n = 2, 4,$ or 8), which becomes a p-harmonic morphism for $p = 3, 4, 7, 8,$ or 15, and thus solving globally the over-determined system

$$\operatorname{div}\left(|d\varphi|^{p-2}\, d\varphi\right) = 0$$
$$g^{ij}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j} = \lambda^2(x)h^{\alpha\beta}$$

of p-Laplace equations for maps between compact manifolds, after a suitable conformal change of the metric on the domain or target manifold is made.

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A map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between two Riemannian manifolds is called a harmonic map if the divergence of its differential vanishes. Such maps are the extrema of the energy functional

$$E_2(\varphi, \Omega) = \frac{1}{2}\int_{\Omega} |d\varphi|^2 \, dx$$

for all compact subsets Ω in M. For a detailed account of harmonic maps we refer to [10],[11],[12],[27] and the references therein.

Harmonic morphisms are a special subclass of harmonic maps which preserve solutions of Laplace's equation in the sense that, for any harmonic function $f : U \longrightarrow \mathbb{R}$, defined on an open subset U of N with $\varphi^{-1}(U)$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \longrightarrow \mathbb{R}$ is a harmonic function. In other words, φ pulls back germs of harmonic functions on N to germs of harmonic functions on M. In stochastic theory, harmonic morphisms are known as Brownian path preserving maps meaning that they send a Brownian motion on M to a Brownian motion on N(see [6],[21]). The following well-known characterization of harmonic morphisms is due to independently B. Fuglede and T. Ishihara.

Theorem 1.1. ([13],[18]) *A non-constant map $\varphi : (M, g) \longrightarrow (N, h)$ is a harmonic morphism if and only if it is a horizontally weakly conformal harmonic map.*

In case of equal dimensions, B. Fuglede obtained the following classification:

Theorem 1.2. ([13]) *Let $\dim M = \dim N = n$, and $\varphi : (M, g) \longrightarrow (N, h)$ be a non-constant C^2 map. Then:*

1) *For $2 = n$, φ is a harmonic morphism if and only if it is weakly conformal.*
2) *For $2 \neq n$, φ is a harmonic morphism if and only if it is weakly conformal with constant dilation.*

In recent years, much work has been done in constructing and classifying harmonic morphisms between certain model spaces. Many interesting results (see the recent monograph [5]) have shown that the study of such maps can help us to understand the topology and geometry of the domain and target manifolds. For a complete bibliography of harmonic morphisms, see [15].

A natural generalization of harmonic maps are p-harmonic maps which are critical points of the p-energy functional (see Definition 2.1). Likewise, it is natural to define and study maps between Riemannian manifolds which preserve the local solutions of p-Laplace's equations. This was first done in [25], in which the authors prove the following:

Theorem 1.3. ([25]) *Let $\varphi : U \longrightarrow V$ be a non-constant p-harmonic morphism between two domains of the Euclidean space R^n.*

1) *If $p = n$ and φ is sense-preserving, then φ is 1-quasiregular mapping. In particular, φ is the restriction of a Möbius transformation when $n \geq 3$.*
2) *If $1 < p < \infty$ and $p \neq n$, and if φ is discrete, then it is the restriction of a Euclidean similarity, i.e., $\varphi(x) = \lambda Ax + b$ for some $\lambda \in (0, \infty)$, $b \in R^n$ and $A \in O(n)$.*

Very recently, E. Loubeau [22] obtained the following characterization of a p-harmonic morphism similar to that of harmonic morphisms given by B. Fuglede and T. Ishihara.

Theorem 1.4. ([22], [8]) *A non-constant C^2-map $\varphi : (M,g) \longrightarrow (N,h)$ is a p-harmonic morphism with $p \in (1,\infty)$ if and only if it is a horizontally weakly conformal p-harmonic map.*

Clearly, a harmonic morphism is a p-harmonic morphism with $p = 2$. It is natural and important to ask if there can be any example of a p-harmonic morphism which is not a harmonic morphism. It seems that few such examples have been found in the literature (see [7], [19], [22], [23], and [26] for recent papers on p-harmonic morphisms). In this paper, we first give a classification of p-harmonic morphisms between Riemannian manifolds of the same dimension. Then we use the compositions of weakly conformal maps and horizontally homothetic harmonic maps to construct many interesting p-harmonic morphisms which are not harmonic morphisms.

2. Preliminaries

2.1. p-harmonic maps and morphisms

Definition 2.1. *Let (M,g) and (N,h) be two Riemannian manifolds, Ω a compact subset in M, and $p \in (1,\infty)$. For $\varphi \in L^{1,p}(\Omega, N)$, we define the p-energy of φ by*

$$E_p(\varphi, \Omega) = \frac{1}{p}\int_{\Omega} |d\varphi|^p \, dx \, .$$

A map $\varphi : (M,g) \longrightarrow (N,h)$ is called p-harmonic if $\varphi|\Omega$ is a critical point of the p-energy for every compact subset Ω of M.

The striking difference between harmonic and p-harmonic ($p \neq 2$) maps is their behavior with respect to regularity. For the regularity and the existence of p-harmonic maps we refer to [9], [17], [24], [28], [29], [30], [31]. In this paper we assume that all our objects, manifolds, vector fields, and maps are smooth, and p-harmonic maps are understood in the strong sense.

Lemma 2.2. (see e.g. [2]) *A map $\varphi : (M,g) \longrightarrow (N,h)$ is p-harmonic if and only if its p-tension field $\tau_p(\varphi) \equiv 0$, where*

$$\tau_p(\varphi) = |d\varphi|^{p-2}\, \tau_2(\varphi) + (p-2)\, |d\varphi|^{p-3}\, d\varphi(\mathrm{grad}\, |d\varphi|) \tag{1}$$

with $\tau_2(\varphi) = \mathrm{trace}\nabla d\varphi$ denoting the tension field of φ.

Notice that when $|d\varphi| \neq 0$ we can write

$$\tau_p(\varphi) = |d\varphi|^{p-2}\, [\tau_2(\varphi) + (p-2) d\varphi(\mathrm{grad}\, \ln |d\varphi|)] \, . \tag{2}$$

As an immediate consequence, we have:

Corollary 2.3. *If $\varphi : (M, g) \longrightarrow (N, h)$ is both a p_1-harmonic and p_2-harmonic map for $p_1 \neq p_2$, then it is a p-harmonic map for any $p \in (1, \infty)$.*

Definition 2.4. *A map $\varphi : (M, g) \longrightarrow (N, h)$ is called a p-harmonic morphism if for any p-harmonic function $f : U \subset N \longrightarrow \mathbb{R}$, defined on an open subset U of N with $\varphi^{-1}(U)$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \subset M \longrightarrow \mathbb{R}$ is a p-harmonic function.*

From the definition we easily have:

Corollary 2.5. *If $\varphi : (M, g) \longrightarrow (N, h)$ and $\psi : (N, h) \longrightarrow (Q, k)$ are p-harmonic morphisms, then their composition $\psi \circ \varphi$ is a p-harmonic morphism.*

A function $f : (M, g) \longrightarrow R$ is called *p-subharmonic* (*p-superharmonic*, resp.) if $\tau_p(f) \geq 0$ (≤ 0, resp.). We can prove the following:

Proposition 2.6. *A map $\varphi : (M, g) \longrightarrow (N, h)$ is a p-harmonic morphism if and only if it pulls back germs of p-subharmonic (p-superharmonic) functions on N to germs of p-subharmonic (p-superharmonic) functions on M.*

Proof. If φ is a p-harmonic morphism, then by [22] we have $\tau_p(f \circ \varphi) = \lambda^p \tau_p(f) \circ \varphi$ which gives the "only if" part of the statement. To prove the "if" part of the statement, take any p-harmonic function f locally defined on N then, by definition, it is p-subharmonic hence we have, $\tau_p(f \circ \varphi) \geq 0$. Notice that $-f$ is also p-harmonic, hence p-subharmonic. By assumption, $-f \circ \varphi$ is p-subharmonic, thus

$$\begin{aligned}
&\tau_p(-f \circ \varphi)\\
&= |d(-f \circ \varphi)|^{p-2} \left[\tau_2(-f \circ \varphi) + (p-2) d(-f \circ \varphi)(\text{grad} \ln |-f \circ \varphi|)\right]\\
&= -\tau_p(f \circ \varphi) \geq 0.
\end{aligned}$$

It follows that $\tau_p(f \circ \varphi) \leq 0$. These two inequalities show that $\tau_p(f \circ \varphi) = 0$ for any p-harmonic function f locally defined on N. Therefore, by definition, φ is a p-harmonic morphism. □

2.2 Horizontally weakly conformal maps

Let $\varphi : (M, g) \longrightarrow (N, h)$ be a map and $x \in M$. We call $V_x = \ker d\varphi_x$ the *vertical space* at x and its orthogonal complement H_x in $T_x M$ the *horizontal space* at x, to have $T_x M = V_x \oplus H_x$. The map φ is called *horizontally weakly conformal* if, for each $x \in M$ at which $d\varphi_x \neq 0$, the restriction $d\varphi_x|_{H_x} : H_x \longrightarrow T_{\varphi(x)} N$ is conformal and surjective. Thus it follows that there is a number $\lambda(x) \in (0, \infty)$ such that $h(d\varphi(X), d\varphi(Y)) = \lambda^2(x) g(X, Y)$ for any $X, Y \in H_x$. Note that, at the point $x \in M$ where $d\varphi_x = 0$, we let $\lambda(x) = 0$ to obtain a continuous function $\lambda : M \longrightarrow R$ called the *dilation* of the horizontally weakly conformal map φ. A non-constant horizontally weakly conformal map φ is called *horizontally homothetic* if the gradient of $\lambda^2(x)$ is vertical i.e., $\mathcal{H}(\text{grad}\, \lambda^2) \equiv 0$, where $\mathcal{H}$ denotes the horizontal distribution. One can easily check that (i) if $\dim M = \dim N$, horizontally weakly conformal is equivalent to weakly conformal, meaning that φ is conformal away from the points at which $d\varphi = 0$, (ii) a Riemannian submersion is horizontally homothetic with $\lambda \equiv 1$, and (iii) if $\dim M < \dim N$ any horizontally weakly conformal map is constant.

3. A classification of p-harmonic morphisms

In this section we prove the following theorem which gives a classification of p-harmonic morphisms between Riemannian manifolds of the same dimension.

Theorem 3.1. *Let* $\varphi : (M, g) \longrightarrow (N, h)$ *be a non-constant map between Riemannian manifolds with* $\dim M = \dim N = n \geq 2$. *Then*
1) φ *is a p-harmonic morphism with* $p = n$ *if and only if* φ *is weakly conformal,*
2) φ *is a p-harmonic morphism with* $p \neq n$ *if and only if* φ *is weakly conformal with constant dilation.*

Remark 3.2. As special p-harmonic maps, p-harmonic morphisms for different values of $p \in (1, \infty)$ can be very different. Theorem 3.1 gives a complete classification of all p-harmonic morphisms between Riemannian manifolds of equal dimensions. That is, n-harmonic morphisms, which are precisely weakly conformal mappings, or, "trivial" p-harmonic morphisms, meaning that they are p-harmonic morphisms for any $p \in (1, \infty)$, i.e. weakly conformal mappings with constant dilations.

To prove Theorem 3.1 we need the following:

Lemma 3.3. *Let* $\varphi : (M, g) \longrightarrow (N, h)$ *be a non-constant map with* $\dim M = \dim N = n \geq 2$. *Then* φ *is an n-harmonic morphism if and only if it is weakly conformal.*

Proof. As we remark in Section 2.2, horizontal weak conformality is equivalent to weak conformality since $\dim M = \dim N$. It follows from Theorem 1.4 that to prove the Lemma it suffices to show that a weak conformal map is a p-harmonic map with $p = \dim M = \dim N = n$. To this end, let $\varphi : (M, g) \longrightarrow (N, h)$ be a non-constant map which is weakly conformal with $\varphi^* h = \lambda^2 g$. We need to show that $\tau_n(\varphi) \equiv 0$. For any $x \in M$, if $d\varphi_x = 0$, then by (1) we have $\tau_n(\varphi) = 0$ at x. If $d\varphi_x \neq 0$, then by the weak conformality of φ and the continuity of $\lambda = \frac{|d\varphi|}{\sqrt{n}}$ we can choose a coordinate neighborhood U of x such that $\varphi : (U, g) \longrightarrow (N, h)$ is a conformal diffeomorphism. This means that $\varphi^* h = \lambda^2 g$ is a Riemannian metric on U. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections on U with respect to g and $\varphi^* h = \lambda^2 g$, respectively. Let ∇^φ denote the pull-back connection on the pull-back bundle $\varphi^{-1}TN \longrightarrow M$. By definition of tension field, we have,

$$\tau_2(\varphi) = \sum_{i=1}^{n} \{\nabla^{\varphi}_{e_i}\, d\varphi(e_i) - d\varphi(\nabla_{e_i}\, e_i)\} \tag{3}$$

for an orthonormal frame field $\{e_i\}$ in $U \subseteq M$. Since the map $\varphi : (U, \varphi^* h = \lambda^2 g) \longrightarrow (N, h)$ becomes an isometry between manifolds of the same dimension it is totally geodesic. It follows that

$$\nabla^{\varphi}_{e_i}\, d\varphi(e_j) = d\varphi(\tilde{\nabla}_{e_i}\, e_j) \tag{4}$$

for any $i, j = 1, 2, \ldots, n$. On the other hand, since $\varphi^* h = \lambda^2 g$ is conformal to g on U we can easily check that

$$\tilde{\nabla}_X\, Y - \nabla_X\, Y = (X(\ln\lambda))Y + (Y(\ln\lambda))X - g(X, Y)\mathrm{grad}(\ln\lambda) \tag{5}$$

holds for any vector fields X, Y on U. Substituting (4) into (3) and applying (5) we have

$$\tau_2(\varphi) = (2-n)d\varphi(\text{grad } \ln\lambda) \tag{6}$$

on U. Since $|d\varphi| = \lambda\sqrt{n} \neq 0$ on U, we see from (2) and (6) that

$$\tau_n(\varphi) = |d\varphi|^{n-2}\left[\tau_2(\varphi) + (n-2)d\varphi(\text{grad } \ln|d\varphi|)\right] \equiv 0$$

on U. Thus in either case we have $\tau_n(\varphi) = 0$ at x. Since $x \in M$ is arbitrary we have $\tau_n(\varphi) \equiv 0$ on M. This proves the lemma. □

Proof of Theorem 3.1. Clearly, statement 1) follows immediately from Lemma 3.3.
To prove statement 2), let us suppose that φ is a p-harmonic morphism with $p \neq n$. It follows from Theorem 1.4 that φ is a horizontally weakly conformal p-harmonic map. In particular, φ is weakly conformal since $\dim M = \dim N = n$. By Lemma 3.3, φ is an n-harmonic morphism and hence an n-harmonic map. Since φ is both p-harmonic and n-harmonic with $p \neq n$, we know from Corollary 2.3 that φ is harmonic. Therefore, it follows from (2) that

$$\begin{aligned} 0 \equiv \tau_p(\varphi) &= |d\varphi|^{p-2}\left[\tau_2(\varphi) + (p-2)d\varphi(\text{grad } \ln|\lambda|)\right] \\ &= (p-2)\,|d\varphi|^{p-2}\, d\varphi(\text{grad } \ln|\lambda|). \end{aligned} \tag{7}$$

Now if $p = 2$ then $2 \neq n$ for $p \neq n$, and in this case we get statement 2) from B. Fuglede's Theorem 1.2. If $p \neq 2$, it follows from (6) that $d\varphi(\text{grad } \ln|\lambda|) = 0$, which implies that λ is a constant since $\dim M = \dim N$ and the horizontal subspace equals the whole tangent space.

Conversely, if $\varphi : (M, g) \longrightarrow (N, h)$ is a weakly conformal map with constant dilation λ, then it follows that

$$\begin{aligned} \tau_p(\varphi) &= |d\varphi|^{p-2}\left[\tau_2(\varphi) + (p-2)d\varphi(\text{grad } \ln|\lambda|)\right] \\ &= |d\varphi|^{p-2}\,\tau_2(\varphi). \end{aligned}$$

This, together with Theorem 1.2 shows that φ is a p-harmonic morphism for any p, which completes the proof of 2). Therefore, we obtain Theorem 3.1. □

Remark 3.4. 1) For $p = 2$, our Theorem 3.1 gives exactly B. Fuglede's classification Theorem 1.2 for harmonic morphisms between Riemannian manifolds of equal dimensions.

2) It is also clear that our Theorem 3.1 generalizes Manfredi and Vespri's results (Theorem 1.3) which gave a partial classification of p-harmonic morphisms between domains of a Euclidean space.

From Theorem 3.1 we can easily deduce the following two corollaries.

Corollary 3.5. *Two Riemannian metrics g and h on the same manifold M determine the same p-harmonic sheaf on M (that is, the same p-harmonic functions on open subsets of M) if and only if they are conformally related, i.e., $h = \lambda^2 g$, with a constant $\lambda > 0$ in case* $\dim M \neq p$.

Corollary 3.6. a) *A non-constant map $\varphi : M^2 \longrightarrow N^2$ between two Riemann surfaces is a p-harmonic morphism if and only if:*

(i) *for $p = 2$, φ is holomorphic or anti-holomorphic;*

(ii) *for $p \neq 2$, φ is a homothety.*

b) *For $n \geq 3$, any non-constant p-harmonic morphism with $1 < p < \infty$ from a domain of R^n into R^n is the restriction of a Möbius transformation.*

4. Constructions of p-harmonic morphisms

In this section, we will use the following composition theorem to construct examples of p-harmonic morphisms which are not harmonic morphisms.

Theorem 4.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ $(n \geq 2)$ be a horizontally homothetic harmonic map, and let $\varphi_1 : (P^m, g_1) \longrightarrow (M^m, g)$ and $\varphi_2 : (N^n, h) \longrightarrow (Q^n, h_2)$ be weakly conformal maps. Then $\varphi \circ \varphi_1 : (P^m, g_1) \longrightarrow (N^n, h)$ is an m-harmonic morphism, and $\varphi_2 \circ \varphi : (M^m, g) \longrightarrow (Q^n, h_2)$ is an n-harmonic morphism.*

Proof. It follows from [22] that a horizontally homothetic harmonic map φ is also a p-harmonic morphism for $p \neq 2$. Hence it is a p-harmonic morphism for any $p \in (1, \infty)$ by Corollary 2.3. From this and Corollary 2.5 we get Theorem 4.1. □

As a consequence, we have:

Corollary 4.2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ $(n \geq 2)$ be a horizontally homothetic harmonic map with dilation λ_1. Let $i_1 : (M^m, (\lambda_2 \circ \varphi)^{-2} g) \longrightarrow (M^m, g)$ (resp. $i_2 : (N^n, h) \longrightarrow (N^n, \lambda_2^2 h)$) be the identity map on M (resp. N), where $\lambda_2 : N \longrightarrow (0, \infty)$ be a non-constant function. Then,*

(i) *$\varphi_1 = \varphi \circ i_1$ is an m-harmonic morphism which is not a harmonic morphism,*

(ii) *$\varphi_2 = i_2 \circ \varphi$ is an n-harmonic morphism which is not a harmonic morphism.*

Proof. For Statement (i), we first note that φ_1 is an m-harmonic morphism by Theorem 4.1. The dilation of φ_1 is $\lambda_1(\lambda_2 \circ \varphi)$, since $h(d\varphi_1(X), d\varphi_1(Y)) = {\lambda_1}^2 g(di_1(X), di_1(Y)) = \lambda_1(\lambda_2 \circ \varphi)^2 g(X, Y)$ for any horizontal vector fields X and Y. To show that φ_1 is not a harmonic morphism is equivalent to check that it is not horizontally homothetic by (1). To see this, it is enough to show that there is a horizontal vector field X such that $X(\lambda_1(\lambda_2 \circ \varphi)) \neq 0$ at some point. Now λ_2 is not constant by our choice, there exists a vector field Y on N such that $Y(\lambda_2)$ does not vanish identically. We notice that φ is submersive, and it is well known (cf. e.g. [2]) that there is a horizontal vector field X on M which is φ-related to Y i.e. $d\varphi(X) = Y$. It follows that $X(\lambda_1(\lambda_2 \circ \varphi)) = \lambda_1 d(\lambda_2 \circ \varphi)(X) = \lambda_1 d\varphi(X)(\lambda_2) = \lambda_1 Y(\lambda_2)$, which is not identically zero by the assumption on λ_2 and the choice of Y. Therefore, φ_1 is not horizontally homothetic and hence it can not be a harmonic morphism.

The proof of statement (ii) is similar and is omitted. □

Since the Hopf fibration $S^{2n-1} \longrightarrow S^n$ ($n = 2, 4$, or 8) is a harmonic morphism with constant dilation (hence a horizontally homothetic harmonic map) we get:

Proposition 4.3. *For $n = 2, 4$ or 8, the composition of any weakly conformal map of S^{2n-1} or of S^n with the Hopf fibration $S^{2n-1} \longrightarrow S^n$ is a p-harmonic morphism with $p = 2n-1$ or n. In particular, the Hopf fibration $H : S^{2n-1} \longrightarrow S^n$ becomes a p-harmonic morphism for $p = 3, 7, 15, 4$ or 8 which is not a harmonic morphism after a suitable conformal change of the standard metric on the domain or target sphere is made.*

Example 4.4. We identify (S^3, g_3) with $(\hat{R}^3, \frac{4}{(1+|u|^2)^2} \sum_{i=1}^{i=3} d{u_i}^2)$ through stereographic projection, then it is not difficult to check that $\sigma : (S^3, g_3) \longrightarrow (S^3, g_3)$ given by $\sigma(u_1, u_2, u_3) = (u_1 - u_2, u_1 + u_2, \sqrt{2}u_3)$ is a conformal diffeomorphism such that $\sigma^* g_3 = e^{2\psi} g_3$, where $\psi = \frac{1}{2} \ln\{\frac{2(1+\sum_{i=1}^{i=3} u_i^2)^2}{(1+2\sum_{i=1}^{i=3} u_i^2)^2}\}$. Then by Proposition 4.3 we have a 3-harmonic morphism $H \circ \sigma : (S^3, g_3) \longrightarrow (S^2, g_2)$. Using Corollary 4.2 we see that the Hopf fibration $H : (S^3, (\lambda_2 \circ H)^2 g_3) \longrightarrow (S^2, g_2)$ is a 3-harmonic morphism which is not a harmonic morphism for any non-constant function $\lambda_2 : S^2 \longrightarrow (0, \infty)$.

Remark 4.5. Note that in their effort to construct non-trivial (i.e., not harmonic) examples of p-harmonic morphisms, the authors in [8] start with a horizontally weakly conformal map $\phi : (S^3, g_3) \longrightarrow (S^2, g_2)$ given by

$$(\cos s e^{ia}, \sin s e^{ib}) \longrightarrow (\cos \alpha(s), \sin \alpha(s) e^{i(ka+lb)}),$$

where the function $\alpha(s)$ is chosen such that ϕ is horizontally weakly conformal, where $s \in [0, \pi/2], a, b \in [0, 2\pi[$. Then they reduce the partial differential equations for p-harmonic map to an ordinary differential equation which is solved to produce p-harmonic morphisms $\phi : (S^3, (\frac{\sin \alpha}{\sin^2 2s})^{2(p-2)/(p-3)} (k^2 \sin^2 s + l^2 \cos^2 s)^{(p-1)/(p-3)} g_3) \longrightarrow (S^2, g_2)$. Notice that in their examples, if $p \neq 2$ then the metric, and hence the map, is not globally defined. By comparison, our examples are globally defined 3-harmonic morphisms $(S^3, g_3) \longrightarrow (S^2, g_2)$.

Example 4.6. Let $F : R^4 \times R^4 \longrightarrow R^4$ denote the standard multiplication in the real algebra of quaternionic numbers. It is proved in [3] that this harmonic morphism restricts to a harmonic morphism $f : S^3 \times S^3 \longrightarrow S^3$, where the target sphere is given the standard metric g_3 and the domain manifold is given the product metric $g_3 \otimes g_3$. One can check that the map f is non-trivial, i.e., it is not a projection, and that it has dilation $\lambda = \sqrt{2}$. Let $\sigma : (S^3, g_3) \longrightarrow (S^3, g_3)$ be given as in Example 4.4. Then, by Theorem 4.1, we have a submersive 3-harmonic morphism between compact manifolds: $\sigma \circ f : (S^3 \times S^3, g_3 \otimes g_3) \longrightarrow (S^3, g_3)$. Here, $p = n = 3$, hence it follows from [23] that all the fibres of $\sigma \circ f$ are minimal submanifolds of $(S^3 \times S^3, g_3 \otimes g_3)$. Therefore $\sigma \circ f$ determines a conformal foliation of $S^3 \times S^3$ with minimal leaves.

Using a result of N. H. Kuiper [20] on the existence of a conformal immersion from a simply connected, conformally flat manifold (M^n, g) into the standard sphere S^n we have:

Corollary 4.7. *For $n = 2$, 4 or 8, there exist $(2n-1)$-harmonic morphisms from any simply connected, conformally flat space (M^{2n-1}, g) into the standard sphere S^n. There also exist n-harmonic morphisms from S^{2n-1} into any simply connected, conformally flat space (M^n, g).*

To construct more examples of p-harmonic morphisms which are not harmonic morphisms, we first note that for $k \leq n$, the map $\rho : R^n \longrightarrow R^k$ given by $\rho(x) = \lambda Ax + b$ with $\lambda \in (0, \infty)$, $b \in R^k$ and A a $(k \times n)$-matrix satisfying $AA^t = Id_k$ is a p-harmonic morphism for any p. On the other hand, we know that the stereographic projection $P : S^n \backslash \{N\} \longrightarrow R^n$ is a conformal map. Therefore, we have:

Example 4.8. 1) The map $\rho \circ P : S^n \backslash \{N\} \longrightarrow R^k$ is an n-harmonic morphism for $n \geq 3$, where P denotes the stereographic projection and ρ the map defined above. In particular, $\varphi : S^4 \setminus \{N\} \longrightarrow R^2$ with

$$\varphi(x) = (1 - x_5)^{-1} (x_1 + 2x_2 + 3x_3 - 2x_4,\ -2x_1 + x_2 + 2x_3 + 3x_4)$$

is a 4-harmonic morphism.

2) The map $P^{-1} \circ \rho : R^n \longrightarrow S^k$ is a k-harmonic morphism for $k \geq 3$, where P^{-1} denotes the inverse of the stereographic projection and ρ the map defined above. In particular, for $\rho : R^4 :\longrightarrow R^3$ with $\rho(x_1, x_2, x_3, x_4) = (x_1 - x_3, x_1 + x_3, x_2 + x_4)$ and $P^{-1} : R^3 \longrightarrow S^3$ the inverse of the stereographic projection, we have a 3-harmonic morphism $R^4 \longrightarrow S^3$ given by

$$\varphi(x_1, x_2, x_3, x_4) = \nu^{-1}(2x_1 - 2x_3,\ 2x_1 + 2x_3,\ 2x_2 + 2x_4,\ \nu - 2),$$

where $\nu = 1 + 2x_1^2 + 2x_3^2 + (x_2 + x_4)^2$.

Example 4.9. Let $\pi : R^n \times R^k \longrightarrow R^n$ be an orthogonal projection, $\sigma_n : R^n \longrightarrow R^n$ be a weakly conformal map. Then $\sigma \circ \pi$ is an n-harmonic morphism.

Finally, we remark that there are several other classes of horizontally homothetic harmonic maps (see e.g. [1], [14]) including Riemannian submersions with minimal fibers and the radial projection $R^{n+1} \backslash \{0\} \longrightarrow S^n$ which can be used to construct non-trivial p-harmonic morphisms.

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