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Groups in which the Bounded Nilpotency of Two-generator Subgroups is a Transitive Relation

Costantino Delizia **Primož Moravec** * **Chiara Nicotera**

[1,3] *Dipartimento di Matematica e Informatica, Università di Salerno*
Via Ponte don Melillo, 84084 - Fisciano (SA), Italy
[1] *e-mail: cdelizia@unisa.it* [3] *e-mail: cnicotera@unisa.it*

[2] *Fakulteta za Matematiko in Fiziko, Univerza v Ljubljani*
Jadranska 19, 1000 Ljubljana, Slovenia
e-mail: primoz.moravec@fmf.uni-lj.si

Abstract. In this paper we describe the structure of locally finite groups in which the bounded nilpotency of two-generator subgroups is a transitive relation. We also introduce the notion of (nilpotent of class c)-transitive kernel. Our results generalize several known results related to the groups in which commutativity is a transitive relation.

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1. Introduction

Let c be a positive integer and let $\mathfrak{N}_c$ denote the class of all groups which are nilpotent of class $\leq c$. A group G is said to be an $\mathfrak{N}_cT$-*group* if for all $x, y, z \in G\backslash\{1\}$ the relations $\langle x, y\rangle \in \mathfrak{N}_c$ and $\langle y, z\rangle \in \mathfrak{N}_c$ imply $\langle x, z\rangle \in \mathfrak{N}_c$. In the case $c = 1$ these groups are known as commutative-transitive groups (also CT-groups

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or CA-groups) and have been studied by several authors [2, 3, 4, 8, 11, 14, 15]. It is not difficult to see that CT-groups are precisely the groups in which centralizers of non-identity elements are abelian. The study of these groups was initiated by Weisner [14] in 1925, but there are some fallacies in his proofs. Nevertheless, it turns out that finite CT-groups are either soluble or simple. Finite nonabelian simple CT-groups have been classified by Suzuki [11]. He proved that every finite nonabelian simple CT-group is isomorphic to some $\mathrm{PSL}(2, 2^f)$, where $f > 1$. The complete description of finite soluble CT-groups has been given by Wu [15] (see also a paper of Lescot [8]), who has also obtained information on locally finite CT-groups and polycyclic CT-groups. At roughly the same time Fine et al. [4] introduced the notion of the commutative-transitive kernel of a group. This topic has been further explored by the first and the third author; see [2] and [3].

Passing to finite $\mathfrak{N}_c T$-groups with $c > 1$ we first note that in these groups centralizers of non-identity elements are nilpotent. The converse is not true, however, as the example of $\mathrm{PSL}(2, 9)$ shows (see Proposition 4.5). Compared to the CT-case, this may seem to be a certain disadvantage at first glance, but nevertheless we obtain satisfactory information on the structure of locally finite $\mathfrak{N}_c T$-groups. We show that soluble locally finite $\mathfrak{N}_c T$-groups are either Frobenius groups or belong to the class of groups in which every two-generator subgroup is nilpotent of class $\leq c$. Furthermore, we prove that finite $\mathfrak{N}_c T$-groups are either soluble or simple. This provides a generalization of results in [15]. Additionally, we show that the groups $\mathrm{PSL}(2, 2^f)$, where $f > 1$, and Suzuki groups $\mathrm{Sz}(q)$, with $q = 2^{2n+1} > 2$, are the only finite nonabelian simple $\mathfrak{N}_c T$-groups for $c > 1$. This result is probably the strongest evidence showing the gap between CT-groups and $\mathfrak{N}_c T$-groups with $c > 1$. We also show that locally finite $\mathfrak{N}_c T$-groups are either locally soluble or simple. In the latter case we give a classification of these groups.

Another notion closely related to CT-groups is the commutative-transitive kernel of a group. Given a group G, we can construct a characteristic subgroup $T(G)$ as the union of a chain $1 = T_0(G) \leq T_1(G) \leq \cdots$ in such way that $G/T(G)$ is a CT-group [4]. In [2] it is proved that if G is locally finite, then $T(G) = T_1(G)$. Similar results have also been obtained in [3] for other classes of groups, such as supersoluble groups. In analogy with this we introduce the notion of the $\mathfrak{N}_c$-transitive kernel of a group and prove that it has similar properties like the commutative-transitive kernel.

In the final section we present some examples of $\mathfrak{N}_2 T$-groups. In particular, we present Frobenius $\mathfrak{N}_2 T$-groups with nonabelian kernel and Frobenius $\mathfrak{N}_2 T$-groups with noncyclic complement. We also show that some finite linear groups with nilpotent centralizers are in a certain sense far from being $\mathfrak{N}_c T$-groups.

2. $\mathfrak{N}_c T$-groups

In this section we investigate the structure of locally finite $\mathfrak{N}_c T$-groups. In the beginning we exhibit some basic properties of these groups. For positive integers $r > 1$ and n denote by $\mathfrak{N}(r, n)$ the class of all groups in which every r-generator subgroup is nilpotent of class $\leq n$. Every finite $\mathfrak{N}(r, n)$-group is nilpotent by Zorn's theorem (see Theorem 12.3.4 in [10]). It is now clear that every locally nilpotent $\mathfrak{N}_cT$-group is also an $\mathfrak{N}(2,c)$-group. In fact, every $\mathfrak{N}_cT$-group with nontrivial center is an $\mathfrak{N}(2,c)$-group. On the other hand, the property $\mathfrak{N}_cT$ behaves badly under taking quotients and forming direct products. For, it is known that every free (soluble) group is a CT-group [15]. Moreover if G and H are $\mathfrak{N}_cT$-groups and there exist $x, y \in G$ such that $\langle x, y\rangle$ is not nilpotent, then it is easy to see that $G \times H$ is not an $\mathfrak{N}_dT$-group for any $d \in \mathbb{N}$.

Our first result shows that the classes of $\mathfrak{N}_cT$-groups form a chain.

Proposition 2.1. *Let c and d be integers, $c \geq d \geq 1$. Then every $\mathfrak{N}_dT$-group is also an $\mathfrak{N}_cT$-group.*

Proof. Let G be an $\mathfrak{N}_dT$-group. Let $x, y, z \in G\backslash\{1\}$ and suppose that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class $\leq c$. By the above remarks $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class $\leq d$. As G is an $\mathfrak{N}_dT$-group, it follows that $\langle x, z\rangle$ is nilpotent of class $\leq d$, hence it is nilpotent of class $\leq c$. □

The following lemma is crucial for the description of soluble locally finite $\mathfrak{N}_cT$-groups.

Lemma 2.2. *Let G be a locally finite $\mathfrak{N}_cT$-group with nontrivial Hirsch-Plotkin radical H. Then the factor group G/H acts fixed-point-freely on H by conjugation.*

Proof. As the Hirsch-Plotkin radical H is a locally nilpotent $\mathfrak{N}_cT$-group, it is also an $\mathfrak{N}(2,c)$-group. Let y be a nontrivial element in H. Suppose there exists $a \in C_G(y)\backslash H$. Since the group $\langle a, y\rangle$ is abelian and H is an $\mathfrak{N}(2,c)$-group, we conclude that the group $\langle a, h\rangle$ is nilpotent of class $\leq c$ for every $h \in H$, since G is an $\mathfrak{N}_cT$-group. By conjugation we get that $\langle a^g, h\rangle$ is also nilpotent of class $\leq c$ for all $g \in G$ and $h \in H$. As G is an $\mathfrak{N}_cT$-group, this implies that the group $\langle a, a^g\rangle$ is nilpotent of class $\leq c$ for every $g \in G$. In particular, we have $1 = [a^g, {}_c a] = [a, g, {}_c a]$ for all $g \in G$, hence a is a left $(c+1)$-Engel element of G. As G is locally finite, this implies that $a \in H$ (see, for instance, Exercise 12.3.2 of [10]), which is a contradiction. □

Theorem 2.3. *Every locally finite soluble $\mathfrak{N}_cT$-group is either an $\mathfrak{N}(2,c)$-group or a Frobenius group whose kernel and complement are both $\mathfrak{N}(2,c)$-groups. Conversely, every locally finite Frobenius group in which kernel and complement are both $\mathfrak{N}(2,c)$-groups is an $\mathfrak{N}_cT$-group.*

Proof. Let G be a locally finite soluble $\mathfrak{N}_cT$-group and suppose G is not in $\mathfrak{N}(2,c)$. Let N be its Hirsch-Plotkin radical. As N is also an $\mathfrak{N}_cT$-group, it is an $\mathfrak{N}(2,c)$-group. By Lemma 2.2 G/N acts fixed-point-freely on N, hence G is a Frobenius group with the kernel N and a complement H; see, for instance, Proposition 1.J.3 in [7]. Since H has a nontrivial center [7, Theorem 1.J.2], we have that $H \in \mathfrak{N}(2,c)$. Besides, N is nilpotent by the same result from [7].

Conversely, let G be a locally finite Frobenius group with the kernel N and a complement H and suppose that both N and H are $\mathfrak{N}(2,c)$-groups. Let $x, y, z \in G\backslash\{1\}$ and let the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ be nilpotent of class $\leq c$. Suppose

$x \in N$ and $y \notin N$. Then the equation $[x, {}_c y] = 1$ implies $[x, {}_{c-1} y] = 1$, since H acts fixed-point-freely on N. By the same argument we get $x = 1$, which is not possible. This shows that if $x \in N$ then $y \in N$ and similarly also $z \in N$. But in this case $\langle x, z\rangle$ is clearly nilpotent of class $\leq c$, since N is an $\mathfrak{N}(2, c)$-group. Thus we may assume that $x, y, z \notin N$. Let $x \in H^g$ and $y \in H^k$ for some $g, k \in G$ and suppose $H^g \neq H^k$. We clearly have $C_G(x) \leq H^g$ and $C_G(y) \leq H^k$. Let α be any simple commutator of weight c with entries in $\{x, y\}$. As $\langle x, y\rangle$ is nilpotent of class $\leq c$, we have $\alpha \in C_G(x) \cap C_G(y) = 1$. This implies that $\langle x, y\rangle$ is nilpotent of class $\leq c - 1$. Continuing with this process, we end at $x = y = 1$ which is impossible. Hence we conclude that $\langle x, y\rangle \leq H^g$ and similarly also $\langle y, z\rangle \leq H^g$. Therefore we have $\langle x, z\rangle \leq H^g$. But H^g is an $\mathfrak{N}(2, c)$-group, hence the group $\langle x, z\rangle$ is nilpotent of class $\leq c$. This concludes the proof. □

Theorem 2.3 can be further refined when we restrict ourselves to finite groups.

Theorem 2.4. *Let G be a finite group. Then G is a soluble $\mathfrak{N}_cT$-group if and only if it is either an $\mathfrak{N}(2, c)$-group or a Frobenius group with the kernel which is an $\mathfrak{N}(2, c)$-group and a complement which is nilpotent of class $\leq c$.*

Proof. By Theorem 2.3 we only need to show that if G is a finite soluble $\mathfrak{N}_cT$-group which is not an $\mathfrak{N}(2, c)$-group, then every complement H of the Frobenius kernel N of G is nilpotent of class $\leq c$. Suppose N is not abelian. Then the order of H is odd, hence all Sylow subgroups of H are cyclic. This implies that H is cyclic. Assume now that N is abelian. Then all the Sylow p-subgroups of H are cyclic for $p \neq 2$, whereas the Sylow 2-subgroup is either cyclic or a generalized quaternion group Q_{2^n} [5]. Moreover, since $H \in \mathfrak{N}(2, c)$, we obtain $n \leq c+1$. As H is nilpotent and all its Sylow subgroups are nilpotent of class $\leq c$, the nilpotency class of H does not exceed c. □

Let G be a finite $\mathfrak{N}_cT$-group and suppose $G \notin \mathfrak{N}(2, c)$. If the Fitting subgroup of G is nontrivial, then Lemma 2.2 together with Theorem 2.4 shows that G is soluble and so its structure is completely determined by Theorem 2.4. The complete classification of finite insoluble $\mathfrak{N}_cT$-groups is described in our next result. Note that it has been shown in [11] that the groups $\mathrm{PSL}(2, 2^f)$, where $f > 1$, are the only finite insoluble $\mathfrak{N}_1T$-groups. Passing to finite $\mathfrak{N}_cT$-groups with $c > 1$, we obtain an additional family of simple groups.

Theorem 2.5. *Let G be a finite $\mathfrak{N}_cT$-group with $c > 1$. Then G is either soluble or simple. Moreover, G is a nonabelian simple $\mathfrak{N}_cT$-group if and only if it is isomorphic either to $\mathrm{PSL}(2, 2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$, the Suzuki group with parameter $q = 2^{2n+1} > 2$.*

Proof. It is easy to see that in every finite $\mathfrak{N}_cT$-group G the centralizers of nontrivial elements are nilpotent, i.e., G is a CN-group. Suppose that G is not soluble. By a result of Suzuki [12, Part I, Theorem 4], G is a CIT-group, i.e., the centralizer of any involution in G is a 2-group. Let P and Q be any Sylow p-subgroups of G and suppose that $P \cap Q \neq 1$. Since P and Q are $\mathfrak{N}(2, c)$-groups and G is an $\mathfrak{N}_c T$-group, we conclude that $\langle P, Q\rangle$ is an $\mathfrak{N}(2,c)$-group, hence it is nilpotent. This shows that $\langle P, Q\rangle$ is a p-group, which implies $P = Q$. Therefore Sylow subgroups of G are independent. Combining Theorem 1 in Part I and Theorem 3 in Part II of [12], we conclude that G has to be simple. Additionally, we also obtain that G is a ZT-group, that is, G is faithfully represented as a doubly transitive permutation group of odd degree in which the identity is the only element fixing three distinct letters. The structure of these groups is described in [13]. It turns out that G is isomorphic either to $\mathrm{PSL}(2, 2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$ with $q = 2^{2n+1} > 2$.

It remains to prove that $\mathrm{PSL}(2, 2^f)$ and $\mathrm{Sz}(q)$ are $\mathfrak{N}_c T$-groups. For projective special linear groups this has been done in [11]. Now, let $G = \mathrm{Sz}(q)$ where $q = 2^{2n+1} > 2$. By Theorem 3.10 c) in [6] G has a nontrivial partition $(G_i)_{i\in I}$, where for every $i \in I$ the group G_i is either cyclic or nilpotent of class ≤ 2. Moreover, the proof of result 3.11 in [6] implies that for all $g \in G\backslash\{1\}$ the relation $g \in G_i$ implies that $C_G(g) \le G_i$. Let $x, y, z \in G\backslash\{1\}$ and suppose that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class ≤ 2. Let a and b be nontrivial elements in $Z(\langle x, y\rangle)$ and $Z(\langle y, z\rangle)$, respectively, and suppose that $a \in G_i$ and $b \in G_j$ for some $i, j \in I$. Then $y \in C_G(a) \cap C_G(b) \le G_i \cap G_j$, hence $i = j$. But now we get $x, z \in G_i$ and since G_i is nilpotent of class ≤ 2, the same is true for the group $\langle x, z\rangle$. Hence G is an $\mathfrak{N}_2 T$-group. By Proposition 2.1 G is an $\mathfrak{N}_c T$-group for every $c > 1$. □

It is proved in [15] that every locally finite insoluble CT-group is isomorphic to $\mathrm{PSL}(2, F)$ for some locally finite field F. For $\mathfrak{N}_c T$-groups, where $c > 1$, we have the following result.

Theorem 2.6. *Let $c > 1$ and let G be a locally finite $\mathfrak{N}_c T$-group which is not locally soluble. Then there exists a locally finite field F such that G is isomorphic either to* $\mathrm{PSL}(2, F)$ *or to* $\mathrm{Sz}(F)$.

Proof. Let G be a locally finite $\mathfrak{N}_c T$-group and suppose that G is not locally soluble. Then G contains a finite insoluble subgroup, hence every finite subgroup of G is contained in some finite insoluble subgroup of G. Using Theorem 2.5, we conclude that every finitely generated subgroup of G has a faithful representation of degree 4 over some field of even characteristic. By Mal'cev's representation theorem [7, Theorem 1.L.6], G has a faithful representation of the same degree over a field which is an ultraproduct of some finite fields. Hence G is a linear periodic group. It is not difficult to see that G has to be simple. Namely, the set of all finite nonabelian simple subgroups of G is a local system of G. By a theorem of Winter [7] the group G is countable. Thus we obtain a chain $(G_i)_{i\in\mathbb{N}}$ of nonabelian finite simple subgroups in G such that G is the union of this chain. By Theorem 2.5 we have either $G_i \cong \mathrm{PSL}(2, F_i)$ or $G_i \cong \mathrm{Sz}(F_i)$ for suitable finite fields F_i, $i \in \mathbb{N}$. On the other hand, $PSL(2, F)$ does not contain any Suzuki group as a subgroup and vice versa (this follows from [13] and Dickson's theorem in [5]). Therefore we either have $G_i \cong \mathrm{PSL}(2, F_i)$ for all $i \in \mathbb{N}$ or $G_i \cong \mathrm{Sz}(F_i)$ for all $i \in \mathbb{N}$. By a theorem of Kegel [7, Theorem 4.18] there exists a locally finite field F such that either $G \cong \mathrm{PSL}(2, F)$ or $G \cong \mathrm{Sz}(F)$. □

Let the group G be locally finite and locally soluble. If G is an $\mathfrak{N}_2T$-group, then Theorem 2.5 implies that every finitely generated subgroup of G is either a 2-Engel group or a Frobenius group with the kernel which is a 2-Engel group and a complement which is nilpotent of class ≤ 2. As every 2-Engel group is nilpotent of class ≤ 3 (see [9, p. 45]), the derived length of finitely generated subgroups of G is bounded, so G is actually soluble. Therefore we have:

Corollary 2.7. *Let G be a locally finite $\mathfrak{N}_2T$-group. Then G is either soluble or simple.*

The structure of locally finite $\mathfrak{N}_cT$-groups, where $c > 2$, is more complicated. Namely, Bachmuth and Mochizuki [1] constructed an insoluble $\mathfrak{N}(2,3)$-group of exponent 5. This is a locally finite $\mathfrak{N}_3T$-group in which all finite subgroups are nilpotent. Therefore the result of Corollary 2.7 is no longer true for $\mathfrak{N}_cT$-groups with $c > 2$.

3. $\mathfrak{N}_c$-transitive kernel

Let G be a group and let c be a positive integer. Put $T_0^{(c)}(G) = 1$ and let $T_1^{(c)}(G)$ be the group generated by all commutators $[x_1, x_2, \ldots, x_{c+1}]$ for $x_i \in \{a, b\}$, where a and b are nontrivial elements of G such that there exist $t \in \mathbb{N}_0$ and $y_1, \ldots, y_t \in G\backslash\{1\}$ with $\langle a, y_1\rangle \in \mathfrak{N}_c, \langle y_1, y_2\rangle \in \mathfrak{N}_c, \ldots, \langle y_t, b\rangle \in \mathfrak{N}_c$. It is clear that $T_1^{(c)}(G)$ is a characteristic subgroup of G. For $n > 1$ we define $T_n^{(c)}(G)$ inductively by $T_n^{(c)}(G)/T_{n-1}^{(c)}(G) = T_1^{(c)}(G/T_{n-1}^{(c)}(G))$. So we get a chain $1 = T_0^{(c)}(G) \leq T_1^{(c)}(G) \leq \cdots \leq T_n^{(c)}(G) \leq \cdots$ of characteristic subgroups of the group G. We define

$$T^{(c)}(G) = \bigcup_{n\in\mathbb{N}_0} T_n^{(c)}(G)$$

to be the *(nilpotent of class c)-transitive kernel* or, shorter, *$\mathfrak{N}_c$-transitive kernel* of the group G. In the case $c = 1$ this definition coincides with the usual definition of the commutative-transitive kernel given in [4]. From the definition it also follows that $T^{(c)}(G)$ is a characteristic subgroup of G and that $T^{(c)}(G) = 1$ if and only if G is an $\mathfrak{N}_cT$-group. Moreover, $G/T^{(c)}(G)$ is an $\mathfrak{N}_cT$-group for every group G. Additionally, notice that $T^{(c)}(G) = T_n^{(c)}(G)$ for some $n \in \mathbb{N}_0$ if and only if $G/T_n^{(c)}(G)$ is an $\mathfrak{N}_cT$-group. We use the notation $\Gamma_t(G) = \langle \gamma_t(\langle a, b\rangle) \mid a, b \in G\rangle$. It is easy to see that $T^{(c)}(G) \leq \Gamma_{c+1}(G)$.

In [2] it is proved that if G is a locally finite group, then $T^{(1)}(G) = T_1^{(1)}(G)$. In this section we shall show that we have an analogous result for the $\mathfrak{N}_c$-transitive kernel.

Proposition 3.1. *Let G be a group and H a subgroup of G. Let c be a positive integer and suppose that the set $\mathcal{S} = \{h \in H \mid \langle h, k\rangle \in \mathfrak{N}_c$ for all $k \in H\}$ contains a nontrivial element. Then the group $HT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group.*

Proof. Let $z \in \mathcal{S}\backslash\{1\}$. For all $a, b \in H\backslash\{1\}$ we have $\gamma_{c+1}(\langle a, b\rangle) \leq T_1^{(c)}(H)$, since the groups $\langle a, z\rangle$ and $\langle z, b\rangle$ are nilpotent of class $\leq c$. This implies that $\Gamma_{c+1}(H) = T_1^{(c)}(H) \leq T_1^{(c)}(G)$, so $HT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group. □

Note that Proposition 3.1 implies that if G is a finite group, then every Sylow subgroup of $G/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group. In particular, if G is finite then the Fitting subgroup of $G/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group.

Proposition 3.2. *The class of finite $\mathfrak{N}_cT$-groups is closed under taking quotients.*

Proof. By Theorem 2.5 it suffices to consider finite soluble $\mathfrak{N}_cT$-groups. So suppose that G is a finite soluble $\mathfrak{N}_cT$-group. If $G \in \mathfrak{N}(2, c)$, then we are done. Otherwise, G is a Frobenius group with the kernel $F = \mathrm{Fitt}(G)$ which is an $\mathfrak{N}(2, c)$-group and a complement H which is nilpotent of class $\leq c$ by Theorem 2.4. If N is a normal subgroup of G, then we have either $N \leq F$ or $F \leq N$. If $F \leq N$, then G/N is nilpotent of class $\leq c$, hence it is an $\mathfrak{N}_cT$-group. Assume now that N is a proper subgroup of F. Then $G/N = F/N \rtimes H$, where the action of H on F/N is induced by the conjugation on F with elements of H. Since the subgroup N is invariant under the action of H, we conclude that H acts fixed-point-freely on F/N by Satz 8.10 in [5]. Therefore G/N is an $\mathfrak{N}_cT$-group by Theorem 2.4. □

The following result is a generalization of Theorem 3 in [2]:

Theorem 3.3. *Let G be a finite group. Then $T^{(c)}(G) = T_1^{(c)}(G)$ for every positive integer c.*

Proof. If $T_1^{(c)}(G) = 1$ or $T_1^{(c)}(G) = \Gamma_{c+1}(G)$, then we have nothing to prove. So we may assume that $1 \neq T_1^{(c)}(G) < \Gamma_{c+1}(G)$. Additionally, we may suppose that $T^{(c)}(H) = T_1^{(c)}(H)$ for every proper subgroup H of G. Let $\mathcal{F} = \{1 \neq H \triangleleft G \,|\, \Gamma_{c+1}(H) \leq T_1^{(c)}(G)\}$. Then this set is not empty since $T_1^{(c)}(G) \in \mathcal{F}$. So $\mathcal{F}$ has a maximal element N. First of all, it is clear that $N \neq G$, since $T_1^{(c)}(G) \neq \Gamma_{c+1}(G)$. Furthermore, since $NT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group, the group $NT_1^{(c)}(G)$ also belongs to $\mathcal{F}$, so we have $T_1^{(c)}(G) \leq N$ by the maximality of N. Let $F/T_1^{(c)}(G)$ be the Fitting subgroup of $G/T_1^{(c)}(G)$. Since $N/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group, it is nilpotent, hence $N/T_1^{(c)}(G) \leq F/T_1^{(c)}(G)$. On the other hand, since $F/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group, we have that $\Gamma_{c+1}(F) \leq T_1^{(c)}(G)$. Thus $F \in \mathcal{F}$, hence $F = N$ by the maximality of N in $\mathcal{F}$. Consider now the set $\mathcal{S} = \{h \in N \,|\, \langle h, k\rangle \in \mathfrak{N}_c \text{ for all } k \in N\}$. Here we have to consider the following two cases.

Case 1. Suppose that $\mathcal{S} \neq \{1\}$ and let h be a nontrivial element of $\mathcal{S}$. Let $y \in N\backslash\{1\}$ and let $a \in C_G(y)$. For every $b \in N$ we have $\gamma_{c+1}(\langle a, b\rangle) \leq T_1^{(c)}(G)$, since $\langle a, y\rangle$, $\langle y, h\rangle$ and $\langle h, b\rangle$ are in $\mathfrak{N}_c$. Additionally we have that $\langle a^g, y^g\rangle$, $\langle y^g, h\rangle$, $\langle h, y^k\rangle$ and $\langle y^k, a^k\rangle$ are in $\mathfrak{N}_c$ for all $g, k \in G$. Hence $\gamma_{c+1}(\langle a^g, a^k\rangle) \leq T_1^{(c)}(G)$ for all $g, k \in G$. In particular, this implies that $aT_1^{(c)}(G)$ is a left $(c + 1)$-Engel element of the group $G/T_1^{(c)}(G)$, hence it is contained in the Fitting subgroup of $G/T_1^{(c)}(G)$ by Theorem 12.3.7 in [10]. This gives that $a \in N$. By Satz 8.5 in [5] G is a Frobenius group and N is its kernel. Let A be a complement of N in G. Since $T_1^{(c)}(A) \leq A \cap T_1^{(c)}(G) \leq A \cap N = 1$, it follows that A is an $\mathfrak{N}_cT$-group. Moreover the center of A is nontrivial by [5, Satz 8.18], so A is an $\mathfrak{N}(2,c)$-group. Therefore G is soluble. If the nilpotency class of N does not exceed c, then G is an $\mathfrak{N}_cT$-group by Theorem 2.3 and $T_1^{(c)}(G) = 1$, which is a contradiction. Hence we may suppose that the nilpotency class of N is greater than c. Consider the group $G/T_1^{(c)}(G) = N/T_1^{(c)}(G) \rtimes AT_1^{(c)}(G)/T_1^{(c)}(G)$. This is a Frobenius group with the kernel $N/T_1^{(c)}(G) \in \mathfrak{N}(2,c)$ and complement $AT_1^{(c)}(G)/T_1^{(c)}(G)$ which is also an $\mathfrak{N}(2,c)$-group. By Theorem 2.3 the group $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group, hence $T^{(c)}(G) = T_1^{(c)}(G)$ in this case.

Case 2. Suppose now that $\mathcal{S} = \{1\}$. Let $\Phi(G)$ be the Frattini subgroup of G. If $T_1^{(c)}(G) \leq \Phi(G)$, then the nilpotency of the group $N/T_1^{(c)}(G)$ implies that N is nilpotent, which is a contradiction. Hence $T_1^{(c)}(G) \not\leq \Phi(G)$, so there exists a maximal subgroup M of G such that $T_1^{(c)}(G) \not\leq M$. Then $G = MT_1^{(c)}(G)$ and $T_1^{(c)}(M) = T^{(c)}(M)$ since $M < G$. From $T_1^{(c)}(M) \leq T_1^{(c)}(G) \cap M$ we now obtain that $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group, since it is a homomorphic image of the $\mathfrak{N}_cT$-group $M/T_1^{(c)}(M)$. So $T^{(c)}(G) = T_1^{(c)}(G)$, as required. □

Corollary 3.4. *Let G be a locally finite group. Then $T^{(c)}(G) = T_1^{(c)}(G)$ for every positive integer c.*

Proof. It suffices to show that if G is locally finite, then $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group. Let $x, y, z \in G \setminus T_1^{(c)}(G)$ and suppose that the groups $\langle x, y\rangle T_1^{(c)}(G)/T_1^{(c)}(G)$ and $\langle y, z\rangle T_1^{(c)}(G)/T_1^{(c)}(G)$ are nilpotent of class $\leq c$. This means that $\gamma_{c+1}(\langle x, y\rangle) \leq T_1^{(c)}(G)$ and $\gamma_{c+1}(\langle y, z\rangle) \leq T_1^{(c)}(G)$. Let $\{\alpha_1, \dots, \alpha_r\}$ and $\{\overline{\alpha}_1, \dots, \overline{\alpha}_{r'}\}$ be the sets of all simple commutators of weight $c+1$ with entries from $\{x, y\}$ and $\{y, z\}$, respectively. For every $i = 1, \dots, r$ we have

$$\alpha_i = \prod_{t=1}^{n_i} [x_{i,t,1}, \dots, x_{i,t,c+1}]^{\epsilon_{i,t}},$$

where $\epsilon_{i,t} = \pm 1$, $x_{i,t,j} \in \{a_{i,t}, b_{i,t}\}$ for some $a_{i,t}, b_{i,t} \in G$ for which there exist $y_{i,t,1}, \dots, y_{i,t,s_{i,t}}$ in G such that $\langle a_{i,t}, y_{i,t,1}\rangle, \langle y_{i,t,1}, y_{i,t,2}\rangle, \dots, \langle y_{i,t,s_{i,t}}, b_{i,t}\rangle$ are nilpotent of class $\leq c$, for all $i = 1, \dots, r$, $j = 1, \dots, c+1$ and $t = 1, \dots, n_i$. Similarly,

$$\overline{\alpha}_{i'} = \prod_{t'=1}^{m_{i'}} [\overline{x}_{i',t',1}, \dots, \overline{x}_{i',t',c+1}]^{\overline{\epsilon}_{i',t'}},$$

where $\overline{\epsilon}_{i',t'} = \pm 1$, $\overline{x}_{i',t',j} \in \{\overline{a}_{i',t'}, \overline{b}_{i',t'}\}$ for some $\overline{a}_{i',t'}, \overline{b}_{i',t'} \in G$ for which there exist $\overline{y}_{i',t',1}, \dots, \overline{y}_{i',t',s'_{i',t'}}$ in G such that $\langle \overline{a}_{i',t'}, \overline{y}_{i',t',1}\rangle, \langle \overline{y}_{i',t',1}, \overline{y}_{i',t',2}\rangle, \dots, \langle \overline{y}_{i',t',s'_{i',t'}}, \overline{b}_{i',t'}\rangle$ are nilpotent of class $\leq c$, for all $i' = 1, \dots, r'$, $j = 1, \dots, c+1$ and $t' = 1, \dots, m_{i'}$. Let H be the subgroup of G generated by all

$$x,\ y,\ z,\ x_{i,t,j},\ \overline{x}_{i',t',j},\ a_{i,t},\ \overline{a}_{i',t'},\ y_{i,t,k},\ \overline{y}_{i',t',k'},$$

where $i = 1, \dots, r$, $i' = 1, \dots, r'$, $t = 1, \dots, n_i$, $t' = 1, \dots, m_{i'}$, $j = 1, \dots, c+1$, $k = 1, \dots, s_{i,t}$ and $k' = 1, \dots, s'_{i',t'}$. Then $\gamma_{c+1}(\langle x, y\rangle) \leq T_1^{(c)}(H)$ and $\gamma_{c+1}(\langle y, z\rangle) \leq T_1^{(c)}(H)$. Since $H/T_1^{(c)}(H)$ is an $\mathfrak{N}_cT$-group by Theorem 3.3, we have $\gamma_{c+1}(\langle y, z\rangle) \leq T_1^{(c)}(H) \leq T_1^{(c)}(G)$. This concludes the proof. □

Remark 3.5. Let G be a locally nilpotent group, and let $c \geq 1$ be any positive integer. It easily follows from Proposition 3.1 that $T_1^{(c)}(G) = T^{(c)}(G) = \Gamma_{c+1}(G)$.

Remark 3.6. Let G be a supersoluble group. It is proved in [3] that $T^{(1)}(G) = T_1^{(1)}(G)$. It is to be expected that the same holds true for $\mathfrak{N}_c$-transitive kernel where $c > 1$, and that the proofs require only suitable modifications of those in [3].

4. Examples and non-examples

Theorem 2.4 completely describes the structure of finite soluble $\mathfrak{N}_cT$-groups. At least in the case $c \leq 2$ we are able to obtain more detailed information about these groups, using the descriptions of fixed-point-free actions on finite abelian groups obtained by Zassenhaus [16].

Example 4.1. Let G be a finite soluble $\mathfrak{N}_1T$-group (or CT-group) which is not abelian. Then $G = F \rtimes \langle x\rangle$ where F is abelian and $\langle x\rangle$ acts fixed-point-freely on F (see Theorem 2.4 or Theorem 10 of [15]). Suppose $F = \bigoplus_{i=1}^m F_i$ where $F_i \cong \mathbb{Z}_{p_i^{e_i}}^{n_i}$ and $e_i \neq e_j$ if $p_i = p_j$. Let k be the order of $\langle x\rangle$. Then it follows from [16] that $x = (x_1, \dots, x_m)$ where $\langle x_i\rangle$ is a fixed-point-free automorphism group of order k on G_i for all $i = 1, \dots, m$. Conversely, for every x with this property the group $\langle x\rangle$ acts fixed-point-freely on F. Note also that a necessary and sufficient condition for the existence of a fixed-point-free automorphism on F is given in Theorem 2 of [15].

As the class of $\mathfrak{N}(2,2)$-groups coincides with the variety of 2-Engel groups, Theorem 2.4 implies that a finite soluble $\mathfrak{N}_2T$-group is either 2-Engel or it is a Frobenius group with the kernel F which is 2-Engel and a complement H which is nilpotent of class ≤ 2. Thus it follows from Levi's theorem (see [9, p. 45]) that F is nilpotent of class ≤ 3. Moreover, if $|H|$ is even, then F is abelian. In this case, H is either a cyclic group or the quaternion group Q_8 of order 8 or $C_m \times Q_8$ where m is odd. Our next example shows that there is essentially only one possibility of having a Frobenius $\mathfrak{N}_2T$-group with the prescribed kernel and a complement isomorphic to Q_8.

Example 4.2. Let F be a finite abelian group and $F = \bigoplus_{i=1}^m F_i$ where $F_i \cong \mathbb{Z}_{p_i^{e_i}}^{n_i}$ and $e_i \neq e_j$ if $p_i = p_j$. Then it follows from [16] that F admits a quaternion fixed-point-free automorphism group H of order 8 if and only if $2 \nmid p_i$ and $2|n_i$

for all $i = 1, \dots, m$. In this case, H is conjugated to the group $\langle x, y\rangle$ where the restrictions of x and y on F_i can be presented by matrices

$$A_i = \bigoplus_{j=1}^{n_i/2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B_i = \bigoplus_{j=1}^{n_i/2} \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & -\alpha_i \end{pmatrix},$$

where $i = 1, \dots, m$ and $\alpha_i^2 + \beta_i^2 \equiv -1 \mod p_i^{e_i}$ for all $i = 1, \dots, m$.

In the following example we present a Frobenius group G with abelian kernel F and a complement H which is isomorphic to $C_p \times Q_8$, where p is an arbitrary odd prime. Of course, in this case G is an $\mathfrak{N}_2T$-group.

Example 4.3. Let q be a prime such that $p|(q-1)$ and let $F = C_q^2$. Let $a, b \in \mathbb{Z}_q$ be such that $a^2+b^2+1 \equiv 0 \mod q$. Consider the automorphisms of C_q^2 represented by the following matrices over $\mathbb{Z}_q$:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad X = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}.$$

Here ζ is a primitive p-th root modulo q. Then we have $\langle A, B, X\rangle \cong C_p \times Q_8$ and it can be verified that $H = \langle A, B, X\rangle$ acts fixed-point-freely on F. The corresponding Frobenius group $F \rtimes H$ is an $\mathfrak{N}_2T$-group, but it is not an $\mathfrak{N}_1T$-group.

On the other hand, if the order of H is odd, then H is cyclic and the group F may be nonabelian. In the next example we show that this is indeed so.

Example 4.4. Let $D = \langle x_1\rangle \times \langle x_2\rangle \times \langle x_3\rangle \times \langle x_4\rangle$ be an elementary group of order 16. Put $D_1 = D \rtimes \langle a\rangle$, where a is an element of order 2 acting on D in the following way: $[x_1, a] = x_3x_4$, $[x_2, a] = x_4$, $[x_3, a] = [x_4, a] = 1$. We make another split extension $F = D_1 \rtimes \langle b\rangle$, where b induces an automorphism of order 2 on D_1 in the following way: $[x_1, b] = x_3$, $[x_2, b] = x_3x_4$ and $[x_3, b] = [x_4, b] = [a, b] = 1$. The group F is nilpotent of class 2 and $|F| = 64$. Consider the following map on F:

$$x_1^\alpha = x_2\ ,\ x_2^\alpha = x_1x_2\ ,\ x_3^\alpha = x_4\ ,\ x_4^\alpha = x_3x_4\ ,\ a^\alpha = ab\ ,\ b^\alpha = a.$$

It can be verified that α is an automorphism of order 3 on F. Moreover, α acts fixed-point-freely on F. The corresponding split extension $G = F \rtimes \langle\alpha\rangle$ is an $\mathfrak{N}_2T$-group of order 192 with the kernel F. One can verify that this is the smallest example of a non-nilpotent soluble $\mathfrak{N}_2T$-group having the nonabelian Frobenius kernel.

Finite simple groups with nilpotent centralizers are classified in [12] and [13]. It turns out that every finite nonabelian simple CN-group is of one of the following types:

(i) $\mathrm{PSL}(2, 2^f)$, where $f > 1$;

(ii) $\mathrm{Sz}(q)$, the Suzuki group with parameter $q = 2^{2n+1} > 2$;

(iii) $\mathrm{PSL}(2,p)$, where p is either a Fermat prime or a Mersenne prime;
(iv) $\mathrm{PSL}(2,9)$;
(v) $\mathrm{PSL}(3,4)$.

By Theorem 2.5 only groups listed under (i) and (ii) are $\mathfrak{N}_cT$-groups for $c > 1$. Our aim is to show that in groups (iii)-(v) we can always find such nontrivial elements x, y and z that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class ≤ 2, yet the group $\langle x, z\rangle$ is not even nilpotent. We call such a triple of elements a *bad triple.*

Proposition 4.5. *In the groups* $\mathrm{PSL}(2,9)$ *and* $\mathrm{PSL}(3,4)$ *there exist bad triples of elements.*

Proof. First we want to show that our proposition holds true for $\mathrm{PSL}(3,4)$. To this end, consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

over the Galois field $\mathrm{GF}(4)$. It is easy to see that A, B and C belong to $\mathrm{SL}(3,4)$. Besides, these matrices are not in the center of $\mathrm{SL}(3,4)$ and a straightforward calculation shows that $[A,B] = [B,C,C] = [C,B,B] = 1$. Let $\overline{A}$, $\overline{B}$ and $\overline{C}$ be the homomorphic images of A, B and C, respectively, under the canonical homomorphism $\mathrm{SL}(3,4) \to \mathrm{PSL}(3,4)$. Then the group $\langle \overline{A}, \overline{B}\rangle$ is abelian and $\langle \overline{B}, \overline{C}\rangle$ is nilpotent of class 2. On the other hand, $\langle \overline{A}, \overline{C}\rangle$ is not nilpotent, since $[A,C], [A,C,C] \notin Z(\mathrm{SL}(3,4))$ and $[A,C,C,C] = [A,C,C]$.

A similar argument also works for the group $\mathrm{PSL}(2,9)$. In this case, we have to consider the following matrices in $\mathrm{SL}(2,9)$:

$$A = \begin{pmatrix} \zeta^3 & 0 \\ 0 & \zeta^5 \end{pmatrix} \quad , \quad B = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^6 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & \zeta^4 \\ \zeta^4 & \zeta^4 \end{pmatrix}.$$

Here ζ is a generator of the multiplicative group of $\mathrm{GF}(9)$. If $\overline{A}$, $\overline{B}$ and $\overline{C}$ are the corresponding elements of $\mathrm{PSL}(2,9)$, then it is a routine to verify that the group $\langle \overline{A}, \overline{B}\rangle$ is abelian and $\langle \overline{B}, \overline{C}\rangle$ is nilpotent of class 2, but $\langle \overline{A}, \overline{C}\rangle$ is not nilpotent. □

Finally we consider the groups $\mathrm{PSL}(2,p)$ where p is a Fermat prime or a Mersenne prime. If $p = 5$, then $\mathrm{PSL}(2,5) \cong \mathrm{PSL}(2,4)$ is an $\mathfrak{N}_1T$-group by [11]. For $p > 5$ the situation is completely different.

Proposition 4.6. *If* p *is a Fermat prime or a Mersenne prime and* $p \neq 5$*, then* $\mathrm{PSL}(2,p)$ *contains a bad triple of elements.*

Proof. First we cover the case of Fermat primes. For this we need the following number-theoretical result:

Claim 1. If p is a Fermat prime, then there exists $x \in \mathbb{Z}_p$ such that $2x^2 \equiv -1 \mod p$.

Proof of Claim 1. Let $p = 2^{2^n} + 1$ for some $n > 1$. It is enough to show that $2^{2^n - 1}$ is a quadratic residue modulo p. Let P be the set of all integers $a \in \{0, \ldots, p-1\}$ which are primitive roots modulo p and let Q be the set of all $a \in \{0, \ldots, p-1\}$ which are not quadratic residues modulo p. We shall show that $P = Q$. First, if $a \notin Q$, then there exists an integer t such that $t^2 \equiv a \mod p$. By Euler's theorem, $a^{\phi(p)/2} \equiv t^{\phi(p)} \equiv 1 \mod p$, hence a is not a primitive root modulo p (here ϕ is the Euler function). This shows that $P \subseteq Q$. To prove the converse inclusion, note that p has exactly $\phi(\phi(p))$ incongruent primitive roots and exactly $(p-1)/2$ quadratic non-residues. Hence

$$|P| = \phi(\phi(p)) = \phi(p-1) = \phi(2^{2^n}) = 2^{2^n - 1} = \frac{p-1}{2} = |Q|$$

and therefore $P = Q$. Since $2^{2^n-1} \notin P = Q$, we have that $2^{2^n-1} \equiv x^2 \mod p$ for some $x \in \mathbb{Z}_p$, hence $2x^2 \equiv -1 \mod p$, as desired.

Now we are ready to finish the proof. Let $c, x \in \mathbb{Z}_p$ be such that $c^2 \equiv -1 \mod p$, $c \not\equiv -c \mod p$ and $2x^2 \equiv -1 \mod p$ (such x exists by Claim 1). Let

$$A = \begin{pmatrix} 2x & 0 \\ 0 & -x \end{pmatrix}, \quad B = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} x & x \\ x & -x \end{pmatrix}$$

be matrices in $\mathrm{SL}(2,p) \backslash Z(\mathrm{SL}(2,p))$. It is clear that A and B commute, and a short calculation shows that $[B, C, C]$ and $[C, B, B]$ belong to $Z(\mathrm{SL}(2,p))$. To prove that $\mathrm{PSL}(2,p)$ is not an $\mathfrak{N}_c T$-group for any $c > 1$ it suffices to show that $[C, {}_n A] \notin Z(\mathrm{SL}(2,p))$ for any $n \in \mathbb{N}$. More precisely, we shall prove that

$$[C, {}_n A] = x^{3 \cdot 2^n - 2} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

where $a_n, b_n, c_n, d_n \in \mathbb{Z}_p$ are such that at least one of b_n, c_n and at least one of a_n, d_n are not zero. First note that this is true for $n = 1$, hence we may assume that $n > 1$. Then

$$[C, {}_{n+1} A] = x^{3 \cdot 2^{n+1} - 2} \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix},$$

where $a_{n+1} = -2a_n d_n - 4b_n c_n$, $b_{n+1} = 3b_n d_n$, $c_{n+1} = 2a_n c_n$ and $d_{n+1} = b_n c_n - 2a_n d_n$. If both b_{n+1} and c_{n+1} are zero, then $a_n = d_n = 0$ which is not possible by the induction assumption. Similarly, if $a_{n+1} = d_{n+1} = 0$, then $a_n d_n = -2b_n c_n$ and $b_n c_n = 2a_n d_n$, hence $5b_n c_n = 0$, a contradiction since $p > 5$. This concludes the proof for Fermat primes.

Assume now that p is a Mersenne prime. In this case we need the following auxiliary result:

Claim 2. If p is a Mersenne prime, then there exist $x, y \in \mathbb{Z}_p$ such that $x^2 - x + 1 \equiv 0 \mod p$ and $xy^4 \equiv 2y^2 + 1 \mod p$.

Proof of Claim 2. First note that since p is a Mersenne prime, $p - 1$ is divisible by 6. The congruence equation $x^3 \equiv -1 \mod p$ is clearly solvable, hence it has $\gcd(3, p-1) = 3$ incongruent solutions. This shows that the equation $x^2 - x + 1 = 0$ is solvable in $\mathbb{Z}_p$. Let x_1 and x_2 be its solutions. Then $x_2 = x_1^{-1} = 1 - x_1$. We claim that at least one of $1+x_1$, $1+x_2$ is a quadratic residue modulo p. For this note that since $(p-1)/2$ is odd, Euler's criterion implies that for every $a \in \mathbb{Z}_p\backslash\{0\}$ we have that precisely one of a and $-a$ is a quadratic residue modulo p. Furthermore, since $\gcd(2^k, p-1) = \gcd(2, p-1)$, every quadratic residue modulo p is also a 2^k-power residue modulo p. Suppose $1+x_1$ is not a square residue modulo p. Then $-1-x_1$ is a quadratic residue modulo p and $1+x_2 = 2-x_1 = 1-x_1^2 = x_1^2(-1-x_1)$ is a square residue modulo p. So from now on we assume x is such that $1-x+x^2 \equiv 0 \mod p$ and $1+x$ is a square residue modulo p. Then the equation $xt^2-2t-1=0$ has two solutions in $\mathbb{Z}_p$, namely $t_{1,2} = x^{-1}(1\pm c) = x^2(-1\mp c)$, where $c^2 = 1+x$ in $\mathbb{Z}_p$. In order to ensure the existence of y it suffices to prove that $-1\mp c$ are square residues modulo p. Since $(-1+c)(-1-c) = -x = x^4$, we have that $-1+c$ and $-1-c$ are either both squares or both non-squares in $\mathbb{Z}_p$. Assume that they are not squares. Then $1+c$ and $1-c$ are squares in $\mathbb{Z}_p$. For every square q in $\mathbb{Z}_p$ denote by $\sqrt{q}$ the square in $\mathbb{Z}_p$ for which $(\sqrt{q})^2 = q$. Let $u = \sqrt{1-c}$ and $v = \sqrt{1+c}$. Then $(u+v)^2 = u^2+v^2+2uv = 2(1+\sqrt{1-c^2}) = 2(1+\sqrt{-x}) = 2(1+x^2)$. Since $p \equiv -1 \mod 8$, 2 is a square residue modulo p, hence $1+x^2$ is a square in $\mathbb{Z}_p$. On the other hand, $-1-x^2 = -x = x^4$ is also a square in $\mathbb{Z}_p$. This leads to a contradiction, hence our claim is proved.

Let x and y be as above and let

$$A = \begin{pmatrix} 0 & x^2 \\ x & 0 \end{pmatrix}, \quad B = \begin{pmatrix} x & x \\ -1 & -x \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix}$$

be matrices in $\mathrm{SL}(2,p)\backslash Z(\mathrm{SL}(2,p))$. It is not difficult to check that $[A,B] = -1$, hence $[A,B] \in Z(\mathrm{SL}(2,p))$. Beside that, we have $[B,C,C] = (a_{ij})_{i,j}$, and a straightforward calculation shows that $a_{11} - a_{22} = x - x^2y^4 - 2x^4y^2 = 0$ by Claim 2. Similarly, we obtain $a_{21} = a_{12} = 0$, hence $[B,C,C]$ belongs to $Z(\mathrm{SL}(2,p))$. Furthermore, it can be checked that the same holds true for $[C,B,B]$. On the other hand, an induction argument shows that

$$[A,{}_nC] = \begin{pmatrix} y^{(-2)^n}x^{2^{n+1}} & 0 \\ 0 & y^{-(-2)^n}x^{2^n} \end{pmatrix}$$

for every $n \in \mathbb{N}$. If $[A,{}_nC] \in Z(\mathrm{SL}(2,p))$ for some $n \in \mathbb{N}$, then $y^{(-2)^m}x^{2^{m+1}} = y^{-(-2)^m}x^{2^m} = 1$ in $\mathbb{Z}_p$ for every $m > n$. Besides we have that x^{2^k} is either $x-1$ or $-x$, depending on whether k is odd or even, respectively. Suppose $m > n$ and let m be even. Then $[A,{}_mC] = 1$ implies $y^{2^m}(x-1) = 1$ and $y^{-2^m}x = -1$. Similarly, from $[A,{}_{m+1}C] = 1$ we obtain $y^{-2^{m+1}}x = -1$ and $y^{2^{m+1}}(x-1) = 1$. This implies $y^{2^m} = 1$ and hence $x = -1$, which contradicts the choice of x. □

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