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On Boundaries of Unions of Sets with Positive Reach

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Abstract. A local regular behavior at almost all boundary points of a set in $\mathbb{R}^d$ representable as a locally finite union of sets with positive reach is shown. As an application, a limit formula for the volume of dilation of such a set with a small convex body is derived.

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1. Introduction

Locally finite unions of sets with positive reach were considered by Zähle [10] as a class extending the convex ring and still admitting the treatment of curvature measures. Recall that the *reach* of a set $X \subseteq \mathbb{R}^d$ (denoted by $\operatorname{reach} X$) is the largest number r such that any point z with $\operatorname{dist}(z, X) < r$ has a unique closest point x in X. We say that $X \subseteq \mathbb{R}^d$ is a $\mathcal{U}_{PR}$-set (or we write $X \in \mathcal{U}_{PR}$) if we can represent X as a locally finite union $X = \bigcup_i X^i$ of sets X^i such that $\bigcap_{i\in I} X^i$ has positive reach for any finite index set I (in particular, $\operatorname{reach} X^i > 0$ for any i). In [8], the curvature measures $C_k(X; \cdot)$ of $X \in \mathcal{U}_{PR}$ are defined by means of integrating suitable differential forms over the unit normal bundle $\operatorname{nor} X := \operatorname{supp} i_X$ with weight factor i_X, where the index function i_X is given by

$$i_X(x, n) := \mathbf{1}_X(x)\Big(1 - \lim_{r\to 0_+} \lim_{s\to 0_+} \chi\big(X \cap B(x + ((r + s)n, r)\big)\Big),$$

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$x \in \mathbb{R}^d$, $n \in S^{d-1}$ ($B(y,t)$ denotes the closed ball of centre y and radius t, S^{d-1} is the unit sphere in $\mathbb{R}^d$ and χ stands for the Euler-Poincaré characteristic).

If $\operatorname{reach} X > 0$ then $(x,n) \in \operatorname{nor} X$ if and only if $x \in \partial X$ and n is a unit outer normal to X, i.e., $n \in \operatorname{Nor}(X,x) \cap S^{d-1}$, where

$$\operatorname{Nor}(X,x) := \{v : v \cdot u \leq 0 \text{ for any } u \in \operatorname{Tan}(X,x)\}$$

and $\operatorname{Tan}(X,x)$ is the tangent cone of X at x, i.e., $0 \neq u \in \operatorname{Tan}(X,x)$ iff there exists a sequence $x_i \to x$, $x_i \in X \setminus \{x\}$, such that $\frac{x_i - x}{|x_i - x|} \to \frac{u}{|u|}$, $i \to \infty$, cf. [2, §3.1.21]. The interpretation of $\operatorname{nor} X$ for a $\mathcal{U}_{PR}$-set X is, however, much more complicated, particularly if the components X^i may osculate. The aim of this note is to show that, nevertheless, at almost all boundary points x, we may interpret unit vectors n with $(x,n) \in \operatorname{nor} X$ again as outer normals, and these are even unique, up to change of orientation. As consequence, we are able to characterize the curvature measure of order $d-1$ of a $\mathcal{U}_{PR}$-set as the surface area measure in the usual sense; if, in particular, X is compact, then $C_{d-1}(X;\cdot)$ is the total surface area of X times the distribution of the unit outer normal over the boundary. This property was already used e.g. in [7] but it seems that the argument given there was not sufficient. This work was motivated also by a remark in [4, Subsection 4.1] which can be obtained as a consequence of Theorem 1.

In the second part of this note, we apply the achieved result together with a Steiner-type formula due to Hug et al. [4] to express the increase of volume of a $\mathcal{U}_{PR}$-set X dilated by an infinitesimal multiple of a convex body (Theorem 3). Such a formula has already been proved in [6, Corollary 4.2] with stronger assumptions and by Hug [3, Theorem 3.3] for polyconvex sets. Its applications in stochastic geometry were considered by Kiderlen and Jensen [5].

2. A boundary property of $\mathcal{U}_{PR}$-sets

The *exoskeleton* $\operatorname{exo}(X)$ of a closed set $X \subseteq \mathbb{R}^d$ is the set of all $z \in \mathbb{R}^d \setminus X$ which do not have a unique nearest point in X. The *metric projection* $\xi_X : \mathbb{R}^d \setminus \operatorname{exo}(X) \to X$ is defined so that $\xi_X(x) \in X$ is the unique nearest point to x in X. The *reduced normal bundle* of X is (see [4])

$$N(X) := \left\{ \left(\xi_X(z), \frac{z - \xi_X(z)}{|z - \xi_X(z)|} \right) : \, z \notin X \cup \operatorname{exo}(X) \right\}.$$

(Note that $N(X)$ is called normal bundle in [4]; we use the adjective 'reduced' in order to avoid confusion with the unit normal bundle $\operatorname{nor} X$. Clearly $N(X) \subseteq \operatorname{nor} X$ if $X \in \mathcal{U}_{PR}$.) The *reach function of* X,

$$\delta(X; x, n) := \inf\{t \geq 0 : \, x + tn \in \operatorname{exo}(X)\}$$

is defined for all $(x,n) \in N(X)$.

Remark. The *local reach* of a set X at $x \in X$, $\operatorname{reach}(X,x)$, is defined by Federer [1] as the supremum of $r \geq 0$ such that to any point y with $y - x < r$ there exists a unique nearest point in X (equivalently, $y \notin \operatorname{exo}(X)$). Note that $\delta(X; x, n) > 0$ does not imply that $\operatorname{reach}(X,x) > 0$.

We say that a set $X \subseteq \mathbb{R}^d$ is locally at $x \in \partial X$ a *Lipschitz subgraph of zero differential* if there exist a unit vector n and a Lipschitz function f defined on the hyperplane $n^\perp$ with zero differential at the projection of x to $n^\perp$ and such that X agrees with the subgraph of f in some neighbourhood of x. Moreover, we say that X is locally at $x \in \partial X$ a *Lipschitz intergraph of zero differential* if there exists a unit vector n and two Lipschitz functions $f \le g$ defined on $n^\perp$ with the same value and zero differentials at the projection of x to $n^\perp$ and such that X agrees in a neighbourhood of x with the intergraph of f and g, i.e., with the set $\{z + sn : z \in n^\perp, f(z) \le s \le g(z)\}$.

In what follows, $\mathcal{H}^k$ will denote the k-dimensional Hausdorff measure.

Theorem 1. *Let $X \in \mathcal{U}_{PR}$. Then for $\mathcal{H}^{d-1}$-almost all $x \in \partial X$ there exists $n \in S^{d-1}$ such that one of the following two situations occurs:*

1. $\operatorname{Tan}(X,x) = \{u : u \cdot n \le 0\}$, *$i_X(x,m) = 1$ if $m = n$ and 0 otherwise, $\delta(X; x, n) > 0$ and X is locally at x a Lipschitz subgraph of zero differential.*
2. $\operatorname{Tan}(X,x) = n^\perp$, *$i_X(x,m) = 1$ if $m = \pm n$ and 0 otherwise, $\delta(X; x, \pm n) > 0$ and X is locally at x a Lipschitz intergraph of zero differential.*

The proof will be based on a few auxiliary results. The first of them is an easy consequence of [1, Remark 4.15 (3)].

Lemma 1. *If* $\operatorname{reach} Y > 0$ *then* $\mathcal{H}^{d-1}(\{x \in \partial Y : \dim \operatorname{Nor}(Y,x) > 1\}) = 0$.

Let $X = \bigcup_i X^i$ be a $\mathcal{U}_{PR}$-representation. Let $\partial^* X$ denote the set of all points $x \in \partial X$ such that

$$\dim \operatorname{Nor}\left(\bigcap_{i \in I} X^i, x\right) \le 1$$

for all nonempty finite index sets I. It follows from Lemma 1 that

$$\mathcal{H}^{d-1}(\partial X \setminus \partial^* X) = 0.$$

Lemma 2. *If $x \in \partial^* X$ then* $\dim \bigcup_i \operatorname{Nor}(X^i, x) \le 1$.

Proof. Assume that $\dim \bigcup_i \operatorname{Nor}(X^i, x) > 1$. Then there exist two linearly independent unit vectors m, n such that $(x,m) \in \operatorname{nor} X^i$ and $(x,n) \in \operatorname{nor} X^j$ for some i,j. But then $\dim \operatorname{Nor}(X^i \cap X^j, x) \ge 2$, hence $x \notin \partial^* X$. □

Proposition 1. *Let X, Y be subsets of $\mathbb{R}^d$ such that all the sets X, Y and $X \cap Y$ have positive reach. Let $x \in X \cap Y$ and $n \in S^{d-1}$ be such that*

$$\begin{aligned} \operatorname{Nor}(X,x) &= \{tn : t \ge 0\}, \\ \operatorname{Nor}(Y,x) &= \{-tn : t \ge 0\}, \\ \operatorname{Nor}(X \cap Y, x) &= \{tn : t \in \mathbb{R}\}. \end{aligned}$$

Then $x \in \operatorname{int}(X \cup Y)$.

Proof. Using Proposition 3 from [9] and its proof we can see that there exists an $\varepsilon > 0$ and Lipschitz functions f, g on $n^\perp$ such that

$$X \cap U_\varepsilon(x) = \operatorname{subgr} f \cap U_\varepsilon(x), \tag{1}$$

$$Y \cap U_\varepsilon(x) = \operatorname{supgr} g \cap U_\varepsilon(x) \tag{2}$$

($U_\varepsilon(x)$ denotes the ε-neighbourhood of x and subgr , supgr stands for the subgraph, supgraph, respectively). By the closeness of $\operatorname{nor}(X \cap Y)$, there exists $\delta > 0$ such that for any $y \in U_\delta(x)$,

$$(y, m) \in \operatorname{nor}(X \cap Y) \implies |m \cdot n| \geq \frac{1}{2}. \tag{3}$$

Assume without loss of generality that $\delta < \min\{\frac{1}{2}, \frac{\varepsilon}{2}, \frac{1}{2}\operatorname{reach}(X \cap Y)\}$. From [1, Theorem 4.8 (7)] we have

$$y \in X \cap Y \implies |(y - x) \cdot n| \leq \frac{|y - x|^2}{2\operatorname{reach}(X \cap Y)}. \tag{4}$$

We shall show that $f \geq g$ on $n^\perp \cap U_{\delta/2}(x)$, hence $U_{\delta/2}(x) \subseteq X \cup Y$ and, consequently, $x \in \operatorname{int}(X \cup Y)$.

Assume, for the contrary, that there is a point $t \in n^\perp$, $|t| < \delta/2$, with $f(t) < g(t)$. Then the segment $S := (t + \operatorname{lin} n) \cap \overline{U_\delta(x)}$ does not hit $X \cap Y$, but any point of S has its unique nearest point in $X \cap Y$. Let $y \in S$ and $z \in X \cap Y$ be such that

$$|y - z| = \min\{|y' - z'| : y' \in S, z' \in X \cap Y\}.$$

Assume first that y is an end point of S, hence, $|y - x| = \delta$ and, say, $(y - x) \cdot n = +\sqrt{\delta^2 - |t|^2}$. Then $(z - y) \cdot n \geq 0$ (otherwise, y would not be the closest point of S from z), and we have

$$\begin{aligned}
(z - x) \cdot n &= (z - y) \cdot n + (y - x) \cdot n \geq \sqrt{\delta^2 - |t|^2} \geq \frac{\sqrt{3}}{2}\delta \\
&= \frac{2\sqrt{3}\delta^2}{4\delta} > \frac{(\frac{3}{2}\delta)^2}{4\delta} \geq \frac{(|y - x| + |t|)^2}{4\delta} \\
&\geq \frac{|z - x|^2}{2\operatorname{reach}(X \cap Y)},
\end{aligned}$$

which contradicts (4). Hence, y must be an inner point of S. But then clearly $z - y \perp S \parallel n$ and $(z, \frac{y-z}{|y-z|}) \in \operatorname{nor}(X \cap Y)$, which is a contradiction to (3). $\square$

Corollary 1. *If $x \in \partial^* X$ then there exists an $n \in S^{d-1}$ such that $n \in \operatorname{Nor}(X^i, x)$ whenever $x \in X^i$.*

Proof. Let $x \in \partial^* X$ be a point for which the assertion is not true. Then there must be two sets X^i, X^j which satisfy the assumptions of Proposition 1. But then x were not a boundary point of X, a contradiction. $\square$

Proof of Theorem 1. Let $x \in \partial^* X$ and let n be the unit vector from Corollary 1. Assume (without loss of generality) that $x \in X^i$ if and only if $i \leq N$ ($N \in \mathbb{N}$). We shall distinguish two cases.

(a) $-n \notin \operatorname{Nor}(X^i, x)$ for some $i \leq N$. Then case 1. of Theorem 1 occurs; it remains to show that X is locally the subgraph of a Lipschitz function with zero differential. Using the method of proof of Proposition 1, each set X^i for $i \leq N$ can be locally represented at x as either a Lipschitz subgraph or a Lipschitz intergraph with zero differential at x. Let X^i be locally the subgraph of f and X^j the intergraph of $g \leq h$. We can apply Proposition 1 to the sets X^i and $X^j \oplus \{\alpha n : \alpha > 0\}$ and we get that $f \geq g$ on a neighbourhood of x, hence, $X^i \cup X^j$ is locally a Lipschitz subgraph again. By induction, we infer that $X^1 \cup \cdots \cup X^N$, and, consequently, also X, is locally at x a Lipschitz subgraph with zero differential at x.

(b) $-n \in \operatorname{Nor}(X^i, x)$ for all $i \leq N$. Then case 2. occurs: each X^i is locally at x a Lipschitz intergraph of functions $f^i \leq g^i$ with zero differentials at x. Applying Proposition 1 to the sets $X^i \oplus \{\pm\alpha n : \alpha > 0\}$ and $X^j \oplus \{\pm\alpha n : \alpha < 0\}$, we get that $f^i \leq g^j$ and $f^j \leq g^i$ on a neighbourhood of x. Consequently, $X^i \cup X^j$ is again locally at x an intergraph of Lipschitz functions with zero differentials at x and, by induction, the same property holds for $X^1 \cup \cdots \cup X^N$ and, consequently, also for X. □

As a corollary, we obtain the following result which has already been used in [7]. See also [4, Proposition 4.1].

Corollary 2. *If $X \in \mathcal{U}_{PR}$ then for any Borel subset $A \subseteq \mathbb{R}^d$ and a Borel subset B of S^{d-1} without antipodal points we have*

$$C_{d-1}(A \times B) = \mathcal{H}^{d-1}(\{x \in A \cap \partial X : \exists n \in B \cap \operatorname{Nor}(X, x)\}).$$

Remark. If B contains antipodal points a similar formula holds but the points $x \in \partial X$ where both n and $-n$ are outer normal to X have to be weighted by factor 2.

Theorem 1 motivates the question whether the reach of a $\mathcal{U}_{PR}$-set is positive at almost all boundary points. The answer is, however, negative, as illustrates the following example.

Example. There exists a set $X \in \mathcal{U}_{PR}$ in $\mathbb{R}^2$ such that

$$\mathcal{H}^1(\{x \in \partial X : \operatorname{reach}(X, x) = 0\}) > 0. \tag{5}$$

Indeed, let f be a real C^2 function on $[0, 1]$ such that its values and one-sided first and second derivatives at the boundary points 0 and 1 vanish, and such that 0 is a cumulation point of points where f vanishes but f' is nonzero. (e.g., we can take $f(x) = x^5(1 - x^5) \sin \frac{1}{x}$). Let further $C = [0, 1] \setminus \bigcup_i I_i$ be a nowhere dense compact set with positive Lebesgue measure obtained by removing countably many pairwise disjoint open intervals $I_1, I_2, \ldots$ from $[0, 1]$. Define a function g on $[0, 1]$ as zero on C and on each I_i, g is a homothetic copy of f (i.e., $g(x) = (b_i - a_i) f(\frac{x - a_i}{b_i - a_i})$ if $I_i = (a_i, b_i)$). Let X^1 be the subgraph of g (in $\mathbb{R}^2$) and X^2 be the lower halfplane in $\mathbb{R}^2$. Then $X = X^1 \cup X^2$ is a $\mathcal{U}_{PR}$-set fulfilling (5).

Remark. It is not difficult to see that a modification of the above example would yield even two convex bodies in $\mathbb{R}^2$ whose union X satisfies (5).

3. Increase of volume by dilation

In the sequel, we shall recall a Steiner-type formula for $\mathcal{U}_{PR}$-sets derived in [4] and derive a consequence strengthening the results from [6]. Let ω_k denote the volume of the unit ball in $\mathbb{R}^k$.

Theorem 2. ([4, Theorem 2.1, Sect. 3]) *If $X \in \mathcal{U}_{PR}$ and f is a measurable bounded function on $\mathbb{R}^d$ with compact support, then*

$$\int_{\mathbb{R}^d \setminus X} f \, d\mathcal{H}^d = \sum_{i=0}^{d-1} \omega_{d-1} \int_0^\infty \int_{N(X)} t^{d-1-i} \mathbf{1}_{\{t<\delta(X;x,n)\}} f(x+tn) \, C_i(X; d(x,n)) \, dt.$$

Given a convex body K, denote $\check{K} = \{-x : x \in K\}$ and let $h(K, \cdot)$ be the support function of K.

The following result strengthens [6, Corollary 4.2], removing some unnecessary assumptions.

Theorem 3. *Let X be a compact $\mathcal{U}_{PR}$-set and K a convex body in $\mathbb{R}^d$. Then*

$$\lim_{\varepsilon \to 0} \frac{\mathcal{H}^d((X \oplus \varepsilon \check{K}) \setminus X)}{\varepsilon} = 2 \int_{\operatorname{nor} X} h_K(-n) \, C_{d-1}(X; d(x,n)).$$

Remark. If, in particular, K is the unit ball, we obtain the formula

$$\lim_{\varepsilon \to 0} \frac{\mathcal{H}^d((X_\varepsilon) \setminus X)}{\varepsilon} = 2 C_{d-1}(X, \mathbb{R}^d \times S^{d-1}),$$

where $X_\varepsilon = \{y : \operatorname{dist}(y, X) \le \varepsilon\}$ is the ε-parallel set to X. The right hand side equals $\mathcal{H}^{d-1}(\partial X)$ if X is full-dimensional and $2\mathcal{H}^{d-1}(\partial X)$ if X is $(d-1)$-dimensional.

Proof. We can assume without loss of generality that K is contained in the unit ball of $\mathbb{R}^d$. We shall apply Theorem 2 to the functions

$$f_\varepsilon(z) = \mathbf{1}_{\{(z+\varepsilon K) \cap X \neq \emptyset\}}, \quad \varepsilon > 0.$$

Since f_ε is bounded and the curvature measures $C_i(X; \cdot)$ are Radon measures, we get

$$\frac{\mathcal{H}^d((X \oplus \varepsilon \check{K}) \setminus X)}{\varepsilon} = 2 \int_{N(X)} \int_0^\infty g_\varepsilon(t, x, n) \, dt \, C_{d-1}(X; d(x,n)) + o(\varepsilon),$$

where

$$g_\varepsilon(t, x, n) = \mathbf{1}_{\{\delta(X;x,n) > t\}} \varepsilon^{-1} f_\varepsilon(x + tn).$$

It follows from Theorem 1 that $C_{d-1}(X; \cdot) = C_{d-1}(X; \cdot \cap N(X))$ and, hence, we can integrate in the last expression over the whole support of $C_{d-1}(X; \cdot)$. We shall show that

$$G_\varepsilon(x, n) := \int_0^\infty g_\varepsilon(t, x, n) \, dt \to h(K, -n), \quad \varepsilon \to 0,$$

for $C_{d-1}(X;\cdot)$-almost all (x,n) and apply the Lebesgue dominated theorem to achieve the assertion.

Fix first any $(x,n) \in N(X)$ and denote for brevity $\delta := \delta(X;x,n)$ (note that $\delta > 0$ since $(x,n) \in N(X)$, see [4]). It follows from the definition of δ that X has no points inside the ball of centre $x + \delta n$ and radius δ. From the definition of the support function, and since K lies in the unit ball, we get that if $t > \varepsilon h(K,-n) + (\delta - \sqrt{\delta^2 - \varepsilon^2})$ then $g_\varepsilon(t,x,n) = 0$. Consequently,

$$G_\varepsilon(x,n) \le h(K,-n) + \frac{\delta - \sqrt{\delta^2 - \varepsilon^2}}{\varepsilon} \to h(K,-n), \quad \varepsilon \to 0. \tag{6}$$

To obtain a lower bound for G_ε, we can assume due to Theorem 1 that there exists a Lipschitz function, say F, defined on $n^\perp$, with zero differential at $\bar{x} := p_{n^\perp} x$, $x = \bar{x} + F(\bar{x})n$, and such that X is locally at x either the subgraph of F or an intergraph with another Lipschitz function on $n^\perp$ smaller or equal to F. Let $y \in \partial K$ be such that $y \cdot (-n) = h(K,-n)$ (clearly, $|y| \le 1$), and denote $\bar{y} = p_{n^\perp} y$. If $t > 0$ is such that $x + tn + \varepsilon K$ does not hit X then

$$F(\bar{x} + \varepsilon \bar{y}) - F(\bar{x}) < t - \varepsilon h(K,-n).$$

Since $dF(\bar{x}) = 0$, the left hand side is $o(\varepsilon)$ and we have

$$G_\varepsilon(x,n) \ge h(K,-n) - o(\varepsilon).$$

Together with (6) we get that $\lim_{\varepsilon \to 0} G_\varepsilon(x,n) = h(K,-n)$ for $C_{d-1}(X;\cdot)$-almost all (x,n). Note that (6) implies that $0 \le G_\varepsilon(x,n) \le h(K,-n) + 1 \le 2$ and, consequently, the Lebesgue dominated theorem may be applied to conclude the proof. □

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