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Multigraded Regularity: Syzygies and Fat Points

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Abstract. The Castelnuovo-Mumford regularity of a graded ring is an important invariant in computational commutative algebra, and there is increasing interest in multigraded generalizations. We study connections between two recent definitions of multigraded regularity with a view towards a better understanding of the multigraded Hilbert function of fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

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Introduction

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. If M is a finitely generated graded module over a $\mathbb{Z}$-graded polynomial ring over $\mathbf{k}$, its Castelnuovo-Mumford regularity, denoted $\mathrm{reg}(M)$, is an invariant that measures the difficulty

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of computations involving M. Recently, several authors (cf. [1, 18, 19]) have proposed extensions of the notion of regularity to a multigraded context.

Taking our cue from the study of the Hilbert functions of fat points in $\mathbb{P}^n$ (cf. [5, 11, 23]), we apply these new notions of multigraded regularity to study the coordinate ring of a scheme of fat points $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with the goal of understanding both the nature of regularity in a multigraded setting and what regularity may tell us about the coordinate ring of Z. This paper also complements the investigation in [16] of the Castelnuovo-Mumford regularity of fat points in multiprojective spaces.

In the study of the coordinate ring of a fat point scheme in $\mathbb{P}^n$ many authors have found beautiful relationships between algebra, geometry, and combinatorics (cf. [17] for a survey when $n = 2$). Extensions and generalizations of such results to the multigraded setting are potentially of both theoretical and practical interest. Schemes of fat points in products of projective spaces arise in algebraic geometry in connection with secant varieties of Segre varieties (cf. [3, 4]). More generally, the base points of rational maps between higher dimensional varieties may be non-reduced schemes of points, and in the case of maps between certain surfaces, the regularity of the ideals that arise may have implications for the implicitization problem in computer-aided design (cf. [8, 26]).

It is well known that the Castelnuovo-Mumford regularity of a finitely generated $\mathbb{Z}$-graded module M can be defined either in terms of degree bounds for the generators of the syzygy modules of M or in terms of the vanishing of graded pieces of local cohomology modules. (See [12].) Aramova, Crona, and De Negri define a notion of regularity based on the degrees appearing in a free bigraded resolution of a finitely generated module over a bigraded polynomial ring in [1]. (See also [20].) We extend this notion to the more general case in which M is a finitely generated multigraded module over the $\mathbb{Z}^k$-graded homogeneous coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ by assigning to M a *resolution regularity vector* $\underline{r}(M) \in \mathbb{N}^k$ (Definition 2.1) which gives bounds on the degrees of the generators of the multigraded syzygy modules of M. By contrast, Maclagan and Smith [19] use the local cohomology definition of regularity as their starting point in defining multigraded regularity for toric varieties. Since $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a toric variety, their definition specializes to a version of multigraded regularity given in terms of the vanishing of $H^i_B(M)_{\underline{p}}$, the degree $\underline{p} \in \mathbb{N}^k$ part of the ith local cohomology module of M, where B is the irrelevant ideal of the homogeneous coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. In this context, the multigraded regularity, which we will denote by $\operatorname{reg}_B(M)$, is a subset of $\mathbb{Z}^k$. It can be shown that the degrees of the generators of M lie outside the set $\bigcup_{1 \le i \le k}(e_i + \operatorname{reg}_B(M))$, and if M is actually $\mathbb{N}^k$-graded then the degrees of the generators must lie in $\mathbb{N}^k$.

If $k \ge 2$, the complement of $\bigcup_{1 \le i \le k}(e_i + \operatorname{reg}_B(M))$ in $\mathbb{N}^k$ may be unbounded; we thus lose a useful feature of regularity in the standard graded case. However, when $k = 2$, Hoffman and Wang [18] introduced a notion of strong regularity (their definition of weak regularity essentially agrees with [19]) which requires the vanishing of graded pieces of additional local cohomology modules and gives a bounded subset of $\mathbb{N}^2$ that contains the degrees of the generators of M. It would be interesting to develop a notion of strong regularity for other multigraded rings. Since the completion of this paper, the authors, together with Wang, have developed a way of thinking about multigraded regularity via $\mathbb{Z}$-graded coarsenings of $\mathbb{Z}^r$-gradings. (See [21] for details.)

We now give an outline of the paper and describe our results. In Section 1 we briefly introduce multigraded regularity as defined in [19]. We also recall basic notions related to fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and show that the degree of a fat point scheme can be computed directly from the multiplicities of the points in Proposition 1.7.

In Section 2, Proposition 2.2 shows how $\underline{r}(M)$ can be used to find a large subset of $\operatorname{reg}_B(M)$. We also extend the work of [20] to show how to use an almost regular sequence to compute $\underline{r}(M)$ if M is generated in degree $\underline{0}$ in Theorem 2.7. We study the connections between $\underline{r}(M)$ and the $\mathbb{Z}$-regularity of modules associated to the factors of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ in Section 3. In Section 4 and Section 5, we study $\operatorname{reg}_B(R/I_Z)$ when R/I_Z is the coordinate ring of a fat point scheme Z in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Theorem 4.2 shows that $\underline{r}(R/I_Z) = (r_1, \ldots, r_k)$ where $r_i = \operatorname{reg}(\pi_i(Z)) \subseteq \mathbb{P}^{n_i}$. Furthermore, we show that $\underline{r}(R/I_Z) + \mathbb{N}^k \subseteq \operatorname{reg}_B(R/I_Z)$ in Proposition 4.4, which improves on bounds that follow from Proposition 2.2. Moreover, if $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is also arithmetically Cohen-Macaulay, we show that we have equality of sets: $\underline{r}(R/I_Z) + \mathbb{N}^k = \operatorname{reg}_B(R/I_Z)$ (Theorem 4.7). In Section 5, we restrict our attention to fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with support in generic position and combine results of [10] and [14] to show that

$$\{(i,j) \in \mathbb{N}^2 \mid (i,j) \geq (m_1 - 1, m_1 - 1) \text{ and } i + j \geq \max\{m - 1, 2m_1 - 2\}\} \subseteq \operatorname{reg}_B(Z),$$

where $m = \sum m_i$ and $m_1 \geq m_2 \geq \cdots \geq m_s$ (Theorem 5.1). At the end of the paper we have included an appendix containing some modified versions of results found in [19].

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1. Setup

1.1. The homogeneous coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. Let $\mathbb{N}$ denote the natural numbers $0, 1, \ldots$. The coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is the multigraded polynomial ring $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ where $\deg x_{i,j} = e_i$, the ith standard basis vector of $\mathbb{Z}^k$. Because R is an $\mathbb{N}^k$-graded ring, $R = \bigoplus_{\underline{i} \in \mathbb{N}^k} R_{\underline{i}}$

and $R_{\underline{i}}$ is a finite dimensional vector space over $\mathbf{k}$ with a basis consisting of all monomials of multidegree $\underline{i}$. Thus, $\dim_{\mathbf{k}} R_{\underline{i}} = \binom{n_1+i_1}{n_1}\binom{n_2+i_2}{n_2}\cdots\binom{n_k+i_k}{n_k}$, where $\underline{i} = (i_1, \dots, i_k)$.

Note that R is the homogeneous coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ viewed as a toric variety of dimension $N := n_1 + \cdots + n_k$. (See [7] for a comprehensive introduction to this point of view.) The homogeneous coordinate ring of a toric variety is modeled after the homogeneous coordinate ring of $\mathbb{P}^n$. The space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is the quotient of $\mathbb{A}^{N+k} - V(B)$, where $B = \bigcap_{i=1}^{k} \langle x_{i,j} \mid j = 0, \dots, n_i \rangle$ is its square-free monomial "irrelevant" ideal. Note that if $k = 1$, then B is just the irrelevant maximal ideal of the coordinate ring of projective space.

The $\mathbb{N}^k$-homogeneous ideals of R define subschemes of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. As in the standard graded case, the notion of saturation plays an important role.

Definition 1.1. *Let $I \subseteq R$ be an $\mathbb{N}^k$-homogeneous ideal. The saturation of I with respect to B is* $\operatorname{sat}(I) = \{f \in R \mid fB^j \subseteq I,\ j \gg 0\}$.

Two homogeneous ideals define the same subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ if and only if their saturations with respect to the irrelevant ideal are equal. (See Corollary 3.8 in [7].)

1.2. Multigraded modules and regularity

We shall work throughout with finitely generated $\mathbb{Z}^k$-graded R-modules $M = \bigoplus_{\underline{t} \in \mathbb{Z}^k} M_{\underline{t}}$. Without loss of generality, we may restrict our attention to $\mathbb{N}^k$-graded modules, since the $\underline{t} \in \mathbb{Z}^k$ with $M_{\underline{t}} \neq 0$ must be contained in $\underline{p} + \mathbb{N}^k$ for some $\underline{p} \in \mathbb{Z}^k$ if M is finitely generated. Write $\underline{p} = \underline{p}^+ - \underline{p}^-$ where $\underline{p}^+, \underline{p}^- \in \mathbb{N}^k$. Shifting degrees by $-\underline{p}^-$ yields a finitely generated $\mathbb{N}^k$-graded module.

When M is a finitely generated $\mathbb{N}^k$-graded R-module, it is useful to view M as both an $\mathbb{N}^1$-graded module and an $\mathbb{N}^k$-graded module. We introduce some notation and conventions for translating between the $\mathbb{N}^k$ and $\mathbb{N}^1$ gradings of a module. Let $\underline{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$, and let $\underline{1} = (1, \dots, 1)$. If $m \in M$ has multidegree $\underline{a} \in \mathbb{N}^k$, define its $\mathbb{N}^1$-degree to be $\underline{a} \cdot \underline{1}$.

We will use $\mathcal{H}_M$ to denote the multigraded Hilbert function $\mathcal{H}_M(\underline{t}) := \dim_{\mathbf{k}} M_{\underline{t}}$, and H_M to denote the $\mathbb{N}^1$-graded Hilbert function $H_M(t) := \dim_{\mathbf{k}} M_t$. Because $M_t = \bigoplus_{t_1+\cdots+t_k=t} M_{(t_1,\dots,t_k)}$, we have the identity

$$H_M(t) = \sum_{t_1+\cdots+t_k=t} \mathcal{H}_M(t_1, \dots, t_k) \qquad \text{for all } t \in \mathbb{N}.$$

If I_Y is the B-saturated ideal defining a subscheme $Y \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then we sometimes write $\mathcal{H}_Y$ (resp. H_Y) for $\mathcal{H}_{R/I_Y}$ (resp. H_{R/I_Y}).

We use the notion of multigraded regularity developed in [19]. To discuss this notion of regularity, we require a preliminary definition.

Definition 1.2. *Let $i \in \mathbb{Z}$ and set*

$$\mathbb{N}^k[i] := \bigcup (\operatorname{sign}(i)\underline{p} + \mathbb{N}^k) \subset \mathbb{Z}^k$$

where the union is over all $\underline{p} \in \mathbb{N}^k$ whose coordinates sum to $|i|$. (In the notation of [19],§ 4*, we have taken $\mathcal{C}$ to be the set of standard basis vectors of $\mathbb{Z}^k$.)*

Note that $\mathbb{N}^k[i]$ may not be contained in $\mathbb{N}^k$. The generality of Definition 1.2 is necessary because $\mathbb{N}^k[i]$ will be used to describe the degrees in which certain local cohomology modules of $\mathbb{N}^k$-graded modules vanish, and these local cohomology modules may be nonzero in degrees with negative coordinates.

Definition 1.3. (Definition 4.1 in [19]) *Let M be a finitely generated $\mathbb{N}^k$-graded R-module. If $\underline{m} \in \mathbb{Z}^k$, we say that M is $\underline{m}$-regular if $H^i_B(M)_{\underline{p}} = 0$ for all $\underline{p} \in \underline{m} + \mathbb{N}^k[1-i]$ for all $i \geq 0$. The multigraded regularity of M, denoted* $\operatorname{reg}_B(M)$*, is the set of all $\underline{m}$ for which M is $\underline{m}$-regular.*

When $M = R/I_Y$, the $\mathbb{N}^k$-graded coordinate ring associated to a scheme $Y \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, we shall write $\operatorname{reg}_B(Y)$ to denote $\operatorname{reg}_B(R/I_Y)$. If $k = 1$ and $M \neq 0$, then $\operatorname{reg}_B(M)$ is a subset of $\mathbb{N}^1$, and there exists some $r \in \mathbb{N}^1$ such that $\operatorname{reg}_B(M) = \{i \mid i \geq r\}$. In this case, we will simply write $\operatorname{reg}(M) = \operatorname{reg}_B(M) = r$. Note that $\operatorname{reg}(M)$ is the standard Castelnuovo-Mumford regularity. When $k = 2$, Definition 1.3 is essentially the same as the notion of weak regularity (Definition 3.1 in [18]) of Hoffman and Wang.

Remark 1.4. As one might expect, $\operatorname{reg}_B(R) = \mathbb{N}^k$. Indeed, it follows from Example 6.5 in [19] that it is enough to show the corresponding fact for a notion of multigraded regularity for the sheaf $\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}}$. This can be done using the Künneth formula generalizing the proof for the case $k = 2$ in Proposition 2.5 of [18]. A topological approach is given in Proposition 6.10 of [19]. For a related result see Proposition 4.3 in [18].

It will also be useful to have the following weaker condition of *multigraded regularity from level ℓ*:

Definition 1.5. (Definition 4.5 in [19]) *Given $\ell \in \mathbb{N}$, the module M is $\underline{m}$-regular from level ℓ if $H^i_B(M)_{\underline{p}} = 0$ for all $i \geq \ell$ and all $\underline{p} \in \underline{m} + \mathbb{N}^k[1-i]$. The set of all $\underline{m}$ such that M is $\underline{m}$-regular from level ℓ is denoted* $\operatorname{reg}^\ell_B(M)$.

Note that $\operatorname{reg}^\ell_B(M) \supseteq \operatorname{reg}_B(M)$ for any finitely generated multigraded R-module M. However, even when $M = R$, the inequality may be strict.

Example 1.6. Let R be the homogeneous coordinate ring of $\mathbb{P}^2 \times \mathbb{P}^2$. We will show that $(-1, 0) \in \operatorname{reg}^4_B(R)$. By Definition 1.5 we only need to check vanishings of graded pieces of $H^i_B(R)$ for $i \geq 4$ which is equivalent to checking vanishings of $H^i(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a, b))$ for $i \geq 3$. (See §6 in [19].) We let $H^i(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a, b))$ denote $H^i(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a, b))$ and $H^i(\mathcal{O}_{\mathbb{P}^2}(a))$ denote $H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a))$. By the Künneth formula, $H^3(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a, b))$ is the direct sum

$$\bigoplus_{i+j=3,\ i,j\geq 0} H^i(\mathcal{O}_{\mathbb{P}^2}(a)) \otimes H^j(\mathcal{O}_{\mathbb{P}^2}(b)).$$

Since

$$H^1(\mathcal{O}_{\mathbb{P}^2}(d)) = H^3(\mathcal{O}_{\mathbb{P}^2}(d)) = 0$$

for all integers d, each of the terms in the direct sum has a factor that is zero.

Similarly, if we compute $H^4(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a,b))$ using the Künneth formula, the only possible nonzero contribution to the direct sum comes from $H^2(\mathcal{O}_{\mathbb{P}^2}(a)) \otimes H^2(\mathcal{O}_{\mathbb{P}^2}(b))$, which is nonzero if and only if both $a, b \leq -3$. However, the vanishing conditions needed for $(-1,0)$ to be in $\operatorname{reg}^4_B(R)$ only require vanishing of $H^4(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a,b))$ for $(a,b) \geq (-5,0), (-4,-1), (-3,-2), (-2,-3), (-1,-4)$.

All of the cohomology groups $H^i(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a,b))$ vanish for $i \geq 5$ since

$$H^i(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a,b)) = \bigoplus_{j_1+j_2=i} H^{j_1}(\mathcal{O}_{\mathbb{P}^2}(a)) \otimes H^{j_2}(\mathcal{O}_{\mathbb{P}^2}(b))$$

and $i \geq 5$ implies that at least one of j_1 and j_2 is at least 3. Therefore, $(-1,0) \in \operatorname{reg}^4_B(R) \supsetneq \operatorname{reg}_B(R)$.

1.3. Hilbert functions of points

We recall some facts about points in multiprojective spaces. If $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a point, then the ideal $I_P \subseteq R$ associated to P is the prime ideal $I_P = \langle L_{1,1}, \ldots, L_{1,n_1}, \ldots, L_{k,1}, \ldots, L_{k,n_k}\rangle$ with $\deg L_{i,j} = e_i$. Let $X = \{P_1, \ldots, P_s\}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and let $m_1, \ldots, m_s$ be positive integers. Set $I_Z = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s}$ where $I_{P_i} \leftrightarrow P_i$. Then I_Z defines the scheme of fat points $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

The degree of Z is its length as a 0-dimensional subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Proposition 1.7. *The degree of $Z = m_1P_1 + \cdots + m_sP_s$ is*

$$\sum_{i=1}^{s} \binom{N + m_i - 1}{m_i - 1}.$$

Proof. The ideal I_Z is a B-saturated ideal defining a finite length subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We will compute the degree of Z by computing the lengths of the stalks of the structure sheaf of Z at each of the points P_i. The stalk of $\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}}$ at a point P_i is a local ring isomorphic to $O = k[x_1, \ldots, x_N]_{\langle x_1,\ldots,x_N\rangle}$ where the x_i are indeterminates and $\langle x_1, \ldots, x_N\rangle$ is a maximal ideal. The length of $\mathcal{O}_{Z,P_i} = O/\langle x_1, \ldots, x_N\rangle^{m_i}$ is

$$\sum_{j=0}^{m_i-1} \binom{N + j - 1}{j},$$

so the result follows once we apply the identity $\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$. □

Short exact sequences constructed by taking a hyperplane section arise frequently in proofs involving regularity in the standard graded case. In the multigraded generalization, we will employ the use of hypersurfaces of each multidegree e_i. Algebraically, we need the following lemma, which generalizes the reduced case of Lemma 3.3 in [24].

Lemma 1.8. *If I_Z is the defining ideal of Z, a set of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then, for each $i = 1, \ldots, k$, there exists an $L \in R_{e_i}$ that is a nonzerodivisor on R/I_Z.*

Proof. We will show only the case $i = 1$. Since the primary decomposition of I_Z is $I_Z = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s}$, the set of zerodivisors of R/I_Z, consists of the set $\bigcup_{i=1}^{s} I_{P_i}$. It will suffice to show $\displaystyle\bigcup_{i=1}^{s} (I_{P_i})_{e_1} \subsetneq R_{e_1}$. It is clear that $(I_{P_i})_{e_1} \subsetneq R_{e_1}$ for each $i = 1, \ldots, s$. Because the field $\mathbf{k}$ is infinite, the vector space R_{e_1} cannot be expressed as a finite union of vector spaces, and hence, $\bigcup_{i=1}^{s} (I_{P_i})_{e_i} \subsetneq R_{e_1}$. $\square$

Using Lemma 1.8 we can describe rules governing the behavior of the multigraded Hilbert function of a set of fat points.

Proposition 1.9. *Let Z be a set of fat points of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $\mathcal{H}_Z$. Then*

(i) *for all $\underline{i} \in \mathbb{N}^k$ and all $1 \leq j \leq k$, $\mathcal{H}_Z(\underline{i}) \leq \mathcal{H}_Z(\underline{i} + e_j)$,*
(ii) *if $\mathcal{H}_Z(\underline{i}) = \mathcal{H}_Z(\underline{i} + e_j)$, then $\mathcal{H}_Z(\underline{i} + e_j) = \mathcal{H}_Z(\underline{i} + 2e_j)$,*
(iii) *$\mathcal{H}_Z(\underline{i}) \leq \deg(Z)$ for all $\underline{i} \in \mathbb{N}^k$.*

Proof. To prove (i) and (ii) use the nonzerodivisors of Lemma 1.8 to extend the proofs of Proposition 1.3 in [14] for fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.
For (iii), if $Z = m_1P_1 + \cdots + m_sP_s$, then $\deg(Z)$ is an upper bound on the number of linear conditions imposed on the forms that pass through the points $P_1, \ldots, P_s$ with multiplicity at least m_i at each point P_i. $\square$

If I_Z defines a set of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then the computation of $\operatorname{reg}_B(Z)$, as defined by Definition 1.3, depends only upon knowing $\mathcal{H}_Z$. Indeed:

Theorem 1.10. (Proposition 6.7 in [19]) *Let Z be a set of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then $\underline{i} \in \operatorname{reg}_B(Z)$ if and only if $\mathcal{H}_Z(\underline{i}) = \deg(Z)$.*

Remark 1.11. The set of reduced points $Z = P_1 + P_2 + \cdots + P_s$ is said to be in *generic position* if $\mathcal{H}_Z(\underline{i}) = \min\{\dim_{\mathbf{k}} R_{\underline{i}}, s\}$ for all $\underline{i} \in \mathbb{N}^k$. Hence, if Z is in generic position, $\operatorname{reg}_B(Z) = \{\underline{i} \mid \dim_{\mathbf{k}} R_{\underline{i}} \geq s\}$.

2. Regularity and syzygies

In the $\mathbb{N}^1$-graded case, the definition of regularity can be formulated in terms of the degrees that appear as generators in the minimal free graded resolution of M. In this section we discuss a multigraded version of this definition extending the bigraded generalization that was given in [1] and studied further in [20].

We define a multigraded version of the notions of x- and y-regularity from [1].

Definition 2.1. *Let*

$$r_\ell := \max\{a_\ell \mid \operatorname{Tor}_i^R(M, \mathbf{k})_{(a_1, \ldots, a_\ell + i, \ldots, a_k)} \neq 0\}$$

for some i and for some $a_1, \ldots, a_{\ell-1}, a_{\ell+1}, \ldots, a_k$. We will call $\underline{r}(M) := (r_1, \ldots, r_k)$ the resolution regularity vector of M.

Note that if $\underline{r}(M) = (r_1, \dots, r_k)$ is the resolution regularity vector of a module M, then the multidegrees appearing at the ith stage in the minimal graded free resolution of M have ℓth coordinate bounded above by $r_\ell + i$. Indeed,

$$R(-b_1, \dots, -b_k)_{(a_1, \dots, a_\ell + i, \dots, a_k)} \neq 0$$

exactly when

$$(a_1 - b_1, \dots, a_\ell + i - b_\ell, \dots, a_k - b_k)$$

has nonnegative coordinates, and

$$\operatorname{Tor}_i^R(M, \mathbf{k})_{(a_1, \dots, a_\ell + i, \dots, a_k)} \neq 0$$

when $R(-a_1, \dots, -a_{\ell-1}, -a_\ell - i, -a_{\ell+1}, \dots, -a_k)$ appears as a summand of the module at the ith stage in the resolution.

The resolution regularity vector of a module allows us to compute a lower bound on the multigraded regularity of a module.

Proposition 2.2. *Let M be a finitely generated $\mathbb{N}^k$-graded R-module with resolution regularity vector $\underline{r}(M) = (r_1, \dots, r_k)$. If $\underline{p} \in \underline{r}(M) + \mathbb{N}^k$, then*

$$\bigcup_{\underline{a} \in \mathbb{N}^k,\, \underline{a} \cdot \underline{1} = m-1} \underline{p} + m \cdot \underline{1} - \underline{a} + \mathbb{N}^k \subseteq \operatorname{reg}_B(M)$$

where $m = \min\{N+1, \operatorname{proj-dim} M\}$. Note that this set equals

$$\underline{p} + m \cdot \underline{1} + \mathbb{N}^k[-(m-1)]$$

if $m > 0$.

Proof. Let $E.$ be the minimal free multigraded resolution of M where $E_i = \bigoplus R(-\underline{q}_{ij})$ with $\underline{q}_{ij} \in \mathbb{N}^k$ and $\underline{q}_{ij} \leq \underline{p} + i \cdot \underline{1}$ for $i = 0, \dots, \operatorname{proj-dim} M$. Therefore, by Remark 1.4 we have the following bound on the multigraded regularity of E_i,

$$\operatorname{reg}_B(E_i) \supseteq \bigcap (\underline{q}_{ij} + \mathbb{N}^k) \supseteq \underline{p} + i \cdot \underline{1} + \mathbb{N}^k.$$

If $\operatorname{proj-dim} M = 0$, then it is immediate that $\underline{p} + \mathbb{N}^k \subseteq \operatorname{reg}_B(M)$. So assume $\operatorname{proj-dim} M > 0$. Let K_0 be the syzygy module of M. So we have a short exact sequence

$$0 \to K_0 \to E_0 \to M \to 0.$$

By Lemma A.1 we then have

$$\bigcap_{1 \leq j \leq k} (-e_j + \operatorname{reg}_B^1(K_0)) \cap \operatorname{reg}_B^0(E_0) \subseteq \operatorname{reg}_B^0(M) = \operatorname{reg}_B(M).$$

Since $\mathrm{reg}^1_B(K_0) \subseteq (-e_j + \mathrm{reg}^1_B(K_0))$ for each j, we have

$$\begin{aligned} \mathrm{reg}^1_B(K_0) \cap (\underline{p} + \mathbb{N}^k) &\subseteq \mathrm{reg}^1_B(K_0) \cap \mathrm{reg}^0_B(E_0) \\ &\subseteq \bigcap_{1 \leq j \leq k} (-e_j + \mathrm{reg}^1(K_0)) \cap \mathrm{reg}^0_B(E_0) \subseteq \mathrm{reg}_B(M). \end{aligned}$$

Applying the revised Corollary 7.3 of [19] (see Theorem A.2) implies that

$$\bigcup_{\phi:[m]\to[k]} \left(\bigcap_{1 \leq i \leq m} -e_{\phi(2)} - \cdots - e_{\phi(i)} + \mathrm{reg}^i(E_i) \right) \subseteq \mathrm{reg}^1_B(K_0) \tag{1}$$

from which we deduce that

$$\bigcup_{\phi:[m]\to[k]} \left(\bigcap_{1 \leq i \leq m} -e_{\phi(2)} - \cdots - e_{\phi(i)} + \underline{p} + i \cdot \underline{1} + \mathbb{N}^k \right) \subseteq \mathrm{reg}^1_B(K_0). \tag{2}$$

Here we are using the fact that $\mathrm{reg}_B(E_i)$ is contained in $\mathrm{reg}^\ell_B(E_i)$ for all $\ell \geq 0$. Suppose that $\phi : [m] \to [k]$. Consider

$$\bigcap_{1 \leq i \leq m} -e_{\phi(2)} - \cdots - e_{\phi(i)} + \underline{p} + i \cdot \underline{1} + \mathbb{N}^k. \tag{3}$$

The maximum value of the jth coordinate of $-e_{\phi(2)} - \cdots - e_{\phi(i)} + \underline{p} + i \cdot \underline{1}$ over $i = 1, \ldots, m$ occurs when $i = m$. Indeed, if the maximum value of the jth coordinate occurs for some $i < m$, then consider $-e_{\phi(2)} - \cdots - e_{\phi(i)} - e_{\phi(i+1)} + \underline{p} + (i+1) \cdot \underline{1}$. Since the difference

$$-e_{\phi(2)} - \cdots - e_{\phi(i)} - e_{\phi(i+1)} + \underline{p} + (i+1) \cdot \underline{1} - (-e_{\phi(2)} - \cdots - e_{\phi(i)} + \underline{p} + i \cdot \underline{1})$$

is $-e_{\phi(i+1)} + \underline{1}$, we see that if $\phi(i+1) = j$, then the two vectors are equal in the jth coordinate. Otherwise, the vector $-e_{\phi(2)} - \cdots - e_{\phi(i)} - e_{\phi(i+1)} + \underline{p} + (i+1) \cdot \underline{1}$ has a bigger jth coordinate. So we see that the maximum value of each of the coordinates must occur when $i = m$ (and possibly earlier as well). Therefore, the intersection in (3) is equal to $\underline{p} + m \cdot \underline{1} - \sum_{i=2}^m e_{\phi(i)} + \mathbb{N}^k$. As ϕ varies over all possible functions from $[m]$ to $[k]$, the set of all vectors $\sum_{i=2}^m e_{\phi(i)}$ is just the set of all $\underline{a} \in \mathbb{N}^k$ such that $\underline{a} \cdot \underline{1} = m - 1$. (2) is just

$$\bigcup_{\underline{a} \in \mathbb{N}^k,\, \underline{a} \cdot \underline{1} = m-1} \underline{p} + m \cdot \underline{1} - \underline{a} + \mathbb{N}^k.$$

is also a subset of $(\underline{p} + \mathbb{N}^k)$ we have

$$\bigcup_{\underline{a} \in \mathbb{N}^k,\, \underline{a} \cdot \underline{1} = m-1} \underline{p} + m \cdot \underline{1} - \underline{a} + \mathbb{N}^k \subseteq \mathrm{reg}^1_B(K_0) \cap (\underline{p} + \mathbb{N}^k) \subseteq \mathrm{reg}_B(M)$$

as desired. □

The $\mathbb{N}^1$-graded regularity of a multigraded module M also gives the following rough bound on $\mathrm{reg}_B(M)$.

Corollary 2.3. *Let M be a finitely generated $\mathbb{N}^k$-graded R-module. If* $\operatorname{reg}(M) \leq r$*, then*

$$\operatorname{reg}_B(M) \supseteq \bigcup_{\underline{a}\cdot\underline{1}=m-1} (r+m)\cdot\underline{1} - \underline{a} + \mathbb{N}^k$$

where $m = \min\{N+1, \operatorname{proj-dim} M\}$.

Proof. Let $E.$ be the minimal free multigraded resolution of M. Since $\operatorname{reg}(M) \leq r$, we know that $\operatorname{reg}(E_i) \leq r+i$ for all $i \geq 0$. Since $E_i = \bigoplus R(-\underline{q}_{ij})$ with $\underline{q}_{ij} \in \mathbb{N}^k$, this means $\underline{q}_{ij} \cdot \underline{1} \leq r+i$. Therefore, the resolution regularity vector $\underline{r}(M) \leq r \cdot \underline{1}$. The result now follows from Proposition 2.2. □

Remark 2.4. Note that if $k=1$ Proposition 2.2 is equivalent to the statement that if an $\mathbb{N}^1$-graded module is p-regular, then it is also q-regular for all $q \in p+\mathbb{N}^1$. When $k>1$, Proposition 2.2 may not give all of $\operatorname{reg}_B(M)$. Indeed, Theorem 7.2 of [19] will not give all of $\operatorname{reg}_B(M)$ even using more detailed information about multidegrees in a resolution than given by $\underline{r}(M)$. (See Example 7.6 in [19].)

Remark 2.5. Let M be a finitely generated $\mathbb{N}^k$-graded R-module with resolution regularity vector $\underline{r}(M)$. Under extra hypotheses on M, the bound of Proposition 2.2 can be improved to $\operatorname{reg}_B(M) \supseteq \underline{r}(M) + \mathbb{N}^k$. For example, in Proposition 4.4 we will show that $\underline{r}(M) + \mathbb{N}^k \subseteq \operatorname{reg}_B(M)$ if $M = R/I_Z$ is the coordinate ring of a set of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

It seems, therefore, natural to ask the following question:

Question 2.6. *Let M be a finitely generated $\mathbb{N}^k$-graded R-module with resolution regularity vector $\underline{r}(M)$. What extra conditions on M imply* $\operatorname{reg}_B(M) \supseteq \underline{r}(M) + \mathbb{N}^k$?

(Since the submission of this paper, Hà has given an example showing that this inclusion may not hold, as well as other related results in [15].)

As we have observed, the resolution regularity vector of the $\mathbb{N}^k$-graded R module M gives us partial information about $\operatorname{reg}_B(M)$. We close this section by describing how to compute the resolution regularity vector for some classes of M. This procedure is a natural extension of the bigraded case as given by [20], which itself was a generalization of the graded case [2]. If M is any finitely generated $\mathbb{N}^k$-graded R-module, then we shall use $M_a^{[\ell]}$ to denote the $\mathbb{N}^{k-1}$-graded module

$$M_a^{[\ell]} := \bigoplus_{(j_1,\dots,j_{\ell-1},j_{\ell+1},\dots,j_k)} M_{(j_1,\dots,j_{\ell-1},a,j_{\ell+1},\dots,j_k)}.$$

Observe that $M_a^{[\ell]}$ is a $\mathbf{k}[x_{1,0},\dots,\hat{x}_{\ell,0},\dots,\hat{x}_{\ell,n_\ell},\dots,x_{k,n_k}]$-module, where ˆ means the element is omitted.

An element $x \in R_{e_\ell}$ is a *multigraded almost regular element* for M if

$$\langle 0 :_M x\rangle_a^{[\ell]} = 0 \quad \text{for } a \gg 0.$$

A sequence $x_1,\dots,x_t \in R_{e_\ell}$ is a *multigraded almost regular sequence* if for $i = 1,\dots,t$, x_i is a multigraded almost regular element for $M/\langle x_1,\dots,x_{i-1}\rangle M$. A multigraded almost regular element need not be almost regular in the usual sense, even for bigraded rings since we may have $\langle 0 :_M x\rangle_a^{[1]} = 0$ for $a \geq a_0$, but $\langle 0 :_M x\rangle_{(a_0-1,j)} \neq 0$ for infinitely many j. (Note that in the single graded case, almost regular elements were studied in [22] under the name of filter regular elements.)

Now suppose that for each $\ell = 1,\dots,k$ we have a basis $y_{\ell,0},\dots,y_{\ell,n_\ell}$ of R_{e_ℓ} that forms a multigraded almost regular sequence for M. Because $\mathbf{k}$ is infinite, it is always possible to find such a basis; one can derive a proof of this fact by adapting the proof of Lemma 2.1 of [20] for the bigraded case to the multigraded case. Set

$$s_{\ell,j} := \max\left\{a \;\middle|\; \left\langle 0 :_{M/\langle y_{\ell,0},\dots,y_{\ell,j-1}\rangle M} y_{\ell,j}\right\rangle_a^{[\ell]} \neq 0\right\},$$

and $s_\ell := \max\{s_{\ell,0},\dots,s_{\ell,n_\ell}\}$. Theorem 2.2 in [20] then extends to the $\mathbb{N}^k$-graded case as follows:

Theorem 2.7. *Let M be a finitely generated multigraded R-module generated in degree $\underline{0}$, for $\ell = 1,\dots,k$, let $y_{\ell,0},\dots,y_{\ell,n_\ell}$ be a basis R_{e_ℓ} that forms a multigraded almost regular sequence for M. Then $\underline{r}(M) = (s_1,\dots,s_k)$.*

3. Resolution regularity and projections of varieties

It is natural to ask if the $\mathbb{N}^1$-regularity of the projections of a subscheme V of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ onto the factors $\mathbb{P}^{n_i}$ are related in a nice way to the coordinates appearing in the resolution regularity vector of R/I_V. We show in Theorem 4.2 that if V is a set of fat points, then the ith coordinate in the resolution regularity vector of R/I_V is precisely the $\mathbb{N}^1$-regularity of the projection of V to $\mathbb{P}^{n_i}$. However, Example 3.1 below shows that in general no such relationship can hold for arbitrary subschemes of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Example 3.1. Let $R = \mathbf{k}[x_0,x_1,x_2,y_0,y_1,y_2]$, and let

$$I = \langle x_0,x_1\rangle \cap \langle x_0-x_1,x_2\rangle \cap \langle y_0,y_1\rangle \cap \langle y_0-y_1,y_2\rangle$$

be the defining ideal of a union of 4 planes in $\mathbb{P}^2 \times \mathbb{P}^2$. The vector $\underline{r}(R/I)$ must be strictly positive in both coordinates since I has a minimal generator of bidegree $(2,2)$. However, the projection of the scheme onto either factor of $\mathbb{P}^2$ is surjective. Therefore, the regularity of the projections of the scheme defined by I is zero.

We consider some circumstances where the resolution regularity vector of a module M is given by the regularities of modules associated to the factors of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We have the following proposition which generalizes Lemma 6.2 in [20].

Proposition 3.2. *Let $R_i = \mathbf{k}[x_{i,0},\dots,x_{i,n_i}]$ and let $M_i \neq 0$ be an $\mathbb{N}^1$-graded R_i-module. The ith coordinate of the resolution regularity vector of $M_1 \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} M_k$ is* $\operatorname{reg}(M_i)$.

Proof. The proof proceeds as in the case $k = 2$ in [20]. The point is that the tensor product (over $\mathbf{k}$) of minimal free graded resolutions of the modules M_i is the minimal free multigraded resolution of $M_1 \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} M_k$. We can read off the resolution regularity vector from the multidegrees appearing in this resolution. □

We have the following corollary.

Corollary 3.3. *Let I_i be a proper homogeneous ideal in R_i, the coordinate ring of $\mathbb{P}^{n_i}$. Set $I := I_1 + \cdots + I_k \subset R$. Then the ith coordinate of the resolution regularity vector of R/I is* $\operatorname{reg}(R_i/I_i)$.

Proof. Let $M_i = R_i/I_i$ and apply Proposition 3.2. □

The R-modules M that are products of modules over the factors of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ have the property that they are $\underline{r}(M) \cdot \underline{1}$-regular as $\mathbb{N}^1$-graded modules.

Corollary 3.4. *Suppose that $M = M_1 \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} M_k$ as in Proposition 3.2 and $\underline{r}(M) = (r_1, \ldots, r_k)$. Then M is $\sum r_i$-regular as an $\mathbb{N}^1$-graded module.*

Proof. Construct a resolution of M by tensoring together minimal free graded resolutions of the M_is. The free module at the jth stage in the resolution is a direct sum of modules $F_{1,\ell_1} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} F_{k,\ell_k}$ where $\sum \ell_i = j$ and F_{i,ℓ_i} is the module at the ℓ_ith stage in the minimal free graded resolution of M_i over the ring R_i defined as in Proposition 3.2. Since F_{i,ℓ_i} is generated by elements of degree $\leq r_i + \ell_i$, the total degree of any generator of $F_{i,\ell_1} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} F_{k,\ell_k}$ is $\leq \sum(r_i + \ell_i) = (\sum r_i) + j$. □

Corollary 3.4 is not true for resolution regularity vectors of arbitrary modules. For example, let $R = \mathbf{k}[x_0, x_1, y_0, y_1]$ and let $I = \langle x_0y_1, x_1y_0\rangle$. (The saturation of I with respect to B is the defining ideal of two points in $\mathbb{P}^1 \times \mathbb{P}^1$.) The resolution of the ideal I is given by the Koszul complex

$$0 \to R(-2,-2) \to R^2(-1,-1) \to \langle x_0y_1, x_1y_0\rangle \to 0.$$

So $\underline{r}(I) = (1,1)$, but the ideal I is not 2-regular as an $\mathbb{N}^1$-graded ideal.

4. Multigraded regularity for points

Let $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a scheme of fat points, and let $Z_i = \pi_i(Z)$ denote the projection of Z into $\mathbb{P}^{n_i}$ by the ith projection morphism $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_i}$. We show how the resolution regularity vector of R/I_Z is related to $\operatorname{reg}(Z_i)$, the regularity of Z_i as a subscheme of $\mathbb{P}^{n_i}$ for $i = 1, \ldots, k$. We then improve upon Proposition 2.2 and show $\underline{r}(R/I_Z) + \mathbb{N}^k \subseteq \operatorname{reg}_B(Z)$. As a corollary, rough estimates of $\operatorname{reg}_B(Z)$ are obtained for any set of fat points by employing well known bounds for fat points in $\mathbb{P}^n$. We also show that if Z is ACM, $\operatorname{reg}_B(Z)$ is in fact determined by $\operatorname{reg}(Z_i)$ for $i = 1, \ldots, k$.

Lemma 4.1. *Let $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let the ith coordinate of the jth point be P_{ji} so that the ideal I_{P_j} defining the point P_j is the sum of ideals $I_{P_{j1}} + \cdots + I_{P_{jk}}$ where $I_{P_{ji}}$ defines the ith coordinate of P_j. Set $Z_i := \pi_i(Z)$ for $i = 1, \ldots, k$. Then*

(i) *Z_i is the set of fat points in $\mathbb{P}^{n_i}$ defined by the ideal* $I_{Z_i} = \bigcap_{j=1}^{s} I_{P_{ji}}^{m_j}$,

(ii) *for all $t \in \mathbb{N}$, $H_{Z_i}(t) = \mathcal{H}_Z(te_i)$.*

Proof. The proof of the reduced case found in Proposition 3.2 in [24] can be adapted to the nonreduced case. □

Theorem 4.2. *Suppose $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a set of fat points. Then*

$$\underline{r}(R/I_Z) = (r_1, \ldots, r_k)$$

where $r_i = \operatorname{reg}(Z_i)$ for $i = 1, \ldots, k$.

Proof. We shall use Theorem 2.7 to compute the resolution regularity vector. Because $\mathbf{k}$ is infinite, for each ℓ there exists a basis $y_{\ell,0}, \ldots, y_{\ell,n_\ell}$ for R_{e_ℓ} that is a multigraded almost regular sequence. Furthermore, by Lemma 1.8 we can also assume that $y_{\ell,0}$ is a nonzerodivisor on R/I_Z.

Since $y_{\ell,0}$ is a nonzerodivisor, $\langle 0 :_{R/I_Z} y_{\ell,0}\rangle = 0$, which implies that $s_{\ell,0} \leq 0$. We now need to calculate $s_{\ell,i}$ for $i = 1, \ldots, n_\ell$.

Because $y_{\ell,0}$ is a nonzerodivisor on R/I_Z, we have the short exact sequence

$$0 \to R/I_Z(-e_\ell) \overset{\times y_{\ell,0}}{\to} R/I_Z \to R/\langle I_Z, y_{\ell,0}\rangle \to 0. \tag{4}$$

Since $r_\ell = \operatorname{reg}(Z_\ell)$, the sequence (4) and Lemma 4.1 give

$$\begin{aligned} \mathcal{H}_{R/\langle I_Z, y_{\ell,0}\rangle}((r_\ell+1)e_\ell) &= \mathcal{H}_Z((r_\ell+1)e_\ell) - \mathcal{H}_Z(r_\ell e_\ell) \\ &= H_{Z_\ell}(r_\ell+1) - H_{Z_\ell}(r_\ell) = \deg Z_\ell - \deg Z_\ell = 0. \end{aligned}$$

Thus $\langle I_Z, y_{\ell,0}\rangle_{ae_\ell} = R_{ae_\ell}$ if $a \geq r_\ell + 1$. Hence for any $\underline{j} \geq (r_\ell+1)e_\ell$, $R_{\underline{j}} = \langle I_Z, y_{\ell,0}\rangle_{\underline{j}} \subseteq \langle I_Z, y_{\ell,0}, \ldots, y_{\ell,i-1}\rangle_{\underline{j}}$.

Since $\langle 0 :_{R/\langle I_Z, y_{\ell,0}, \ldots, y_{\ell,i-1}\rangle} y_{\ell,i}\rangle$ is an ideal of $R/\langle I_Z, y_{\ell,0}, \ldots, y_{\ell,i-1}\rangle$, and because $R/\langle I_Z, y_{\ell,0}, \ldots, y_{\ell,i-1}\rangle_{\underline{j}} = 0$, if $\underline{j} \geq (r_\ell+1)e_\ell$,

$$\left\langle 0 :_{R/\langle I_Z, y_{\ell,0}, \ldots, y_{\ell,i-1}\rangle} y_{\ell,i}\right\rangle_a^{[\ell]} = 0 \;\text{ if } a \geq r_\ell + 1. \tag{5}$$

Thus, from (5) we have $s_{\ell,j} \leq r_\ell$ for each ℓ and each $j = 1, \ldots, n_\ell$. Since $s_{\ell,0} \leq 0$, it suffices to show that $s_{\ell,1} = r_\ell$ since this gives $s_\ell = \max\{s_{\ell,0}, \ldots, s_{\ell,n_\ell}\} = s_{\ell,1} = r_\ell$. The short exact sequence (4) also implies that

$$\mathcal{H}_{R/\langle I_Z, y_{\ell,0}\rangle}(r_\ell e_\ell) = H_{Z_\ell}(r_\ell) - H_{Z_\ell}(r_\ell - 1) > 0$$

because $H_{Z_\ell}(r_\ell - 1) < H_{Z_\ell}(r_\ell) = \deg Z_\ell$. So there exists $0 \neq \overline{F} \in (R/\langle I_Z, y_{\ell,0}\rangle)_{r_\ell e_\ell}$. Because $\deg \overline{F}\overline{y}_{\ell,1} = (r_\ell+1)e_\ell$, and $(R/\langle I_Z, y_{\ell,0}\rangle)_{(r_\ell+1)e_\ell} = 0$, we must have $\overline{F} \in \langle 0 :_{R/\langle I_Z, y_{\ell,0}\rangle} y_{\ell,1}\rangle$. So, $0 \neq \overline{F} \in \langle 0 :_{R/\langle I_Z, y_{\ell,0}\rangle} y_{\ell,1}\rangle_{r_\ell}^{[\ell]}$, thus implying $s_{\ell,1} = r_\ell$. □

The previous result, combined with Proposition 2.2, gives us a crude bound on $\operatorname{reg}_B(Z)$. However, we can improve upon this bound.

Lemma 4.3. *Let $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a point with defining ideal $I_P \subseteq R$, and $m \in \mathbb{N}^+$. Then* $\mathrm{reg}_B(R/I_P^m) = (m-1, \ldots, m-1) + \mathbb{N}^k$.

Proof. After a change of coordinates, we can assume $P = [1:0:\cdots:0] \times \cdots \times [1:0:\cdots:0]$. So $I_P^m = \langle x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}\rangle^m$. Since I_P^m is a monomial ideal, $\mathcal{H}_{R/I_P^m}(\underline{i})$ equals the number of monomials of degree $\underline{i}$ in R not in I_P^m.

A monomial $\prod x_{j,\ell}^{a_{j,\ell}} \notin (I_P^m)_{\underline{i}}$ if and only if $a_{1,1} + \cdots + a_{k,n_k} \leq m-1$ and $a_{j,1} + \cdots + a_{j,n_j} \leq i_j$ for each $j = 1, \ldots, k$. The result now follows since

$$\#\left\{(a_{1,1}, \ldots, a_{k,n_k}) \in \mathbb{N}^N \;\middle|\; a_{1,1} + \cdots + a_{k,n_k} \leq m-1,\ \forall j\ a_{j,1} + \cdots + a_{j,n_j} \leq i_j\right\}$$

is equal to

$$\begin{aligned}\#\left\{(a_{1,1}, \ldots, a_{k,n_k}) \in \mathbb{N}^N \;|a_{1,1} + \cdots + a_{k,n_k} \leq m-1\right\} &= \binom{m-1+N}{m-1} \\ &= \deg(mP)\end{aligned}$$

if and only if $\underline{i} = (i_1, \ldots, i_k) \geq (m-1, \ldots, m-1)$. □

Proposition 4.4. *Suppose $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a set of fat points. Then*

$$(r_1, \ldots, r_k) + \mathbb{N}^k \subseteq \mathrm{reg}_B(Z)$$

where $r_i = \mathrm{reg}(Z_i)$ for $i = 1, \ldots, k$.

Proof. The proof is by induction on s, the number of points in the support. If $s = 1$, then the result follows from Lemma 4.3.

So, suppose $s > 1$ and let $X = \{P_1, \ldots, P_s\}$ be the support of Z. We can find an $i \in \{1, \ldots, k\}$ such that $1 < |\pi_i(X)|$, i.e., there exists an i where the projection of X onto its ith coordinates consists of two or more points. Fix a $\tilde{P} \in \pi_i(X)$. We can then write $Z = Y_1 \cup Y_2$ where $Y_1 = \{m_j P_j \in Z \mid \pi_i(P_j) = \tilde{P}\}$ and $Y_2 = \{m_j P_j \in Z \mid \pi_i(P_j) \neq \tilde{P}\}$. By our choice of i, Y_1 and Y_2 are nonempty, $Y_1 \cap Y_2 = \emptyset$, and $\pi_i(Y_1) \cap \pi_i(Y_2) = \emptyset$.

Let I_{Y_1}, resp., I_{Y_2}, denote the defining ideal associated to Y_1, resp., Y_2. Consider the short exact sequence

$$0 \to R/\langle I_{Y_1} \cap I_{Y_2}\rangle \to R/I_{Y_1} \oplus R/I_{Y_2} \to R/\langle I_{Y_1} + I_{Y_2}\rangle \to 0.$$

Since $I_Z = I_{Y_1} \cap I_{Y_2}$, this exact sequence gives rise to the identity

$$\mathcal{H}_Z(\underline{t}) = \mathcal{H}_{Y_1}(\underline{t}) + \mathcal{H}_{Y_2}(\underline{t}) - \mathcal{H}_{R/\langle I_{Y_1}+I_{Y_2}\rangle}(\underline{t}) \text{ for all } \underline{t} \in \mathbb{N}^k. \tag{6}$$

Set $Y_{j,1} := \pi_j(Y_1)$ and $Y_{j,2} := \pi_j(Y_2)$ for each $j = 1, \ldots, k$. Since $Y_{j,1} \subseteq Z_j$ and $Y_{j,2} \subseteq Z_j$, we have $\mathrm{reg}(Y_{j,1}) \leq r_j$ and $\mathrm{reg}(Y_{j,2}) \leq r_j$. By induction and the above identity we therefore have

$$\mathcal{H}_Z(r_1, \ldots, r_k) = \deg Y_1 + \deg Y_2 - \mathcal{H}_{R/\langle I_{Y_1}+I_{Y_2}\rangle}(r_1, \ldots, r_k).$$

Since $\deg Z = \deg Y_1 + \deg Y_2$, it suffices to show $\mathcal{H}_{R/\langle I_{Y_1}+I_{Y_2}\rangle}(r_1,\ldots,r_k) = 0$.

By Lemma 4.1, $\mathcal{H}_Z(r_ie_i) = H_{Z_i}(r_i) = \deg Z_i$, $\mathcal{H}_{Y_1}(r_ie_i) = H_{Y_{i,1}}(r_i) = \deg Y_{i,1}$, and $\mathcal{H}_{Y_2}(r_ie_i) = H_{Y_{i,2}}(r_i) = \deg Y_{i,2}$. But because $Y_{i,1} \cap Y_{i,2} = \emptyset$ by our choice of i, $\deg Z_i = \deg Y_{i,1} + \deg Y_{i,2}$. Substituting into (6) with $\underline{t} = r_ie_i$ then gives $\mathcal{H}_{R/\langle I_{Y_1}+I_{Y_2}\rangle}(r_ie_i) = 0$, or equivalently, $R_{r_ie_i} = \langle I_{Y_1} + I_{Y_2}\rangle_{r_ie_i}$. It now follows that $R_{(r_1,\ldots,r_k)} = \langle I_{Y_1} + I_{Y_2}\rangle_{(r_1,\ldots,r_k)}$ which gives $\mathcal{H}_{R/\langle I_{Y_1}+I_{Y_2}\rangle}(r_1,\ldots,r_k) = 0$. □

Using well known bounds for fat points in $\mathbb{P}^n$ thus gives us:

Corollary 4.5. *Let $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $m_1 \geq \cdots \geq m_s$.*

(i) *Set $m = m_1 + m_2 + \cdots + m_s - 1$. Then $(m,\ldots,m) + \mathbb{N}^k \subseteq \operatorname{reg}_B(Z)$.*

(ii) *Suppose $X = \{P_1,\ldots,P_s\}$ is in generic position. For $i = 1,\ldots,k$ set*

$$\ell_i = \max\left\{m_1 + m_2 + 1, \left\lceil \frac{(\sum_{i=1}^{s} m_i) + n_i - 2}{n_i} \right\rceil\right\}.$$

Then $(\ell_1,\ldots,\ell_k) + \mathbb{N}^k \subseteq \operatorname{reg}_B(Z)$.

Proof. It follows from Davis and Geramita [9] that $r_i = \operatorname{reg}(Z_i) \leq m$ for each i. So $(m,\ldots,m) + \mathbb{N}^k \subseteq (r_1,\ldots,r_k) + \mathbb{N}^k$, and hence (i) follows.

For (ii), because X is in generic position, the support of Z_i is in generic position in $\mathbb{P}^{n_i}$. In [5] it was shown that $r_i = \operatorname{reg}(Z_i) \leq \ell_i$ for each i. □

Recall that a scheme $Y \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is *arithmetically Cohen-Macaulay* (ACM) if $\operatorname{depth} R/I_Y =$K-dim R/I_Y. For any collection of fat points $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ we always have K-dim $R/I_Z = k$, the number of projective spaces. However, for each $\ell \in \{1,\ldots,k\}$ there exist sets of fat points (in fact, reduced points) $X_\ell \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $\operatorname{depth} R/I_{X_\ell} = \ell$. See [25] for more details.

A scheme of fat points, therefore, may or may not be ACM. When Z is ACM, $\operatorname{reg}_B(Z)$ depends only upon knowing $\operatorname{reg}(Z_i)$ for $i = 1,\ldots,k$.

Lemma 4.6. *Let Z be an ACM set of fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then there exist elements $L_i \in R_{e_i}$ such that $L_1,\ldots,L_k$ is a regular sequence on R/I_Z.*

Proof. The nontrivial part of the statement is the existence of a regular sequence whose elements have the specified multidegrees. The proof given for the reduced case (see Proposition 3.2 in [25]) can be adapted to the nonreduced case. □

Theorem 4.7. *Let $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a set of fat points. If Z is ACM, then*

$$(r_1,\ldots,r_k) + \mathbb{N}^k = \operatorname{reg}_B(Z)$$

where $r_i = \operatorname{reg}(Z_i)$ for $i = 1,\ldots,k$.

Proof. Let $L_1,\ldots,L_k$ be the regular sequence from Lemma 4.6, and set $J = \langle I_Z, L_1,\ldots,L_k\rangle$. We require the following claims.

Claim 1. If $\underline{j} \not\leq (r_1,\ldots,r_k)$, then $\mathcal{H}_{R/J}(\underline{j}) = 0$.

Since $\underline{j} \not\leq (r_1,\ldots,r_k)$ there exists $1 \leq \ell \leq k$ such that $j_\ell > r_\ell$. Using the exact sequence (4) of Theorem 4.2, the claim follows if we replace $y_{\ell,0}$ with L_ℓ.

Claim 2. For $i = 1,\ldots,k$, $\mathcal{H}_{R/J}(r_i e_i) > 0$.

By degree considerations, $\mathcal{H}_{R/J}(r_i e_i) = \mathcal{H}_{R/\langle I_Z, L_i\rangle}(r_i e_i)$ for each i. Employing the short exact sequence

$$0 \rightarrow R/I_Z(-e_i) \stackrel{\times L_i}{\rightarrow} R/I_Z \rightarrow R/\langle I_Z, L_i\rangle \rightarrow 0$$

to calculate $\mathcal{H}_{R/\langle I_Z,L_i\rangle}(r_i e_i)$ gives $\mathcal{H}_{R/\langle I_Z,L_i\rangle}(r_i e_i) = \mathcal{H}_Z(r_i e_i) - \mathcal{H}_Z((r_i-1)e_i) = H_{Z_i}(r_i) - H_{Z_i}(r_i - 1)$. The claim now follows since $H_{Z_i}(r_i-1) < \deg Z_i = H_{Z_i}(r_i)$.

We complete the proof. Since $L_1,\ldots,L_k$ is a regular sequence, we have the following short exact sequences

$$0 \rightarrow R/\langle I_Z, L_1,\ldots,L_{i-1}\rangle(-e_i) \stackrel{\times L_i}{\rightarrow} R/\langle I_Z, L_1,\ldots,L_{i-1}\rangle \rightarrow R/\langle I_Z, L_1,\ldots,L_i\rangle \rightarrow 0$$

for $i = 1,\ldots,k$. It then follows that

$$\mathcal{H}_Z(\underline{j}) = \sum_{\underline{0}\leq \underline{i}\leq \underline{j}} \mathcal{H}_{R/J}(\underline{i}) \quad \text{for all } \underline{j} \in \mathbb{N}^k.$$

Now suppose that $\underline{j} \notin (r_1,\ldots,r_k)+\mathbb{N}^k$. So $j_\ell < r_\ell$ for some ℓ. Set $j'_i = \min\{j_i, r_i\}$ and let $\underline{j}' = (j'_1,\ldots,j'_k)$. Note that $\underline{j}' \leq (r_1,\ldots,r_k)$ and $j'_\ell = j_\ell < r_\ell$. By Claim 1 and the above identity

$$\mathcal{H}_Z(\underline{j}) = \sum_{\underline{0}\leq \underline{i}\leq \underline{j}} \mathcal{H}_{R/J}(\underline{i}) = \sum_{\underline{0}\leq \underline{i}\leq \underline{j}'} \mathcal{H}_{R/J}(\underline{i}).$$

Then, by Claim 2,

$$\begin{aligned} \mathcal{H}_Z(\underline{j}) &= \sum_{\underline{0}\leq \underline{i}\leq \underline{j}'} \mathcal{H}_{R/J}(\underline{i}) < \sum_{\underline{0}\leq \underline{i}\leq \underline{j}'} \mathcal{H}_{R/J}(\underline{i}) + H_{R/J}(r_\ell e_\ell) \\ &\leq \sum_{\underline{0}\leq \underline{i}\leq (r_1,\ldots,r_k)} \mathcal{H}_{R/J}(\underline{i}) = \mathcal{H}_Z(r_1,\ldots,r_k) = \deg Z. \end{aligned}$$

So, if $\underline{j} \notin (r_1,\ldots,r_k)+\mathbb{N}^k$, then $\underline{j} \notin \operatorname{reg}_B(Z)$. This fact, coupled with Theorem 4.4, gives the desired result. $\square$

Remark 4.8. The converse of Theorem 4.7 is false because there exist fat point schemes Z such that $\operatorname{reg}_B(Z) = \underline{r}(R/I_Z)+\mathbb{N}^k$, but Z is not ACM. For example, set $P_{ij} := [1:i]\times[1:j] \in \mathbb{P}^1\times\mathbb{P}^1$, and consider $Z = \{P_{11}, P_{12}, P_{13}, P_{21}, P_{22}, P_{31}, P_{33}\}$. Then $\mathcal{H}_Z$ is

$$\mathcal{H}_Z = \begin{bmatrix} 1 & 2 & 3 & 3 & \ldots \\ 2 & 4 & 6 & 6 & \ldots \\ 3 & 6 & 7 & 7 & \ldots \\ 3 & 6 & 7 & 7 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It follows that $\operatorname{reg}_B(Z) = (2,2) + \mathbb{N}^2$. However, the resolution of R/I_Z has length 3, so by the Auslander-Buchsbaum Theorem, $\operatorname{depth} R/I_Z = \operatorname{depth} R - \operatorname{pd} R/I_Z = 4 - 3 = 1 < 2 = \text{K-}\dim R/I_Z$. Alternatively, R/I_Z is not ACM since the first difference function of $\mathcal{H}_Z$ is not the Hilbert function of an $\mathbb{N}^2$-graded Artinian quotient of $\mathbf{k}[x_1, y_1]$. (See Theorem 4.8 in [25].)

5. A bound for fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ with support in generic position

Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$, and furthermore, suppose that $X = \{P_1, \ldots, P_s\}$, the support of Z, is in generic position, i.e., $\mathcal{H}_X(i,j) = \min\{\dim_{\mathbf{k}} R_{(i,j)}, s\}$ for all $(i,j) \in \mathbb{N}^2$. Using Proposition 4.4, we can obtain the bound $(m-1, m-1) + \mathbb{N}^k \subseteq \operatorname{reg}_B(Z)$ where $m = \sum m_i$. However, under these extra hypotheses, we can give a much stronger bound.

Theorem 5.1. *Let $Z = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of fat points whose support is in generic position. Assume $m_1 \geq m_2 \geq \cdots \geq m_s$, and set $m = m_1 + m_2 + \cdots + m_s$. Then*

$$\begin{aligned} \{(i,j) \in \mathbb{N}^2 \mid (i,j) \geq (m_1 - 1, m_1 - 1) \text{ and } i + j &\geq \max\{m-1, 2m_1 - 2\}\} \\ &\subseteq \operatorname{reg}_B(Z). \end{aligned}$$

We require a series of lemmas.

Lemma 5.2. *Assume $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is as in Theorem 5.1. For $j = 0, \ldots, m_1 - 1$, set*

$$c_j := \sum_{i=1}^{s} [m_i + (m_i - 1)_+ + \cdots + (m_i - j)_+] \text{ where } (n)_+ := \max\{0, n\}.$$

Then $c_{m_1-1} = \deg Z$. As well, if we write $\mathcal{H}_Z$ as an infinite matrix, $\mathcal{H}_Z$ has the following eventual behavior:

$$\mathcal{H}_Z = \begin{array}{cc} & \begin{array}{ccccccccc} 0 & & & & m_1-1 & & m-1 & & \end{array} \\ \begin{array}{c} 0 \\ \\ \\ \\ m_1-1 \\ \\ m-1 \\ \\ \\ \end{array} & \begin{bmatrix} & & & & & & c_0 & c_0 & \cdots \\ & & & & & & c_1 & c_1 & \cdots \\ & & & & & & \vdots & & \\ & & * & & & & c_{m_1-2} & c_{m_1-2} & \cdots \\ & & & & & & \deg Z & \deg Z & \cdots \\ & & & & & & \vdots & & \\ c_0 & c_1 & \cdots & c_{m_1-2} & \deg Z & \cdots & \deg Z & \deg Z & \cdots \\ c_0 & c_1 & & c_{m_1-2} & \deg Z & & \deg Z & \deg Z & \cdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots \end{bmatrix} \end{array}.$$

Proof. Because $m_1 \geq \cdots \geq m_s$ we have

$$c_{m_1-1} = \sum_{i=1}^{s} [1 + 2 + \cdots + m_i] = \sum_{i=1}^{s} \binom{m_i + 1}{s} = \deg Z.$$

The eventual behavior of $\mathcal{H}_Z$ can be obtained from Theorem 3.2 in [14]. □

Lemma 5.3. *Let R/I_Z be the coordinate ring of a set of fat points that satisfies the conditions of Theorem* 5.1*.*

(i) *As an* $\mathbb{N}^1$*-graded ring,* $\operatorname{reg}(R/I_Z) \leq m-1$*.*

(ii) *As an* $\mathbb{N}^1$*-graded ring, the Hilbert polynomial of* R/I_Z *is*

$$HP_{R/I_Z}(t) = \sum_{i=1}^{s}\left[\binom{m_i+1}{2}t + \binom{m_i+1}{2}\left(\frac{-2m_i+5}{3}\right)\right].$$

Proof. (i) By using Theorem 4.4 in [10], we obtain the bound $\operatorname{reg}(I_Z) \leq m$. The result now follows since $\operatorname{reg}(R/I_Z) = \operatorname{reg}(I_Z) - 1$.

(ii) If I_{P_i} is the defining ideal of a point P_i in the support, then as an $\mathbb{N}^1$-homogeneous ideal of $R = \mathbf{k}[x_0, x_1, y_0, y_1]$, I_{P_i} defines a line in $\mathbb{P}^3$. Since the points in the support are in generic position, the lines that they correspond to in $\mathbb{P}^3$ must all be skew.

The structure sheaf $\mathcal{O}_Z = \bigoplus_{i=1}^{s} \mathcal{O}_{m_iP_i}$, and hence, the Hilbert polynomial of $\mathcal{O}_Z$ is just the sum of the Hilbert polynomials of $\mathcal{O}_{m_iP_i}$. The ideal $I_{P_i}^{m_i}$ is a power of complete intersection. The resolution of $I_{P_i}^{m_i}$ is thus given by the Eagon-Northcott resolution. Furthermore, since the generators of $I_{P_i}^{m_i}$ all have the same degree, the Eagon-Northcott resolution produces the following minimal graded free resolution of $R/I_{P_i}^{m_i}$:

$$0 \longrightarrow R^{m_i}(-(m_i+1)) \longrightarrow R^{m_i+1}(-m_i) \longrightarrow R \longrightarrow R/I_{P_i}^{m_i} \longrightarrow 0.$$

We can compute the Hilbert polynomial of $R/I_{P_i}^{m_i}$ from this resolution. Since the Hilbert polynomial of R/I_Z and its sheafification agree, we are done. $\square$

Proof. (of Theorem 5.1) Set $\ell = \max\{m-1, 2m_1-2\}$. It suffices to show that $\mathcal{H}_Z(i,j) = \deg Z$ if $(i,j) \geq (m_1-1, m_1-1)$ and $i+j=\ell$. The conclusion then follows from Proposition 1.9.

By Lemma 5.3, $\operatorname{reg}(Z) \leq m-1$. Thus H_Z, the Hilbert function of Z as a graded ring, agrees with HP_{R/I_Z} for all $t \geq m-1$. In particular, since $\ell \geq m-1$,

$$H_Z(\ell) = \sum_{i=1}^{s}\left[\binom{m_i+1}{2}\ell + \binom{m_i+1}{2}\left(\frac{-2m_i+5}{3}\right)\right]. \tag{7}$$

Now $H_Z(\ell) = \sum_{i+j=\ell} \mathcal{H}_Z(i,j)$. If $i+j=\ell$, then there are three cases:

1. If $j \leq m_1-2$, then by Lemma 5.2 and Proposition 1.9 (i), we have $\mathcal{H}_Z(i,j) \leq c_j$.
2. If $i \leq m_1-2$, then we have $\mathcal{H}_Z(i,j) \leq c_i$.
3. If $j \geq m_1-1$ and $i \geq m_1-1$, then $\mathcal{H}_Z(i,j) \leq \deg Z$.

Since there are $\ell+1$ pairs $(i,j) \in \mathbb{N}^2$ such that $i+j=\ell$, and m_1-1 pairs fall into the first case, and m_1-1 are in the second case, we must have $\ell+1-2(m_1-1) = \ell+1-2m_1+2 > 0$ pairs in the third case because $\ell \geq 2m_1-2$.

We thus have

$$H_Z(\ell) = \sum_{i+j=\ell} \mathcal{H}_Z(i,j) \leq 2(c_0 + \cdots + c_{m_1-2}) + (\ell+1-2m_1+2)\deg Z.$$

Claim. $c_0 + \cdots + c_{m_1-2} = \sum_{i=1}^{s} \left[\binom{m_i+1}{2}\left(\frac{2m_i-2}{3}\right) + \binom{m_i+1}{2}(m_1 - m_i) \right].$

Proof of Claim.

$$\begin{aligned}
\sum_{j=0}^{m_1-1} c_j &= \sum_{j=0}^{m_1-1}\sum_{i=1}^{s} [m_i + (m_i-1)_+ + \cdots + (m_i - j)_+] \\
&= \sum_{i=1}^{s} \left[m_i^2 + (m_i-1)^2 + \cdots + 2^2 + 1^2 + (m_1 - m_i)\binom{m_i+1}{2} \right] \\
&= \sum_{i=1}^{s} \left[\frac{m_i(m_i+1)(2m_i+1)}{6} + (m_1 - m_i)\binom{m_i+1}{2} \right] \\
&= \sum_{i=1}^{s} \left[\binom{m_i+1}{2}\left(\frac{2m_i+1}{3}\right) + (m_1 - m_i)\binom{m_i+1}{2} \right].
\end{aligned}$$

By subtracting $c_{m_1-1} = \sum_{i=1}^{s} \binom{m_i+1}{2}$ from both sides of the above expression, we arrive at the claimed result.

We complete the proof:

$$\begin{aligned}
H_Z(\ell) &= \sum_{i+j=\ell} \mathcal{H}_Z(i,j) \\
&\leq 2\sum_{i=1}^{s} \left[\binom{m_i+1}{2}\left(\frac{2m_i-2}{3}\right) + \binom{m_i+1}{2}(m_1 - m_i) \right] \\
&\quad +(\ell + 1 - 2m_1 + 2)\sum_{i=1}^{s} \binom{m_i+1}{2} \\
&= \sum_{i=1}^{s} \left[\binom{m_i+1}{2}\ell + \binom{m_i+1}{2}\left(\frac{4m_i-4}{3} - 2m_i + 3\right) \right] \\
&= \sum_{i=1}^{s} \left[\binom{m_i+1}{2}\ell + \binom{m_i+1}{2}\left(\frac{-2m_i+5}{3}\right) \right] = H_Z(\ell).
\end{aligned}$$

If we view $\mathcal{H}_Z$ as an infinite matrix, then because $\mathcal{H}_Z$ strictly increases along each row and column until it reaches its eventual growth value as given in Lemma 5.2, we must have $\mathcal{H}_Z(i,j)$ equal to this eventual growth value for all (i,j) with $i + j = \ell$. That is, in all three cases, $\mathcal{H}_Z(i,j)$ equals the given upper bound. In particular, $\mathcal{H}_Z(i,j) = \deg Z$ if $i + j = \ell$ and $i \geq m_1 - 1$ and $j \geq m_1 - 1$. □

Remark 5.4. Note that we have in fact proved a stronger result. In conjunction with Lemma 5.2, we can describe $\mathcal{H}_Z(i,j)$ for all (i,j) with $i + j \geq \max\{m - 1, 2m_1 - 2\}$ directly from the multiplicities of the points.

A. Resolutions and B-regularity

In this section we prove some modified versions of results in §7 of [19] for finitely generated $\mathbb{N}^k$-graded R-modules with $R = [x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$, where $\deg x_{i,j} = e_i$, the ith standard basis vector of $\mathbb{Z}^k$. Here, we use the notation of [19].

Lemma A.1. (Lemma 7.1 in [19]) *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated $\mathbb{N}^k$-graded R-modules. If $i \geq 1$, then*

$$\left(\bigcup_{1 \leq j \leq k} (-e_j + \operatorname{reg}^{i+1}(M')) \right) \cap \operatorname{reg}^i(M) \subseteq \operatorname{reg}^i(M'').$$

Otherwise, when $i = 0$

$$\bigcap_{1 \leq j \leq k} (-e_j + \operatorname{reg}^1(M')) \cap \operatorname{reg}^0(M) \subseteq \operatorname{reg}^0(M'').$$

Proof. For the first statement, the proof proceeds as in [19]. The key inclusion is that for any j, $m + \mathbb{N}^k[1-t] \subseteq m + e_j + \mathbb{N}^k[-t]$ for all $t \geq 1$. However, the inclusion fails if $t = 0$. Turning to the second part of the claim, we let

$$m \in \bigcap_{1 \leq j \leq k} (-e_j + \operatorname{reg}^1(M')) \cap \operatorname{reg}^0(M).$$

The long exact sequence in cohomology gives

$$\cdots \to H^0_B(M)_p \to H^0_B(M'')_p \to H^1_B(M')_p \to \cdots$$

We know that the $H^0_B(M)_p = 0$ for all

$$p \in m + \mathbb{N}^k[1] = \bigcup_{j=1}^{k} m + e_j + \mathbb{N}^k.$$

We will be done if we can show that $H^1_B(M')_p = 0$ for all $p \in m + \mathbb{N}^k[1]$ since the vanishings of the higher cohomology modules follow from the first (and stronger) part of the lemma. We know that $H^1_B(M')_p = 0$ for all $p \in \operatorname{reg}^1(M')$. So, it is enough to show that $m + \mathbb{N}^k[1] \subseteq \operatorname{reg}^1(M')$. Since $m \in \bigcap_{1 \leq j \leq k}(-e_j + \operatorname{reg}^1(M'))$, $m + e_j \in \operatorname{reg}^1(M')$ for every j. That is, $m + e_j + \mathbb{N}^k \subseteq \operatorname{reg}^1(M)$ for every j, and since

$$\bigcup_{j=1}^{k} m + e_j + \mathbb{N}^k = m + \mathbb{N}^k[1],$$

and we are done. □

In our situation, we have the following variant of Corollary 7.3 in [19].

Theorem A.2. *Let $0 \to E_r \to \cdots \to E_3 \to E_2 \to E_1 \to E_0 \to 0$ be a free $\mathbb{N}^k$-graded resolution of a finitely generated $\mathbb{N}^k$-graded R-module M. Let $m = \min\{r, N+1\}$. We have*

$$\bigcup_{\phi:[m]\to[k]} \left(\bigcap_{1 \leq i \leq m} (-e_{\phi(2)} - \cdots - e_{\phi(i)} + \operatorname{reg}^i(E_i)) \right) \subseteq \operatorname{reg}^1(K_0)$$

where K_0 is the first syzygy module of M and the union is over all functions $\phi : [m] \to [k]$.

The result follows from the proof of Theorem 7.2 in [19] which is based upon the first conclusion of Lemma A.1 and descending induction on i.

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