Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 48 (2007), No. 2, 443-467.

Virtual Points and Separating Sets in Spherical Circle Planes

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Abstract. In this paper we introduce and investigate virtual points of spherical circle planes and use them to construct several new types of separating sets of circles in such geometries. Two spherical circle planes combine very naturally into another spherical circle plane if they share such a separating set of circles.

MSC2000: 51B10, 51H15
Keywords: spherical circle planes, flat Möbius planes, separating sets, virtual points

1. Introduction

Associated with a strictly convex topological sphere in real projective three-space is the circle geometry of nontrivial plane sections of the sphere. For example, the classical spherical circle plane is the geometry of nontrivial plane sections of the unit sphere and, in this special case, the nontrivial plane sections are just the Euclidean circles contained in the unit sphere. The set of circles corresponding to the planes through a point in space gives rise to an interesting subgeometry of the circle plane. This subgeometry is a flat affine plane if the point is on the sphere, a double cover of a flat projective plane if the point is an inner point of the sphere and a double cover of an $\mathbb{R}^2$-plane if the point is an outer point of the sphere.

General spherical circle planes usually do not come with a surrounding projective space. To generalize the subgeometries that correspond to points off the sphere, we introduce virtual inner and outer points of spherical circle planes that can play the roles of the respective inner and outer points of a strictly convex topological sphere in real projective three-space. Here a virtual inner or outer point of a spherical circle plane is a special involutory homeomorphism of the point space of the circle plane to itself. Its associated geometry of fixed circles is of the same type as the geometry of fixed (Euclidean) circles of the bundle involution associated with an inner or outer point of the unit sphere. The connection with what we said before is established by the fact that the geometry of fixed circles of a bundle involution associated with a point coincides with the nontrivial plane sections of the sphere with the planes containing this point.

Two points in real projective three-space separate their connecting line into two connected components. Every plane in this space either contains the whole line or intersects it in exactly one point. This implies that the set of planes which contain one or both of the distinguished points also separates the set of planes of real projective three-space into two connected components. In turn this implies that the set of planes which contain one or both of the distinguished points corresponds to a set of circles of the circle plane that separates the circle set of this geometry into two connected components. Of course, this set of circles is just the union of the circle sets of the two subgeometries associated with the two points.

The main focus of this paper is to investigate similar separating sets of circles in general spherical circle planes constructed from pairs of actual and virtual points of such planes. These special sets of circles do not only separate the circle sets of the spherical circle planes that they are contained in topologically, but also geometrically: given two spherical circle planes that share such a separating set, it is possible to recombine the separating set and one associated component of circles each taken from the two spherical circle planes, into the circle set of a new spherical circle plane.

This paper is organized as follows. In Section 2 we give a brief introduction to spherical circle planes and separating sets, introduce virtual inner and outer points of general spherical circle planes and derive some basic properties of these virtual points. In Section 3 we introduce seven new types of separating sets in spherical circle planes.

We only note that separating sets of circles similar to the ones investigated in this paper have been shown to exist in other types of geometries on surfaces, such as flat projective planes, cylindrical circle planes and toroidal circle planes; see [1], [2], [3] and, in particular, [4, Chapters 2.7.10, 3.3.4, 4.3.1, 4.3.2, 5.3.3 and 7.3.2] for more detailed information. Furthermore, see [5, Example 31.25(b)] and [6] for information about the closely related Moulton constructions which are based on separating sets contained in the point sets of flat linear spaces.

2. Spherical circle planes and separating sets

In this section we collect some basic information about spherical circle planes, and give definitions of virtual points and separating sets.

Axioms for spherical circle planes

A *spherical circle plane* is a point-circle geometry whose point set is homeomorphic to the unit sphere $\mathbb{S}^2$ and whose circles are simply closed curves embedded in the point set. Furthermore, every spherical circle plane satisfies the following **Axiom of Joining**:

- Three points are contained in exactly one circle.

A spherical circle plane is *nested* if it satisfies the following additional **Axiom of Touching**:

- Given two points and a circle through at least one of the points, there is exactly one circle that contains both points and touches the given circle, that is, coincides with this circle or intersects it in only one point.

Nested spherical circle planes are also known as *flat Möbius planes* and are Möbius planes in the usual incidence-geometric sense.

Ovoidal planes

Consider the geometry of nontrivial plane sections of a strictly convex topological sphere in real projective three-space. This geometry is an *ovoidal* spherical circle plane. It is nested if and only if the surface is differentiable, that is, if the surface has a unique tangent plane at every single one of its points. The spherical circle plane associated with the sphere $\mathbb{S}^2$ is the classical spherical circle plane. Its circles are just the Euclidean circles on $\mathbb{S}^2$. Many non-ovoidal spherical circle planes are known.

Topological geometries

Spherical circle planes are automatically *topological* geometries in the sense that the connecting circle of three points depends continuously on the three points, and, in the case of a nested plane, the touching circle depends continuously on the given two points and circle. More precisely, with respect to the natural topology on the point set and the topology induced by the Hausdorff metric on the circle set, the geometric operations are continuous operations on their respective domains of definition. By the *circle space* of a spherical circle plane we mean the circle set provided with the topology induced by the Hausdorff metric. The circle space is a connected, three-dimensional locally Euclidean space. In fact, it is known that the circle space of a flat Möbius plane is homeomorphic to a real projective 3-space from which one point has been removed.

Here is one more important fact which establishes spherical circle planes as very well-behaved topological geometries. Given a spherical circle plane with point set P, any two of its circles c and d having two points in common *intersect* *transversally* in these two points. This means that parts of c are contained in both connected components of $P \setminus d$. On the other hand, any two simply closed curves c and d on the sphere intersecting in exactly one point clearly touch in this point. This means that c minus this point is contained in one of the connected components of $P \setminus d$.

See [4, Chapter 3] for a comprehensive summary of results about spherical circle planes.

Separating sets

From now on we will assume, without loss of generality, that all spherical circle planes have the unit sphere $\mathbb{S}^2$ as point set. We will describe various types of *separating sets* of spherical circle planes. Here a separating set S is a subset of a circle space C satisfying the following axioms:

(S1) The complement $C \setminus S$ has two path-connected components C^+ and C^-. The circles in C^+ and C^- can be characterized in terms of S.

(S2) Given three points r, s, and t on the sphere, it suffices to look at the position of these three points with respect to the separating set to decide whether the connecting circle is contained in S, C^+, or C^-.

(S3) If we are dealing with a flat Möbius plane, it suffices to look at the position of two points r and s, and a circle c that contains r, to decide whether the circle d through s that touches c in r is contained in S, C^+, or C^-.

Now assume that a certain set of circles S is a separating set in two different spherical circle planes. Then it follows from axiom S2 that S together with C^+ taken from the first circle plane and C^- taken from the second circle plane forms the circle set of a new spherical circle plane. Furthermore, if the two planes we started with are flat Möbius planes, then, as a consequence of axiom S3, the new plane will be a flat Möbius plane as well.

Let us switch back to the spatial model. Fix a line l and two points p and q on this line. Clearly, the set $l \setminus \{p, q\}$ has two connected components, and every plane in the space either contains the line or intersects the line in exactly one point. This means that the set of planes containing p or q also gives rise to a set of circles that separates the circle space of the classical spherical circle plane into two connected components. Depending on the position of the two points and their connecting line with respect to the sphere, we distinguish eleven cases, as indicated in Figure 1.

Note that in the diagrams a solid point, an open inner point of the circle, and an open outer point of the circle represents a point on, an inner point, and an outer point of the sphere, respectively. The tangents to the sphere through an outer point touch the sphere in points that form a circle which we will call the *horizon* of the outer point. In the following, these horizons will play a special role and, in the diagrams, these special circles are represented by dashed lines. In diagram 5a the horizon of the outer point is also a circle associated with the inner point. In diagram 6d the horizon associated with one of the two outer points is

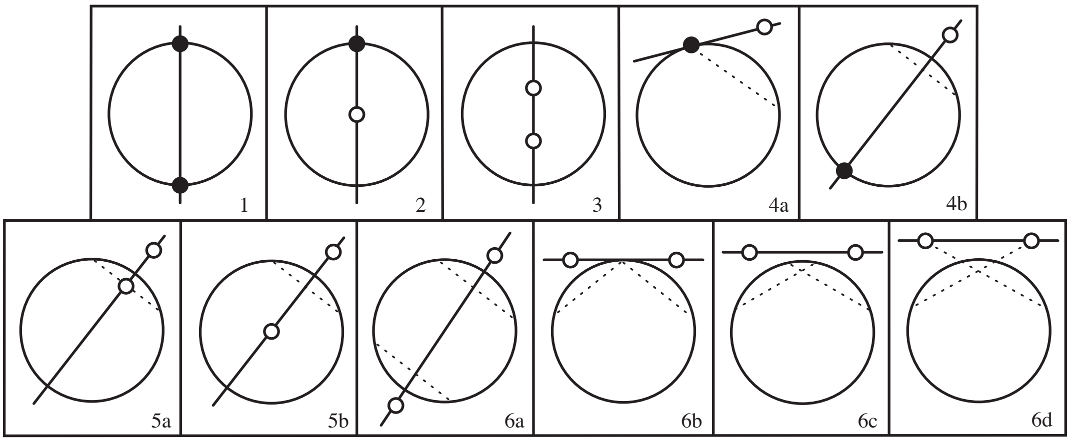

Figure 1. Eleven different positions of a pair of points with respect to a sphere

also one of the circles associated with the respective other outer point. Note that the cases captured by the diagrams present the complete picture in the classical case and that other conceivable scenarios such as a 'mixed' type in between types 6c and 6d do not occur in this case.

To be able to define virtual inner and outer points that can play the role of inner and outer points in general spherical circle planes, consider the *bundle involution* $i : \mathbb{S}^2 \to \mathbb{S}^2$ associated with a point r of the space not contained in $\mathbb{S}^2$; see Figure 2.

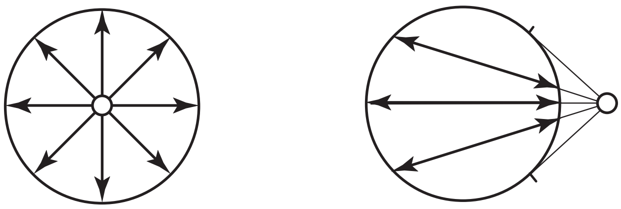

Figure 2. Bundle involutions associated with inner and outer points of $\mathbb{S}^2$

Let s be a point of $\mathbb{S}^2$. Then i fixes s if the line connecting r and s is a tangent of the sphere. Otherwise, it interchanges s with the unique second point of intersection of the connecting line with the sphere. The bundle involution associated with r is a homeomorphism. If r is an inner point of the sphere, it is

(I) fixed-point-free and orientation-reversing.

If r is an outer point of the sphere, it is

(O) orientation-reversing and the set of fixed points $\text{fix}(i)$ of the involution is a circle of the circle plane.

Furthermore, if $\text{FIX}(i)$ denotes the set of circles that are globally but not pointwise fixed by i, then in both cases

(B) two points of the spherical circle plane that are not interchanged by the involution are contained in exactly one circle in $\text{FIX}(i)$.

Note that the circles that are globally fixed but not pointwise fixed by a bundle involution associated with a point r are just the circles that correspond to the planes through r. The set of fixed points of an outer involution forms the horizon that we mentioned earlier.

Now, consider an arbitrary spherical circle plane with point set $\mathbb{S}^2$. Let $i : \mathbb{S}^2 \to \mathbb{S}^2$ be an involutory homeomorphism. If i satisfies conditions (I) and (B) above, we call it an *inner involution* of the circle plane. If i satisfies conditions (O) and (B) above, we call it an *outer involution.* Note that, unlike in the classical case, an inner or outer involution need not be an automorphism of the spherical circle plane.

In the next section we will describe seven different types of separating sets in general spherical circle planes that correspond to all those cases shown in Figure 1 in which the connecting line of the two points intersects the sphere in two points. These are cases 1, 2, 3, 4b, 5a, 5b, 6a. In the definitions of these separating sets inner and outer involutions play the role of inner and outer points. Therefore, we will also refer to inner and outer involutions as *virtual inner* and *outer points.*

Examples. Bundle involutions can also be defined for ovoidal spherical circle planes associated with convex topological spheres that are not Euclidean spheres. Those bundle involutions associated with inner points always give examples of inner involutions. A bundle involution associated with an outer point corresponds to an outer involution only if its set of fixed points is a circle of the circle plane (in general this is not the case). Usually bundle involutions of ovoidal circle planes are not automorphisms of these planes. The eleven cases shown in Figure 1 only give a complete picture in the classical case. In non-classical ovoidal circle planes a number of additional configurations can occur in the case of two outer points. For example, it is possible that two different outer involutions of an ovoidal spherical circle plane have the same set of fixed points. Here is how you construct such a spherical circle plane: Join two copies of a spherical cap along their circular boundaries to create a convex topological sphere. If the cap you started with is not a hemisphere, then the resulting topological sphere has only one symmetry axis, and the bundle involutions associated with infinitely many points on this symmetry axis have the circle at which the two caps are joined together as fixed circle.

It is also easy to construct inner and outer involutions that are inner and outer involutions of two different ovoidal spherical circle planes. Consider, for example, two different strictly convex topological spheres O_1 and O_2, one nested inside the other, and the associated ovoidal spherical circle planes S_1 and S_2. Furthermore, let p be a point in the interior of both. Projecting the circles of S_2 through p onto O_1 yields a spherical circle plane S_2' with point set O_1 isomorphic to S_2. Clearly, the bundle involution with respect to p and O_1 is an inner involution of both S_1 and S_2'. We leave constructing two spherical circle planes that share an outer involution as an exercise.

Examples of outer involutions in nonovoidal spherical circle planes include those involutions of the so-called semi-classical Möbius planes $\mathcal{M}_{id,h}$ (see [4, Chapter 3.3.3]) that correspond to the map $(x, y) \mapsto (-x, y)$. Furthermore, if $h \neq id$, then these outer involutions are not automorphisms of $\mathcal{M}_{id,h}$. These planes also admit each reflection $(x, y) \mapsto (x, 2t - y)$ at the horizontal line $y = t$ as an outer involution. Another class of nonovoidal spherical circle planes that admit outer involutions are certain Ewald planes (see [4, Chapter 3.3.2]). Starting from an oval O in the Euclidean plane that has a line l as a symmetry axis one first dilates O from one point with all possible positive factors and then translates the resulting ovals in all possible directions. These spherical circle planes then admit reflections about each line parallel to l as outer involutions.

Basic properties of inner and outer involutions

Lemma 2.1. (Inner involutions) *Let i be an inner involution of a spherical circle plane with circle set C. Then the following hold:*

1. *The circles through a point p that are contained in* $\mathrm{FIX}(i)$ *are exactly the circles through p and $i(p)$.*
2. *Two distinct circles in* $\mathrm{FIX}(i)$ *have exactly two points in common. These two points get interchanged by i.*
3. *Let $c \in C \setminus \mathrm{FIX}(i)$. Then c is disjoint from its image $i(c)$ under the involution. (Note that $i(c)$ is not necessarily a circle in C.)*
4. *Let r, s, $i(r)$, and $i(s)$ be four distinct points on a circle c in* $\mathrm{FIX}(i)$. *Then r and $i(r)$ are contained in different connected components of $c \setminus \{s, i(s)\}$.*
5. *It is possible to reconstruct i from* $\mathrm{FIX}(i)$.

Proof. Remember that i is fixed-point-free and orientation-reversing. This means that, restricted to one of its fix-circles, it is an orientation-preserving involutory homeomorphism.

(1) By definition, a circle that contains p and is contained in $\mathrm{FIX}(i)$ also contains $i(p)$. On the other hand, let c be a circle through p and $i(p)$, and let q be a third point on this circle. Since i satisfies condition (B), the set $\mathrm{FIX}(i)$ contains a circle d connecting p and q. This circle also contains $i(p)$. Because there is exactly one circle in C containing p, $i(p)$, and q, the circles c and d coincide. Hence every circle through p and $i(p)$ is contained in $\mathrm{FIX}(i)$.

(2) As a consequence of the axiom of joining, two circles in C intersect in 0, 1, or 2 points. Let c and d be two distinct circles in $\mathrm{FIX}(i)$. If the two circles were to intersect in just one point, this point would be a fixed point of i. Since i is fixed-point-free this is not possible. Assume that c and d do not intersect. Then, since the connected components of $\mathbb{S}^2 \setminus c$ are interchanged, d and $i(d)$ are distinct, which is not the case. We conclude that c and d intersect in exactly two points. Also, since i does not have any fixed points, we conclude that these two points are interchanged by i.

(3) Let $c \in C \setminus \mathrm{FIX}(i)$. Assume that there is a point p that is contained in both c and $i(c)$. Then the same is true for $i(p)$, and by (1), the circle c is contained in $\mathrm{FIX}(i)$. This is a contradiction.

(4) follows immediately from the fact that i restricted to c is a fixed-point-free involution.

(5) Given a point p, consider any two circles in $\mathrm{FIX}(i)$ through p. Then, by Lemma 2.1.2., the point $i(p)$ is necessarily the second point of intersection of the two circles. This completely determines i. □

Lemma 2.2. (Outer involutions) *Let i be an outer involution of a spherical circle plane with circle set C. Then the following hold:*

1. *Let p be a point that is not fixed by i. Then the circles through p that are contained in* $\mathrm{FIX}(i)$ *are exactly the circles in the pencil of circles through p and $i(p)$.*
2. *Let $c \in C \setminus \mathrm{FIX}(i)$. Then $c \cap i(c) = c \cap \mathrm{fix}(i)$.*
3. *Let r, s, $i(r)$, and $i(s)$ be four distinct points on a circle c in* $\mathrm{FIX}(i)$. *Then r and $i(r)$ are contained in the same connected component of $c \setminus \{s, i(s)\}$.*
4. *The circles in* $\mathrm{FIX}(i)$ *that contain a point of* $\mathrm{fix}(i)$ *all touch at this point. If we are dealing with a flat Möbius plane, then these circles form a touching pencil of circles.*
5. *It is possible to reconstruct i from* $\mathrm{FIX}(i)$.

We skip the proof of this result since it is very similar to the proof of Lemma 2.1.

The topology of the fixgeometries of inner and outer involutions

To figure out what $\mathrm{FIX}(i)$ looks like topologically, we consider the fixgeometries associated with our special involutions. Depending on whether i is inner or outer, the point set of this fixgeometry is $\mathbb{S}^2$ or $\mathbb{S}^2 \setminus \mathrm{fix}(i)$ with pairs of points that are interchanged by the involution identified. Its line set consists of the elements of $\mathrm{FIX}(i)$ which have been identified accordingly.

A *flat projective plane* is a projective plane whose point set is homeomorphic to the real projective plane and whose lines are subsets of this point set homeomorphic to the circle. The classical example of a flat projective plane is the projective plane over the real numbers. An $\mathbb{R}^2$*-plane* is a linear space whose point set is homeomorphic to $\mathbb{R}^2$ and whose lines are subsets of this point set homeomorphic to $\mathbb{R}$ which separate the point set into two open components. The classical example of an $\mathbb{R}^2$-plane is the Euclidean plane. Now the following result is an easy corollary of property (B) shared by both inner and outer involutions.

Proposition 2.3. (Types of fixgeometries) *The fixgeometry of an inner involution of a spherical circle plane is a flat projective plane. The fixgeometry of an outer involution is an* $\mathbb{R}^2$*-plane.*

For example, the fixgeometry associated with an inner or outer point of the classical spherical circle plane is isomorphic to the classical flat projective plane or the restriction of this plane to the interior of the unit circle, respectively.

Now it is clear that $\mathrm{FIX}(i)$ is homeomorphic to the line set of the fixgeometry of i. Using well-known properties of flat projective planes and $\mathbb{R}^2$-planes (see, e.g., [4, Theorem 2.3.6]), we arrive at the following result:

Corollary 2.4. (Topology) *Let i be an inner or outer involution.*

1. *If i is an inner involution, then* $\mathrm{FIX}(i)$ *is homeomorphic to the real projective plane.*
2. *If i is an outer involution, then* $\mathrm{FIX}(i)$ *is homeomorphic to a Möbius strip (without boundary).*
3. *As long as the points r and s are not interchanged by i, the unique circle in* $\mathrm{FIX}(i)$ *that contains both points depends continuously on the position of these points.*

Combining two involutions or combining an actual point and an involution

Consider a point p on the unit sphere and a point q off the sphere in real projective three-space; see again Figure 1. The connecting line of p and q touches the sphere in exactly one point if the line touches the sphere at p. In this case q is necessarily an outer point of the sphere. Otherwise, the line intersects the sphere in exactly two points. Lemmas 2.1.1, 2.2.1 and 2.2.4 show how these facts generalize if we replace q by an inner or outer involution of a spherical circle plane. We summarize these results and supplement them by some topological interpretations that follow immediately from Corollary 2.4.

Lemma 2.5. (One actual and one virtual point) *Let p be a point on the sphere and let q be an inner or outer involution of a spherical circle plane.*

1. *If p is not fixed by q, then the circles through p that are fixed by q are all the circles through p and $q(p)$. Topologically this set of circles is a non-separating topological circle inside the surface formed by* $\mathrm{FIX}(q)$*. It is also a non-separating circle inside the surface formed by all the circles containing the point p (this surface is homeomorphic to a Möbius strip).*
2. *If p is fixed by q, then the circles through p that are fixed by q all touch at the point p. Topologically this set of circles is homeomorphic to a non-separating open interval inside the surface formed by* $\mathrm{FIX}(q)$*.*

Consider two points p and q off the unit sphere in real projective three-space; see again Figure 1. If their connecting line intersects the sphere in two points p' and q', then p', q' is the only pair of points on the sphere that gets interchanged by both the bundle involutions associated with p and q. This fact generalizes as follows:

Lemma 2.6. (Two virtual points) *Let p be an inner involution and let q be either an inner involution different from p or an outer involution. Or let p and q be outer involutions such that* $\mathrm{fix}(p) \cap \mathrm{fix}(q) = \emptyset$*. Then the following hold:*

1. *There is a unique pair of points p' and q' on the sphere that are interchanged by both p and q.*
2. *The pencil of circles through p' and q' is equal to* $\mathrm{FIX}(p) \cap \mathrm{FIX}(q)$.
3. *It is possible to reconstruct p and q from* $\mathrm{FIX}(p) \cup \mathrm{FIX}(q)$.
4. *Topologically* $\mathrm{FIX}(p) \cap \mathrm{FIX}(q)$ *is a simply closed non-separating curve in both the surfaces formed by* $\mathrm{FIX}(p)$ *and by* $\mathrm{FIX}(q)$.

Proof. (1) Note that pq, as a product of two orientation-reversing homeomorphisms of the sphere to itself, is a continuous orientation-preserving homeomorphism. As such it has at least one fixed point p'. Hence $pq(p') = p'$, and thus $q(p') = p(p')$. Since p and q do not have a common fixed point, the points p' and $q' = q(p') = p(p')$ are distinct. We conclude that p' and q' are interchanged by both involutions.

Assume that there is a second pair of points p'', q'' on the sphere that is interchanged by both involutions. Also, let us first assume that p', q', p'', q'' are not all contained in a circle. Let u be any point different from p', p'', q', q'' and make sure that u is fixed by neither p nor q. Then, by Lemmas 2.1.1 and 2.2.1, the circle containing p', q', u and the circle p'', q'', u are both fixed by p and q. The two circles intersect in u. Since u is not a fixed point of either involution, the circles have to intersect in a second point u', and, consequently, both p and q interchange u and u'. This implies that the two involutions are equal off $\mathrm{fix}(p) \cup \mathrm{fix}(q)$ and therefore, by continuity everywhere, which is not the case. If p', q', p'', q'' are all contained in a circle, then we can conclude as just now that both involutions operate in the same way off this circle. However, by continuity, it then follows that both involutions are also operating in the same way on this circle and are therefore identical. We conclude that p', q' is the only pair of points that gets interchanged by both involutions.

(2) Again, as a consequence of Lemmas 2.1.1 and 2.2.1, the pencil of circles through p' and q' is contained in $\mathrm{FIX}(p) \cap \mathrm{FIX}(q)$. If $\mathrm{FIX}(p) \cap \mathrm{FIX}(q)$ included circles other than those in the pencil, intersecting one of these other circles with a suitably chosen one in the pencil would give two further points that are interchanged by both involutions, which is impossible.

(3) It also follows that the only pencil of circles in $\mathrm{FIX}(p) \cup \mathrm{FIX}(q)$ that cuts $\mathrm{FIX}(p) \cup \mathrm{FIX}(q)$ into two connected halves is $\mathrm{FIX}(p) \cap \mathrm{FIX}(q)$. Therefore, $\mathrm{FIX}(p)$ and $\mathrm{FIX}(q)$ can be reconstructed from $\mathrm{FIX}(p) \cup \mathrm{FIX}(q)$, and, in turn, the involutions p and q can be reconstructed from $\mathrm{FIX}(p)$ and $\mathrm{FIX}(q)$, by Lemmas 2.1.5 and 2.2.5.

(4) This is a simple consequence of the definition of the fixgeometry of one of our involutions. □

Problems. Instead of intersecting the sphere in two points, the connecting line of two outer points of the sphere may only touch the sphere in one point or miss the sphere altogether. This is the case iff the corresponding horizons touch in a point or intersect in two points. Also, as we have already pointed out in our discussion of inner and outer involutions of non-classical ovoidal circle planes, it is possible that the horizons of two outer involutions coincide. Anyway, in the case that the two pointwise fixed circles of two outer involutions intersect, it seems difficult to determine how many pairs of points there are that are interchanged by both involutions. Consequently, we can also not say much about the structure of the set of circles that is fixed by both involutions.

3. Seven new separating sets

In this section we establish the existence of seven new types of separating sets associated with two actual or virtual points p and q of a spherical circle plane. These seven types correspond to all those cases shown in Figure 1 in which the connecting line of the two points intersects the sphere in two points. These are cases 1, 2, 3, 4b, 5a, 5b, and 6a. At the end of this section we will discuss the main obstacles in the way of also generalizing those cases in which the connecting line of the two points only touches or misses the sphere.

If C is the circle set of our spherical circle plane, let $C(p)$ denote the set of circles through p if p is a point of the sphere, or $C(p) := \mathrm{FIX}(p)$ if p is an inner or outer involution. Furthermore $C^{\cap}(p,q) := C(p) \cap C(q)$, and the separating set is $C^{\cup}(p,q) := C(p) \cup C(q)$. We will abbreviate $C^{\cap}(p,q)$ and $C^{\cup}(p,q)$ by $C^{\cap}$ and $C^{\cup}$, whenever this will not lead to confusion.

In all cases that we will be considering in the following, the two path-connected components of $C \setminus C^{\cup}$ will be denoted by $C^{+}(p,q)$ and $C^{-}(p,q)$, or, for short, C^{+} and C^{-}. We describe the components C^{+} and C^{-} in terms of $C^{\cup}$ in the box at the beginning of every subsection. Of course, it suffices to describe just one of the components, for example C^{+}, since $C^{-} = C \setminus (C^{+} \cup C^{\cup})$. Note that $C(p)$ is a surface which is homeomorphic to a Möbius strip if p is a point of the sphere or an outer involution, or the real projective plane if p is an inner involution; see Corollary 2.4. As we have shown in the previous section, in all of the cases under consideration $C^{\cap}(p,q)$ is a pencil of circles through two points. So, the picture to keep in mind is that of C being a three-dimensional space that is cut by the two-dimensional space $C^{\cup}$ into the two three-dimensional chunks C^{+} and C^{-}.

Important additional requirement: For those separating sets $C^{\cup}(p,q)$ that involve outer involutions we will also always require that the fixed-point sets of all involved outer involutions are circles of the circle planes that we are dealing with. To understand why we have to require this extra alignment note the following open problem:

Open Problem. Given two spherical circle planes C and D, and an outer involution i of C such that $\mathrm{FIX}(i)$ is contained in D, is i also an outer involution of D, that is, is $\mathrm{fix}(i)$ also a circle of D?

Procedure for establishing the existence of the new separating sets

In the following, we will work our way through the different types of separating sets/candidates in the order set out in Figure 1. As we proceed, the proofs will get more involved. However, since there are also lots of similarities in what exactly has to be checked for each separating set, we will be able to skip more and more details as we go along. Therefore, if you intend to understand the individual facts that need to be verified for one of the types of separating sets under discussion, it is important that you work your way through all of the discussion of the types that precede the one that you are interested in.

In order to check that a set $C^\cup$ is really a separating set, we have to show that it satisfies axioms S1, S2, and S3. In every single case we verify this by proceeding in the order S2, S3, S1.

Note that the descriptions of C^+ and C^- and our proofs will often make use of the involutions that entered into the original construction of $C^\cup$. This only makes sense because, as we have shown in Lemmas 2.1.5, 2.2.5 and 2.6.4, these involutions can be reconstructed from $C^\cup$.

Whenever we are dealing with an outer involution i, the two connected components of $\mathbb{S}^2 \setminus \mathrm{fix}(i)$ will be denoted by $H^+(i)$ and $H^-(i)$. We will specify in every single case which half carries which name.

The proofs will be accompanied by lots of diagrams. The most important thing to remember when trying to make sense of one of these diagrams is that two circles in a spherical circle plane that intersect in two points always intersect transversally at these points – we mentioned this fact before.

Type 1

Two points *p* and *q* on the sphere

C^+ The circles that separate p and q.

C^- The circles that miss both p and q and do not separate the two points.

To understand where C^+ comes from, consider Figure 3. It shows what is happening in the classical model. We start with the two points p and q on the sphere and their connecting line. Clearly, all planes that intersect the interval pq inside the sphere give rise to circles on the sphere that separate p and q, and every circle like this arises from one of these planes. Similarly, all planes that intersect the interval pq outside the sphere, and intersect the sphere itself nontrivially, give rise to circles on the sphere that do not separate p and q.

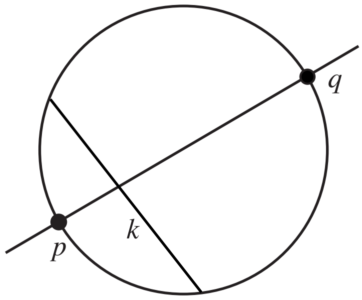

Figure 3. Two points on the sphere

Proof that we are dealing with a separating set.

Connecting circle: Here we start with three distinct points r, s, and t on the sphere and we are interested in determining which of the sets $C^\cup$, C^+ or C^- contains the connecting circle c of the three points. If the three points r, s, and t are contained in a circle in $C^\cup$, then this circle is the connecting circle we are after. Now assume that r, s and t are not all contained in a circle in $C^\cup$. In particular, this implies that all three points are different from p and q. Consider the two circles (in $C^\cup$) determined by r, s, and p, and r, s, and q. If these two circles coincide, that is, if r, s, p and q are concircular[1], then there are essentially two different ways in which the four points can be distributed along their connecting circle; see the first two diagrams in Figure 4. In the first diagram it is clear that any circle through r and s other than the distinguished one will be contained in C^+ (remember that two circles that intersect in two points intersect transversally in both points). In the second diagram it is clear that any circle through r and s other than the distinguished one will be contained in C^-.

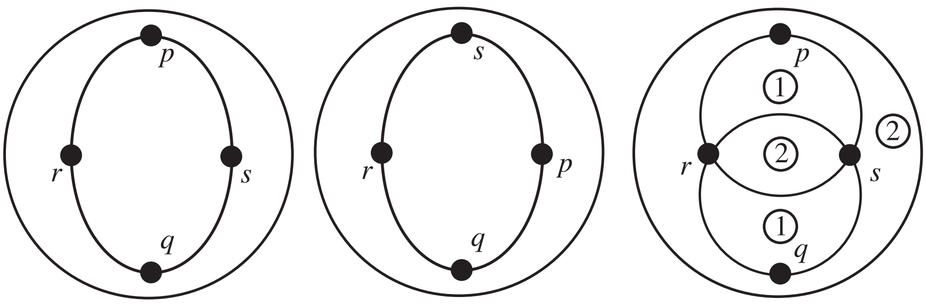

Figure 4. The different relative positions of the points p, q, r and s

If the two circles determined by r, s, and p, and r, s, and q are distinct, then we are in the situation depicted in the last diagram in Figure 4. The two distinguished

[1]At this point we could argue that if r, s, p and q are concircular, then r, t, p and q are not, and we therefore do not have to worry about the case of four concircular points. However, later on in the proof, when it comes to checking that axiom S1 is satisfied, we will again stumble across this case. There we won't have the option of sidestepping this case. Therefore it makes sense to already deal with it at this point.

circles dissect the sphere into regions 1 and 2 as indicated. Both regions consist of two connected components. The point t is contained either in region 1 or region 2. If it is contained in region 1, then, apart from r and s, the connecting circle we are after is also contained in region 1. This implies that this connecting circle is contained in C^-. Similarly, if t is contained in region 2, we conclude that the connecting circle we are after is contained in C^+.

Touching circle: Here we start with a circle c, a point r on c and a point s different from r and we are interested in the circle through s that touches c in r. If either one of the points r or s coincides with p or q, then the touching circle is contained in $C^\cup$. Assume that this is not the case. Then we again consider the three possible configurations of the connecting circles of r, s and p and r, s and q shown in Figure 4. There are a number of different possible scenarios. If the circle c touches one of these two distinguished circles, then this distinguished circle is the touching circle that we are looking for. Otherwise, the circle c intersects the distinguished circle(s) transversally at r. If this is the case in the first diagram or the second diagram in Figure 4, then the touching circle is clearly contained in C^+ or C^-, respectively. If we are dealing with a situation that corresponds to the last diagram, then, close to r, the circle c is either contained in region 1 or it is contained in region 2. If the region under discussion is region 1, then except for the points r and s, the touching circle will be completely contained in region 1, and hence is an element of C^-. Similarly, if, close to r, the circle c is contained in region 2, then the touching circle we are looking for is also contained in this region, and hence is an element of C^+.

Connectivity: The circles through p form a space homeomorphic to a Möbius strip and the pencil of circles through p and q forms a non-separating topological circle in this Möbius strip. This means that any two circles through p that miss q are connected by a path of circles in the Möbius strip that does not contain any circle in the pencil. Let us call such a path a *missing path*. Consider two circles c_0 and c_1 in C^+ and choose points r_0, s_0, t_0 on c_0 and points r_1, s_1 on c_1 such that the connecting circle d_0 of r_0, s_0, p and the connecting circle d_1 of r_1, s_1, p do not pass through q. Choose a missing path of circles $d : [0,1] \to C$, with $d(0) = d_0$ and $d(1) = d_1$. Choose paths $r : [0,1] \to \mathbb{S}^2$ and $s : [0,1] \to \mathbb{S}^2$ such that $r(x), s(x) \in d(x)$ and $r(x) \neq s(x)$, for all $x \in [0,1]$. This means that, as we vary x continuously from 0 to 1, the connecting circle $d(x)$ of the points $r(x), s(x)$ and p and the connecting circle of $r(x), s(x)$ and q will always be distinct, and we can label regions 1 and 2 as in the right diagram in Figure 4 consistent with the movement of the connecting circles. Now, choose a path $t : [0,1] \to \mathbb{S}^2$ with $t(0) = t_0$ and $t(1) \in c_1$ such that $t(x)$ is contained in region 1 for all $x \in [0,1]$. Then the moving connecting circle of $r(x), s(x)$ and $t(x)$ is a path completely contained in C^+ that connects the two circles c_0 and c_1 that we started with. This shows that C^+ is path-connected. The same argument shows that C^- is path-connected.

For the present type of $C^\cup$ it is obvious that any path in C that connects a circle in C^+ with a circle in C^- has to contain a circle that contains p or q. Hence

$C^\cup$ separates C^+ and C^-. However, we also need a formal argument that can be easily adapted to the other types of separating sets that we will be dealing with in the following. Here is one such argument: Consider a path of circles $c : [0,1] \to C$ that starts in a circle $c(0)$ in C^+ and does not contain any circle in $C^\cup$. We need to show that $c(1)$ is also contained in C^+. Choose paths $r, s, t : [0,1] \to \mathbb{S}^2$ such that the three points $r(x), s(x)$ and $t(x)$ are distinct for all $x \in [0,1]$ and contained in the circle $c(x)$. Let us also make sure that the connecting circle of $r(0), s(0)$ and p does not contain q. Let $c_p(x)$ and $c_q(x)$ be the connecting circles of $r(x), s(x)$ and p and $r(x), s(x)$ and q, respectively. Since $c_p(0)$ does not contain q, we are in the situation described by the third diagram in Figure 4, and since $c(0)$ is in C^+, the point $t(0)$ will be contained in region 2. Let us follow our path of circles. If $c_p(x)$ never contains q, then, since our path misses $C^\cup$, $t(x)$ will always remain in region 2 and, therefore, the whole path is contained in C^+. Assume that x_1 is the first time that $c_p(x)$ contains q, then, at this point we are necessarily in the situation depicted in the first diagram in Figure 4, $t(x_1)$ is off $c_p(x_1)$, and, therefore, $c(x_1)$ is still in C^+ (otherwise $t(x)$ would have been squeezed onto the distinguished circle in the second diagram as the third diagram collapses and $c(x_1)$ would end up in $C^\cup$, which is not possible). If $c_p(x)$ returns to not passing through q at a time x_2, then it necessarily does so as in the third diagram in Figure 4, with $t(x_2)$ being again contained in region 2, and so on. We conclude that the whole path is contained in C^+. Obviously, this also implies that any path of circles that starts in C^- and misses $C^\cup$ will also be completely contained in C^- and that, therefore, $C^\cup$ separates C^+ and C^-. □

Type 2

One point p on the sphere and one inner involution q
C^+ All circles k such that p and $q(k)$ are in different components of $\mathbb{S}^2 \setminus k$.

To understand where the definition of C^+ comes from, consider the left diagram in Figure 5. It shows what is happening in the classical model. To start with, we have the point p on the sphere, the point q inside the sphere, and their connecting line. Now we want to find a characterization in terms of the point p and in terms of the bundle involution associated with q of all those circles that correspond to planes that intersect the interval pq inside the sphere. Looking at one such circle k, we see that its image $q(k)$ under this bundle involution is separated (on the sphere) from p by k. Note that our definition of C^+ really makes sense because Lemma 2.1.3 guarantees that for any inner involution q, any circle k that is not contained in FIX(q) is disjoint from $q(k)$.

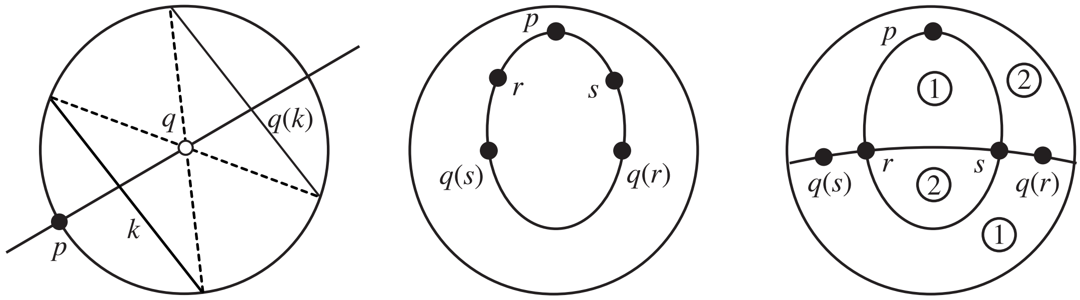

Figure 5. One point on the sphere and one inner involution

Proof that we are dealing with a separating set.

Connecting circle: If the three points r, s, and t are contained in a circle in $C^{\cup}$, then this circle is the connecting circle we are after. In particular, this is the case if any one of these three points coincides with p or if any two of the three points are interchanged by q (see Lemma 2.1.1).

Assume that none of this is the case. Then r and s are contained in exactly one of the circles fixed by q (this is property (B) in action). If p is contained in this fixed circle, then the positions of $q(r)$ and $q(s)$ relative to r and s alone determine whether a circle through r and s other than the distinguished one is contained in C^+ or C^-. For example, in the case shown in the middle diagram of Figure 4 any circle through r and s different from the distinguished one clearly separates its image (through $q(r)$ and $q(s)$) from p and is therefore contained in C^+. (Note that, by Lemma 2.1.4, the points r and $q(r)$ are separated by s and $q(s)$ on the distinguished circle and that, by Lemma 2.1.3, any circle through r and s different from the distinguished one is disjoint from its image under q.)

Now assume that p is not contained in the fixed circle through r and s. Then the right diagram in Figure 5 shows the circle through p, r, and s, the distinguished fixed circle through r and s, and the points r, s, $q(r)$, and $q(s)$. This diagram looks very similar to the third diagram in Figure 4 that we considered for our first separating set. As in that figure, we label regions 1 and 2. Furthermore, since $q(c)$ passes through $q(r)$ and $q(s)$, it is easy to verify that t being contained in region 1 or 2 implies that the connecting circle c of the three points r, s, and t is contained in C^- or C^+, respectively.

Touching circle: If either one of the two points r or s coincides with p, then the touching circle is contained in $C^{\cup}$. Assume the two points are interchanged by q. Then, since by Lemma 2.1.1 every circle through r and s $(=q(r))$ is contained in FIX(q), the touching circle will also be contained in $C^{\cup}$.

Assume that none of this is the case. Consider the connecting circles of r, s and p and the unique element of FIX(q) through r and s. If both circles coincide, that is, if p is contained in the fix-circle, consider again the middle diagram of Figure 5. If the circle c touches the distinguished circles at r, then the touching circle we are looking for is the distinguished circle and is therefore contained in $C^{\cup}$. Otherwise, the positions of p, r, s, $q(r)$ and $q(s)$ on the distinguished circle alone determine whether another circle through r and s is contained in C^+ or C^-.

If the connecting circle of r, s and p and the unique element of FIX(q) through r and s do not coincide, then the right diagram of Figure 5 is what we have to consider again and we can finish our argument as for our type 1 separating set.

Connectivity: Using Lemma 2.5.1, we can adapt the respective arguments for type 1 separating sets to show that both C^+ and C^- are path-connected. □

Type 3

Two inner involutions p and q
C^+ All circles k such that $p(k)$ and $q(k)$ are in different components of $\mathbb{S}^2 \setminus k$.

As in the case of the previous two separating sets, the classical model motivates the definition of C^+; see the left diagram in Figure 6.

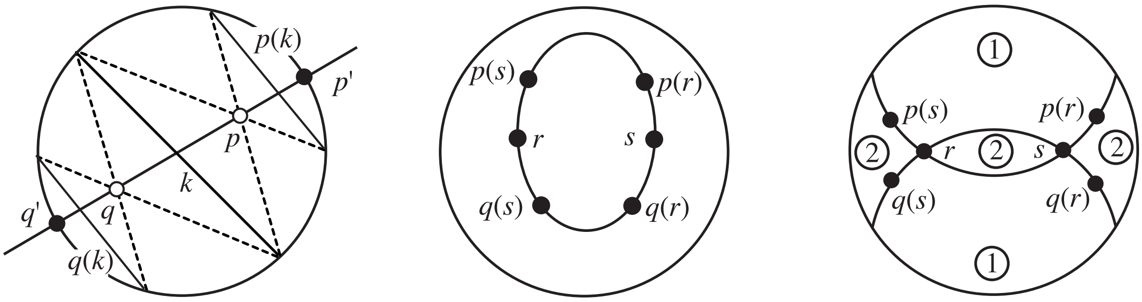

Figure 6. Two inner involutions

Sketch of a proof that we are dealing with a separating set.

Connecting circle: If the three points r, s, and t are contained in a circle in $C^{\cup}$, then this circle is the connecting circle we are after. In particular, this is the case if any two of these three points are interchanged by p or q. Assume that none of this is the case. Then r and s are contained in exactly one of the circles fixed by p and in exactly one of the circles fixed by q. If these two circles coincide, then we can argue as in the case of our type 2 separating sets that the positions of r, s, $p(r)$, $p(s)$, $q(r)$, and $q(s)$ on the distinguished circle alone determine whether another circle through r and s is contained in C^+ or C^-. For example, if these points are situated as in the middle diagram of Figure 6, then the connecting circle c necessarily separates its images under p and q and is therefore contained in C^+.

Similarly, if the two fixed circles are distinct, we can argue as in the corresponding case of our type 2 separating set, using the right diagram of Figure 6.

Touching circle & Connectivity: Again it is a straightforward exercise to adapt the arguments that we used in the case of our type 2 separating set to deal with this situation. □

Expressing type 3 in terms of type 2

Use the left diagram in Figure 6 to convince yourself that in the classical setting the line through the two inner points p and q intersects the sphere in two points p' and q'. In terms of the bundle involutions associated with p and q these two points on the sphere form the only pair of points that are interchanged by both involutions. Also, it is easy to see that

$$C^+(p,q) = C^+(p',q) \cap C^+(q',p)$$

and that

$$C^-(p,q) = C^-(p',q) \cup C^-(q',p).$$

Furthermore, we always have

$$C^\cup(p,q) = C \setminus (C^+(p,q) \cup C^-(p,q)).$$

This means that in the classical setting the separating set $C^\cup(p,q)$ of type 3 and its associated sets $C^+(p,q)$ and $C^-(p,q)$ can be expressed in terms of two separating sets of type 2 and their associated sets. This relationship generalizes to our general setting in which both p and q are inner involutions. Note that Lemma 2.6.1 guarantees the existence of a unique pair of points on the sphere that are interchanged by both p and q. To figure out which of these points should be called p' and which q', consider a circle k in C^+. Then, as in the left diagram of Figure 6,

(O) p' is the point that is separated from k by $p(k)$ and q' is the point that is separated from k by $q(k)$.

Proposition 3.1. (type 3 via type 2) *Let p and q be distinct inner involutions of a spherical circle plane, and let p', q' be the unique pair of points on the sphere interchanged by both involutions and labeled as specified under* (O)*, above. Then*

$$C^+(p,q) = C^+(p',q) \cap C^+(q',p) \text{ and } C^-(p,q) = C^-(p',q) \cup C^-(q',p).$$

Proof. We first show that every circle in $C^+(p,q)$ separates p' and q'. Consider a circle that contains p'. Then the images of this circle under p and q intersect in q'. We conclude that no circle in $C^+(p,q)$ contains p' or q'. Let us consider a circle c in the spherical circle plane that does not contain p' or q' and also does not separate the two points. Then there is a circle d through p' and q' that does not intersect c. Since d is fixed by both p and q, both $p(c)$ and $q(c)$ are contained in the connected component of $\mathbb{S}^2 \setminus d$ that c is not contained in. Hence c does not separate $p(c)$ and $q(c)$ and is therefore not contained in $C^+(p,q)$. We conclude that every circle in $C^+(p,q)$ separates p' and q'.

We now need to show that the labeling of p' and q' as specified under (O) above is independent of the choice of the circle k in $C^+(p,q)$. Let l be a second circle in $C^+(p,q)$. Since $C^+(p,q)$ is path-connected, there is a path of circles in $C^+(p,q)$ connecting k and l. As we move along this path, the relative positions of the two points p', q', of the moving circle, and its two images cannot change.

Hence the labeling of p' and q' is independent of the circle k. Furthermore, if k is any circle in $C^+(p,q)$, then $p(k)$ separates p' from k and $q(k)$ separates q' from k and on the sphere the two points and three curves are therefore separated out in the following order: $p', p(k), k, q(k), q'$.

Now it is clear from the definitions of the C^+ sets for type 2 and type 3 separating sets that $C^+(p,q) = C^+(p',q) \cap C^+(q',p)$.

Also, if k is a circle in $\mathrm{FIX}(p) \setminus \mathrm{FIX}(q) = \mathrm{FIX}(p) \setminus C^\cap(p,q)$, then there is a path that ends in k but is otherwise completely contained in $C^+(p,q)$. By continuity, it then follows easily that $q(k)$ is disjoint from k (touching is impossible because this would give a fixed point of q), and that the two points p', q' and the two curves are separated out on the sphere in the following order: $p', k = p(k), q(k), q'$.

To finish the proof we need to show that $C^-(p,q) = C^-(p',q) \cup C^-(q',p)$. Let us first spell out characterizations for the C^- sets associated with type 2 and type 3 separating sets. First, $C^-(p,q)$ is the set of all circles k such that both $p(k)$ and $q(k)$ are contained in one connected component of $\mathbb{S}^2 \setminus k$, and $C^-(p',q)$ is the set of all circles such that p' and $q(k)$ are contained in the same connected component of $\mathbb{S}^2 \setminus k$.

Since $C^\cup(p',q)$ contains $C^\cap(p,q)$, a circle k in $C^-(p',q)$ is not contained in $C^\cap(p,q)$. If k was contained in $\mathrm{FIX}(p)$, then $p', q(k), k$ would be disjoint and separated out on the sphere in this order. Since this order does not mesh in with that derived above for a circle in $\mathrm{FIX}(p) \setminus C^\cap(p,q)$, we conclude that k is not contained in $C^\cup(p,q)$. Also, since the order $p', p(k), k, q(k), q'$ is definitely not present for this circle it is not contained in $C^+(p,q)$. Hence we can be sure that it is contained in $C^-(p,q)$. Thus $C^-(p',q) \cup C^-(q',p) \subset C^-(p,q)$. Now let c be a circle in $C^-(p,q)$. If c contains q', then $q(c)$ contains p' and therefore p' and $q(c)$ are in the same connected component of $\mathbb{S}^2 \setminus c$. Hence, c is contained in $C^-(p',q)$. We conclude that if c contains either p' or q', then it is contained in $C^-(p',q) \cup C^-(q',p)$. Assume that c contains neither p' nor q'. Since c is contained in $C^-(p,q)$, both $q(c)$ and $p(c)$ are contained in the same connected component of $\mathbb{S}^2 \setminus c$. This component also contains either p' or q'. If it contains p', then it is contained in $C^-(p',q)$, otherwise in $C^-(q',p)$. We conclude that c is contained in $C^-(p',q) \cup C^-(q',p)$. Hence $C^-(p,q) \subset C^-(p',q) \cup C^-(q',p)$. □

Type 4b

One point p on the sphere and one outer involution q, with $p \notin \mathrm{fix}(q)$

Let $H^+(q)$ be the connected component of $\mathbb{S}^2 \setminus \mathrm{fix}(q)$ that contains p. The set C^+ generalizes the set of circles in the classical model corresponding to the planes that intersect the open interval pq in Figure 7. Hence, C^+ consists of all circles not in $C^\cup$ belonging to one of the following categories:

- The circles k completely contained in $\overline{H^+(q)}$ such that the connected component of $\mathbb{S}^2 \setminus k$ completely contained in $H^+(q)$ does not contain the point p; see the left diagram in Figure 7.
- All circles k intersecting $\mathrm{fix}(q)$ in two points such that restricted to $H^+(q)$, $q(k)$ and p are separated by k; see the right diagram in Figure 7.

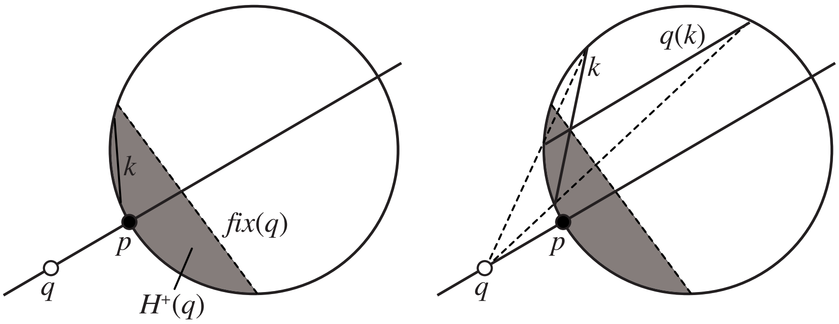

Figure 7. One outer involution q and one point on the sphere p not fixed by the involution

Proof that we are dealing with a separating set.

Connecting circle: If the three points r, s, and t are contained in a circle in $C^\cup$, then this circle is the connecting circle we are looking for. In particular, this is the case if any one of these three points coincides with p or if any two are interchanged by q. Also, if all three points are fixed by q, then their connecting circle is $\mathrm{fix}(q)$, which is contained in C^-.

Assume that none of this is the case and that, w.l.o.g., r is not fixed by q. Then r and s are contained in exactly one circle through p and in exactly one circle fixed by q. If these two circles coincide, we have to consider a number of different cases. Since r is not fixed by q, the two points r and $q(r)$ are contained in different connected components of $\mathbb{S}^2 \setminus \mathrm{fix}(q)$. Figure 8 shows the different possible positions of the points $r, q(r)$ and p with respect to the distinguished fixed circle through r and s and the circle $\mathrm{fix}(q)$. The gray squares indicated the essentially different possible positions of the point s. This means that the every one of the three diagrams corresponds to as many cases as there are gray squares.

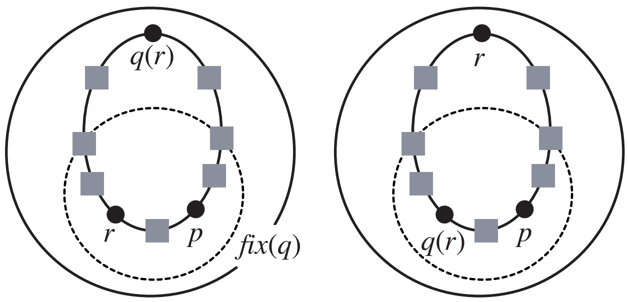

Figure 8. The fixed circle through r and s and the circle through r, s and p coincide

For two of the possible positions of s the two rows of diagrams in Figure 9 illustrate the essentially different ways in which a circle through r and s and its image under q (drawn thick) can be situated with respect to fix(q) and the distinguished circle. In the second row we skipped the limiting cases in between the first in second diagram in which the thick circle through r and s touches fix(q). Similarly, we skipped the limiting case between the second and third diagram.

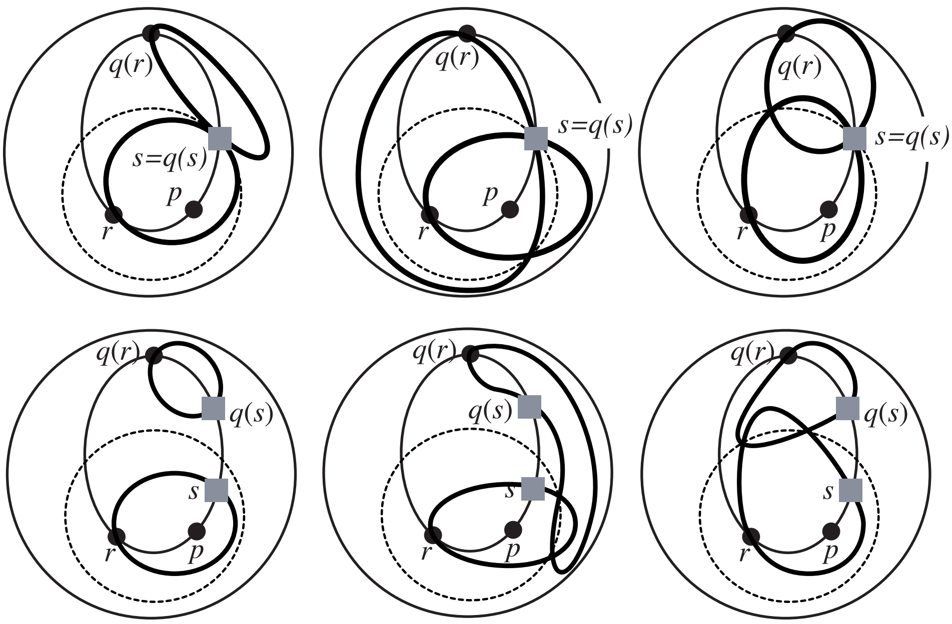

Figure 9. Possible locations of the circle through r and s and its image under q (both drawn thick) for two different positions of r, s, $q(r)$ and $q(s)$ on the distinguished circle

In all diagrams it is easy to see that any circle through r and s different from the distinguished circle is contained in C^-. This means that in the two cases under consideration we can decide by just looking at the positions of r, s, $q(r)$, and $q(s)$ on the distinguished circle alone whether another circle through r and s is contained in C^+ or C^-. This turns out to be true for all cases corresponding to

Figure 8. To check this is a (tedious) routine exercise. As usual, it turns out that all cases that need to be considered are exactly those that pop up in the classical model and there are no surprises.

Now assume that the circle through r, s and p and the fixed circle through r and s are different. Again, corresponding to the different relative positions of the distinguished circles, fix(q), p, r, s, $q(r)$ and $q(s)$ there are lots of different cases that need to be considered. As above this is a routine exercise and there are no surprises. We just give the details for one of the possible cases; see the left diagram in Figure 10.

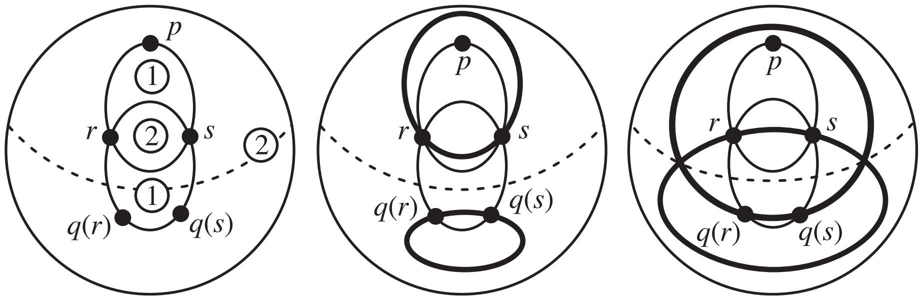

Figure 10.

If t is contained in region 1 or region 2, then no matter whether the connecting circle intersects fix(q) or not, it will be in contained in C^+ or C^-, respectively. For example, the middle diagram of Figure 10 shows what things will look like if the connecting circle is contained in region 2 and does not intersect fix(q). Similarly, the right diagram shows what happens if t is contained in region 2 and intersects fix(q) in two points.

Touching and Connectivity are again just variations of what we did for the first type of separating set that we considered. □

Type 5a

One inner involution p and one outer involution q, with fix$(q) \in$ FIX(p)

Let $H^+(q)$ be any one of the two connected components of $\mathbb{S}^2 \setminus \text{fix}(q)$. C^+ consists of all circles k

- that are contained in $\overline{H^+(q)}$, but are not equal to fix(q); or
- that intersect fix(q) in two points and, when restricted to $H^+(q)$, separate $q(k)$ from $p(k)$; see Figure 11 for different instances of k in the classical setting.

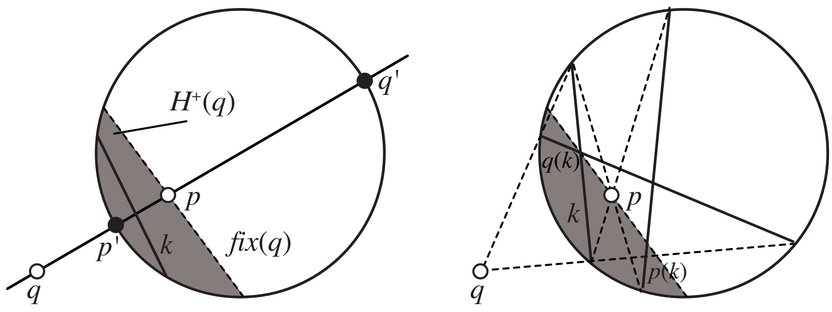

Figure 11. One inner involution p and one outer involution q such that the circle that is pointwise fixed by q is also fixed by p

The proofs that this and the remaining two types of separating sets are really separating sets are fairly straightforward variations (involving lots of cases) of the proofs so far and will be omitted here.

Also, in some sense these new separating sets are not really that new because it turns out (again lots of cases) that they can be reduced, just like the type 3 separating sets (two inner involutions), to type 2 and type 4b separating sets. For example, for the type of separating set at hand, use the left diagram in Figure 11 to convince yourself that in the classical setting the line through the inner and outer points p and q intersects the sphere in two points $p' \in H^+(q)$ and $q' \notin H^+(q)$. In terms of the bundle involutions associated with p and q these two points form the only pair of points that are interchanged by both involutions. Also, it is easy to see that

$$C^+(p, q) = C^-(q', q) \cap C^-(q', p)$$

and that

$$C^-(p, q) = C^-(p', p) \cap C^-(p', q).$$

Furthermore, we always have

$$C^{\cup}(p, q) = C \setminus (C^+(p, q) \cup C^-(p, q)).$$

This relationship generalizes to our general setting. Note that Lemma 2.6.1 guarantees the existence of a unique pair of points on the sphere that are interchanged by both p and q.

Type 5b

One inner involution p and one outer involution q, fix$(q) \notin$ FIX(p)

It is very complicated to describe $C^{\cup}$ from scratch. So, we again restrict ourselves to skipping straight to the end of our arguments and describing this set in terms of the unique pair of points p', q' on the sphere that are interchanged by p and q. It turns out that just like in the classical model only one of the connected components of $\mathbb{S}^2 \setminus \mathrm{fix}(q)$ contains some circles in FIX(p). Let us call this component $H^+(q)$.

If p', q' is the unique pair of points on the sphere that are interchanged by both p and q, label with p' the one of the two points that is contained in $H^+(q)$ and q' the one that is not; see the left diagram in Figure 12 for the classical setup that motivates this definition. Now, it is again possible to prove that $C^+(p,q) = C^-(q',q) \cap C^-(q',p)$ and $C^-(p,q) = C^-(p',p) \cap C^-(p',q)$.

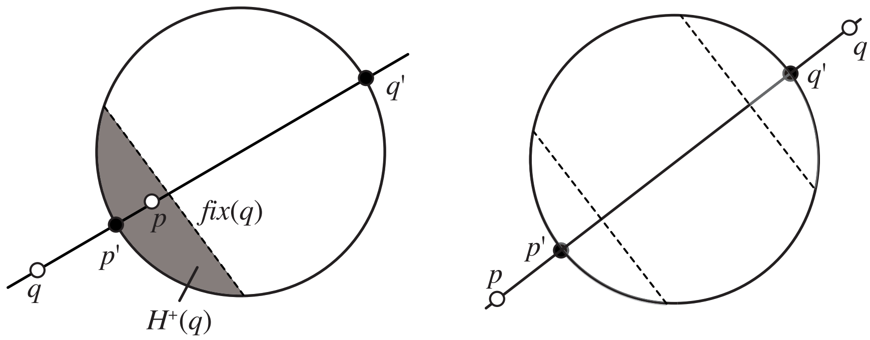

Figure 12. One inner involution and one outer involution in general position on the left, and, on the right, two outer involutions whose pointwise fixed circles are disjoint

Type 6a

Two outer involutions p and q, with $\mathrm{fix}(p) \cap \mathrm{fix}(q) = \emptyset$

We again skip straight to the end of our arguments and describe the set C^+ in terms of the unique pair of points p', q' on the sphere that are interchanged by p and q. To label p' and q', choose a circle k in between $\mathrm{fix}(p)$ and $\mathrm{fix}(q)$. Then p' is the point that is separated from k by $p(k)$ and q' is the point separated from k by $q(k)$; see the right diagram in Figure 12 for the classical setup that motivates this definition. Now, $C^+(p,q) = C^-(p',q) \cup C^-(q',p)$ and $C^-(p,q) = C^+(p',q) \cap C^+(q',p)$.

What about cases 4a, 6b, 6c, 6d, ...?

Why not try and also turn diagrams 4a, 6b, 6c, and 6d in Figure 1 into new separating sets? It should be possible to come up with further new separating sets that generalize these cases. However, apart from the messy splitting up into numerous subcases that outer involutions entail, there are added complications in all these extra cases. For example, diagrams 6b, 6c and 6d correspond to two outer involutions whose pointwise fixed circles intersect, and in these cases we do not know much about the structure of the corresponding set $C^\cap$ to define possible separating sets in the way we have done so far. It may be possible to overcome these problems by adding extra assumptions that guarantee that we stay close to the classical setup, but we have decided not to pursue this any further.

In this context it is also important to point out again that we are interested in separating sets mainly because a separating set contained in two spherical circle planes allows us to combine these two circle planes into a new spherical circle plane. Sadly, unlike for most of the other types of geometries on s urfaces, we have not been able to produce an example of any of the new types of separating sets that is simultaneously contained in two different spherical circle planes. The closest we got to this goal are the examples of pairs of ovoidal spherical circle planes constructed in the previous section that share an inner or outer involution. Also, it is worth pointing out that the derived incidence structure at a point p of an ovoidal spherical circle plane (basically the set $C(p)$) is isomorphic to part of the Euclidean plane. In particular, the derived incidence structure of a ovoidal flat Möbius plane is isomorphic to the Euclidean plane itself. This means that, using a suitable homeomorphism of the sphere to itself, we can always arrange isomorphic copies of any two ovoidal flat Möbius planes such that for one of the points of the sphere the sets $C(p)$ of both planes coincide. So, the closest we have been able to come to our goal of finding two spherical circle planes sharing one of our new separating sets is finding two spherical circle planes that share "half" of such a separating set.

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Received August 13, 2006