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Squaring the Circle via Affine Congruence by Dissection with Smooth Pieces

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Abstract. For every $k \geq 2$, we construct a dissection of a square into finitely many topological discs, each having a piecewise k times differentiable boundary, such that images of these pieces under appropriate maps of the group $\mathbf{Aff}_+$ of orientation preserving affine transformations of the plane form a dissection of a circular disc. The dissections consist of six pieces in the case $k = 2$ and of 14 pieces for every $k \geq 3$.

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1. Introduction and results

Motivated by Tarski's set theoretical circle squaring problem (see [6]), Dubins, Hirsch, and Karush have shown that a circular disc cannot be dissected into finitely many topological discs such that images of these pieces under suitable Euclidean motions form a dissection of a square (see [1]). In contrast with that, positive results are possible if the group of Euclidean motions is replaced by appropriate other groups of affine transformations. In [2] one can find first observations of that kind including a circle squaring result based on the group of homotheties. We recall the corresponding definitions.

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A *topological disc* $D \subseteq \mathbb{R}^2$ is the image of the closed unit disc of the Euclidean plane under a homeomorphism of the plane onto itself. We denote the set of all topological discs by $\mathcal{D}^0$. Let $\mathcal{D}^r$ be the family of all $D \in \mathcal{D}^0$ whose boundary is rectifiable in the sense that its one-dimensional Hausdorff measure is finite. The class $\mathcal{D}^c$ contains all $D \in \mathcal{D}^0$ whose boundary splits into finitely many subarcs each being a subset of the boundary of some convex topological disc. Moreover, $\mathcal{D}^k$, $k \in \{1, 2, \ldots\}$, is to denote the class of all $D \in \mathcal{D}^0$ whose boundaries are finite unions of k times continuously differentiable closed subarcs. Similarly, $\mathcal{D}^\infty$ includes all discs whose boundaries split into a finite number of infinitely differentiable arcs. (In the end-points of an arc one has to consider one-sided differentiability.) Note that $\mathcal{D}^\infty \neq \bigcap_{k\in\mathbb{N}} \mathcal{D}^k$. Indeed, the convex hull of the graph of $f : [-1, 0] \to \mathbb{R}$, $f(x) = \sum_{i=1}^{\infty} \frac{1}{i!}(\max\{x + 2^{-i}, 0\})^i$, belongs to $\bigcap_{k\in\mathbb{N}} \mathcal{D}^k \setminus \mathcal{D}^\infty$, since the restrictions $f|_{[-1,-2^{-1}]}, f|_{[-2^{-1},-2^{-2}]}, \ldots, f|_{[-2^{-(k-1)},-2^{-k}]}$, and $f|_{[-2^{-k},0]}$ are k times continuously differentiable, but f is not k times differentiable at the points $-2^{-1}, -2^{-2}, \ldots, -2^{-k}$.

A disc $D \in \mathcal{D}^0$ is said to be dissected into the pieces $D_1, \ldots, D_n \in \mathcal{D}^0$ if $D = D_1 \cup \cdots \cup D_n$ and $\operatorname{int}(D_i) \cap \operatorname{int}(D_j) = \emptyset$ for $1 \leq i < j \leq n$.

Given a group $\mathbf{G}$ of affine transformations of $\mathbb{R}^2$, two discs $D, E \in \mathcal{D}^0$ are called *congruent by dissection with respect to* $\mathbf{G}$ if there exist an integer $n \geq 1$ and dissections of D and E into subdiscs $D_1, \ldots, D_n \in \mathcal{D}^0$ and $E_1, \ldots, E_n \in \mathcal{D}^0$, respectively, such that D_i and E_i are congruent with respect to $\mathbf{G}$, $1 \leq i \leq n$ (that is; $E_i = \gamma_i(D_i)$ with some $\gamma_i \in \mathbf{G}$).

The paper [3] is devoted to the question if (or under what conditions) two discs of a class $\mathcal{D} \in \{\mathcal{D}^0, \mathcal{D}^r, \mathcal{D}^1, \mathcal{D}^2\}$ admit a congruence by dissection with respect to one of the groups **Isom** (isometries), $\mathbf{Aff_1}$ (equiaffine maps), **Hot** (homotheties), **Sim** (similarities), $\mathbf{Aff}_+$ (affine transformations preserving the orientation), or **Aff** (general affine transformations) such that all pieces of dissection belong to $\mathcal{D}$.

The main results of [4, 5] comprise the following: Given a subgroup $\mathbf{G}$ of **Aff**, any two discs from $\mathcal{D}^0$ are congruent by dissection with respect to $\mathbf{G}$ if and only if $\mathbf{G}$ contains a contraction and every orbit $\mathbf{G}(\mathfrak{x})$, $\mathfrak{x} \in \mathbb{R}^2$, is dense in $\mathbb{R}^2$. In this case any two discs from $\mathcal{D}^0$ admit a congruence by dissection realized by only three pieces of dissection from $\mathcal{D}^0$. The number three is minimal in general, in particular for the circle squaring problem. The same are true if $\mathcal{D}^0$ is replaced by $\mathcal{D}^r$.

The situation is more difficult if one considers dissections into discs with piecewise continuously differentiable boundaries (see [3]): A congruence by dissection of a circular disc and a square with respect to **Sim** cannot be realized with pieces of dissection exclusively from $\mathcal{D}^1$. But for any two discs from $\mathcal{D}^1$ there exists a congruence by dissection with respect to $\mathbf{Aff}_+$ whose pieces of dissection are from $\mathcal{D}^1$. Although the last general statement becomes false if $\mathcal{D}^1$ is replaced by $\mathcal{D}^2$, there exists a solution of the circle squaring problem with respect to $\mathbf{Aff}_+$ based on pieces of dissection from $\mathcal{D}^2$, more specifically even from $\mathcal{D}^2 \cap \mathcal{D}^c$. This is shown in [2]. However, there one uses a large number of pieces (about 100). In the present paper we improve this result in two directions.

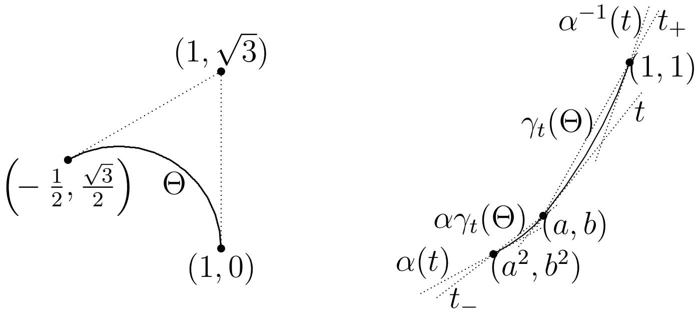

Figure 1. Proof of Lemma 1

Theorem 1. *There is a congruence by dissection of a square S and a circular disc C with respect to the group $\mathbf{Aff}_+$ realized by only six pieces of dissection from $\mathcal{D}^2 \cap \mathcal{D}^c$.*

Theorem 2. *For every $k \geq 3$, there is a congruence by dissection of a square S and a circular disc C with respect to the group $\mathbf{Aff}_+$ realized by 14 pieces of dissection from $\mathcal{D}^k$.*

It is shown in [4] that any congruence by dissection of a square and a circular disc with respect to **Aff** requires at least three pieces of dissection, even if the pieces are allowed to be arbitrary topological discs. We do not know if the number six given in Theorem 1 is optimal if the dissections are restricted to pieces from $\mathcal{D}^2 \cap \mathcal{D}^c$. In contrast with that, we expect that a refined analysis of the following proof of Theorem 2 would allow a reduction of the number 14 to 13.

The question for an affine congruence by dissection of a square and a circular disc with pieces exclusively from $\bigcap_{k\in\mathbb{N}} \mathcal{D}^k$ or even from $\mathcal{D}^\infty$ remains open. We conjecture that such realizations do not exist.

2. Proof of Theorem 1

Lemma 1. *Let $0 < b < a < 1$ and let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2, \alpha(x, y) = (ax, by)$. Then there exists an arc Γ between (a, b) and $(1, 1)$ with the following properties:*

(i) *Γ is the image of $\Theta = \left\{(\cos(\varphi), \sin(\varphi)) : 0 \leq \varphi \leq \frac{2\pi}{3}\right\}$ under a map from $\mathbf{Aff}_+$.*

(ii) *Γ is the graph of a strictly increasing convex function $f : [a, 1] \to [b, 1]$ whose slope in the point $(1, 1)$ is less than $\frac{a(1-b)}{b(1-a)}$.*

(iii) *$\Gamma \cup \alpha(\Gamma)$ is a twice continuously differentiable arc.*

Proof. Let t be a straight line through (a, b) such that $(1, 1)$ and (a^2, b^2) belong to the same open half plane generated by t (see Figure 1).

The lines t and $\alpha^{-1}(t)$ have an intersection point in the interior of the triangle $\triangle((a, b), (1, b), (1, 1))$. We find $\gamma_t \in \mathbf{Aff}_+$ such that $\gamma_t(1, 0) = (a, b)$, $\gamma_t\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (1, 1)$, and $\gamma_t(1, \sqrt{3})$ is the intersection point of t and $\alpha^{-1}(t)$. Then $\gamma_t(\Theta)$ is an arc between (a, b) and $(1, 1)$ that certainly satisfies (i) and the first part of (ii). The slope of $\gamma_t(\Theta)$ in $(1,1)$ agrees with that of the tangent $\alpha^{-1}(t)$. Since the slope of t is strictly less than that of the straight line t_+ through (a,b) and $(1,1)$, the slope of $\alpha^{-1}(t)$ is less then that of $\alpha^{-1}(t_+)$. Passing through $(1,1)$ and (a^{-1},b^{-1}), $\alpha^{-1}(t_+)$ has a slope of $\frac{b^{-1}-1}{a^{-1}-1} = \frac{a(1-b)}{b(1-a)}$. This gives the second part of (ii).

The arcs $\gamma_t(\Theta)$ and $\alpha\gamma_t(\Theta)$ have the end-point (a,b) in common. Since $\alpha^{-1}(t)$ is the tangent of $\gamma_t(\Theta)$ at $(1,1)$, $t = \alpha\alpha^{-1}(t)$ is the tangent of $\alpha\gamma_t(\Theta)$ at the point $(a,b) = \alpha(1,1)$. So $\gamma_t(\Theta)$ and $\alpha\gamma_t(\Theta)$ have the same tangent t in the common end-point; that is, the arc $\gamma_t(\Theta) \cup \alpha\gamma_t(\Theta)$ is continuously differentiable.

The arc $\gamma_t(\Theta)$ would satisfy condition (iii) if the curvatures $\kappa_0(t)$ of $\gamma_t(\Theta)$ and $\kappa_1(t)$ of $\alpha\gamma_t(\Theta)$ at the point (a,b) would agree. This is not the case in general. If t approaches the extremal position t_+ then $\kappa_0(t)$ tends to zero. If t is sufficiently close to t_+, there exists a neighbourhood U_t of (a,b) such that the elliptic disc E containing $\gamma_t(\Theta)$ in its boundary covers the part $U_t \cap \alpha(E)$ of the disc $\alpha(E)$ associated with $\alpha\gamma_t(\Theta)$. Hence $\kappa_0(t) \le \kappa_1(t)$ if t is close to t_+. Similarly, $\kappa_0(t) \ge \kappa_1(t)$ if t approaches the other extremal position t_-, which is the straight line through (a,b) and (a^2,b^2). By the Intermediate Value Theorem there is a position t_0 of the tangent t such that $\kappa_0(t_0) = \kappa_1(t_0)$. Hence the arc $\Gamma = \gamma_{t_0}(\Theta)$ satisfies (iii). $\square$

Corollary 1. *Let a, b, α, and Γ be as in Lemma 1 and suppose that $b < a^2$. Then $\Delta = \{(0,0)\} \cup \bigcup_{i=0}^{\infty} \alpha^i(\Gamma)$ is a twice continuously differentiable arc between $(0,0)$ and $(1,1)$ that is the graph of a strictly increasing convex function $g : [0,1] \to [0,1]$ and is contained in the triangle $\triangle\big((0,0),(1,1),\big(\frac{a-b}{a-ab},0\big)\big)$.*

Proof. The arcs $\alpha^i(\Gamma)$, $i = 0,1,2,\ldots$, are graphs of the increasing convex functions $g_i(x) = b^i f(a^{-i}x) : [a^{i+1},a^i] \to [b^{i+1},b^i]$ with $g_i(a^{i+1}) = b^{i+1}$ and $g_i(a^i) = b^i$. Since $\lim_{i\to\infty}[a^{i+1},a^i] \times [b^{i+1},b^i] = \{(0,0)\}$ in the Hausdorff distance, $\Delta = \{(0,0)\} \cup \bigcup_{i=0}^{\infty} \alpha^i(\Gamma)$ is the graph of a strictly increasing continuous function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$. By claim (iii) of the lemma, $\alpha^{i+1}(\Gamma) \cup \alpha^i(\Gamma)$ is a twice continuously differentiable arc for all $i \ge 0$. So Δ is twice continuously differentiable in all points apart from $(0,0)$. This proves in particular the convexity of g, because all functions g_i are convex.

By (i) and (ii), there is a bound $r > 0$ such that $|f'(x)| \le r$ and $|f''(x)| \le r$ for all $x \in [a,1]$. Hence $|g_i'(x)| = \frac{b^i}{a^i}|f'(a^{-i}x)| \le \big(\frac{b}{a}\big)^i r$ and $|g_i''(x)| = \frac{b^i}{a^{2i}}|f''(a^{-i}x)| \le \big(\frac{b}{a^2}\big)^i r$ for $i \ge 0$, $x \in [a^{i+1},a^i]$. The assumptions $b < a$ and $b < a^2$ yield $\lim_{i\to\infty}\big(\frac{b}{a}\big)^i = \lim_{i\to\infty}\big(\frac{b}{a^2}\big)^i = 0$ and give $\lim_{x\downarrow 0} g'(x) = \lim_{x\downarrow 0} g''(x) = 0$ and $g'(0) = g''(0) = 0$. Hence Δ is twice continuously differentiable in $(0,0)$, too.

Since Δ is the graph of an increasing differentiable convex function, it is contained in the triangle whose vertices are the end-points $(0,0)$ and $(1,1)$ of Δ and the intersection point of the tangents t_0 and t_1 of Δ in $(0,0)$ and $(1,1)$, respectively. We replace t_1 by the straight line t_1^* through $(1,1)$ of the slope $\frac{a(1-b)}{b(1-a)}$. By (ii), the slope of t_1^* exceeds that of t_1. Therefore the replacement of t_1 by t_1^* leads to a triangle covering the previous one. The third vertex of the enlarged triangle is $\big(\frac{a-b}{a-ab},0\big)$, because the slope of t_0 is $g'(0) = 0$. This yields $\Delta \subseteq \triangle\big((0,0),(1,1),\big(\frac{a-b}{a-ab},0\big)\big)$. $\square$

Lemma 2. *Let* $b = \frac{-1+\sqrt{5}}{2}$ *and let* $a, c \in \mathbb{R}$ *be such that the numbers*

$$p = \tfrac{2a+(1-\sqrt{5})a^2+(3-\sqrt{5})(c-1)}{2c} \quad \textit{and} \quad q = \tfrac{(6-2\sqrt{5})a^2+(1-\sqrt{5})a^3+(7-3\sqrt{5})(c-1)}{2a^2+(1-\sqrt{5})a^3c+(3-\sqrt{5})(c-1)}$$

are well-defined and coincide. Then there exists an affine map $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfying* $\beta(\mathfrak{a}_i) = \mathfrak{b}_i$*,* $i = 1, 2, 3, 4$*, where*

$$\begin{array}{llll} \mathfrak{a}_1 = (1-c, 1), & \mathfrak{a}_2 = (1,1), & \mathfrak{a}_3 = (a^2, b^2), & \mathfrak{a}_4 = (a, b), \\ \mathfrak{b}_1 = \left(\frac{a^2-b^2(1-c)}{1-b^2}, 0\right), & \mathfrak{b}_2 = (a^3c, 0), & \mathfrak{b}_3 = (a^2, b^2), & \mathfrak{b}_4 = (a^3, b^3). \end{array}$$

Proof. The vectors $\overrightarrow{\mathfrak{a}_1\mathfrak{a}_2} = (c, 0)$ and $\overrightarrow{\mathfrak{a}_1\mathfrak{a}_3} = (a^2 + c - 1, b^2 - 1)$ are linearly independent, for $c \neq 0$ and $b^2 - 1 \neq 0$. Hence there is an affine map $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\beta(\mathfrak{a}_i) = \mathfrak{b}_i$, $i = 1, 2, 3$. The relations $\overrightarrow{\mathfrak{a}_1\mathfrak{a}_4} = p\,\overrightarrow{\mathfrak{a}_1\mathfrak{a}_2} + b\,\overrightarrow{\mathfrak{a}_1\mathfrak{a}_3}$, $\overrightarrow{\mathfrak{b}_1\mathfrak{b}_4} = q\,\overrightarrow{\mathfrak{b}_1\mathfrak{b}_2} + b\,\overrightarrow{\mathfrak{b}_1\mathfrak{b}_3}$, and $p = q$ yield $\beta(\mathfrak{a}_4) = \mathfrak{b}_4$. (The numbers p and q as well as the following particular value of c were found by the aid of the computer algebra system Maple 9.01.) □

An example of numbers a, b, c satisfying all assumptions of the lemmas and the corollary are

$$a = \tfrac{4}{5} = 0.8, \quad b = \tfrac{-1+\sqrt{5}}{2} = 0.61803\ldots, \quad \text{and}$$
$$c = \tfrac{1}{6400}\left(4809 - 1281\sqrt{5} + \sqrt{17699286 - 5344658\sqrt{5}}\right) = 0.67846\ldots.$$

We fix these values for the further constructions and illustrations. (Other choices of a and c are possible.)

Since $0 < \frac{a-b}{a-ab} < c$, the triangle $\triangle\big((0,0), (1,1), \big(\frac{a-b}{a-ab}, 0\big)\big)$ is a subset of the parallelogram P with the vertices $(0,0), (c,0), (1,1)$, and $(1-c,1)$. Hence, by the corollary, the arc $\Delta = \{(0,0)\} \cup \bigcup_{i=0}^{\infty} \alpha^i(\Gamma)$ is contained in P (see Figure 2).

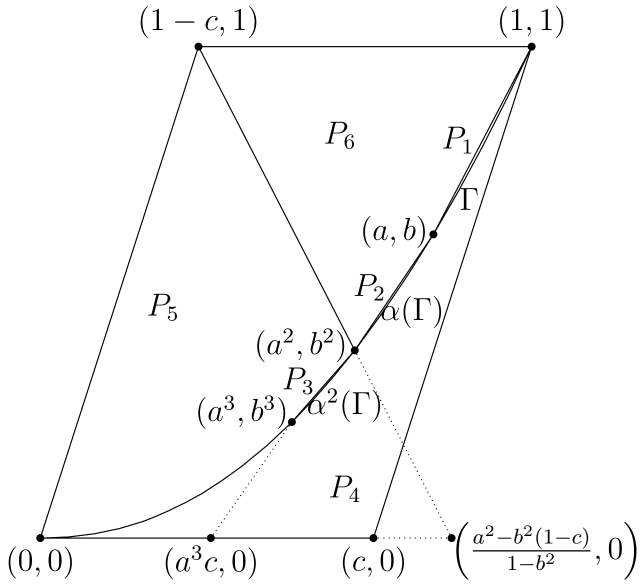

Figure 2. Dissections of the parallelogram P and the triangle T

We obtain a dissection

$$P = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$$

of P into the discs $P_i = \mathrm{conv}(\alpha^{i-1}(\Gamma))$, $i = 1, 2, 3$, the disc P_4 bounded by Δ and the polygonal arc $\overline{(0,0),(c,0),(1,1)}$, the disc P_5 bounded by $\{(0,0)\} \cup \bigcup_{i=3}^{\infty} \alpha^i(\Gamma)$ and the polygonal arc $\overline{(0,0),(1-c,1),(a^2,b^2),(a^3,b^3)}$, and the quadrangle P_6 with the vertices $(1-c,1),(1,1),(a,b)$, and (a^2,b^2).

The triangle $T = \triangle\big((0,0),(1-c,1),\big(\frac{a^2-b^2(1-c)}{1-b^2},0\big)\big)$ splits into P_5, into $\alpha^3(P_4)$, which is the disc bounded by $\{(0,0)\} \cup \bigcup_{i=3}^{\infty} \alpha^i(\Gamma) = \alpha^3(\Delta)$ and the polygonal arc $\overline{(0,0),(a^3c,0),(a^3,b^3)} = \alpha^3\big(\,\overline{(0,0),(c,0),(1,1)}\,\big)$, and into the quadrangle with the vertices $\big(\frac{a^2-b^2(1-c)}{1-b^2},0\big),(a^3c,0),(a^3,b^3)$, and (a^2,b^2) (see the dotted lines in Figure 2). By Lemma 2, there is an affine map β such that the last quadrangle can be written as $\beta(P_6)$. The particular choice of a and c yields $\beta \in \mathbf{Aff}_+$. Thus we have the dissection

$$T = \alpha^3(P_4) \cup P_5 \cup \beta(P_6) \quad \text{with} \quad \alpha, \beta \in \mathbf{Aff}_+.$$

We come to the claim of the theorem (see Figure 3).

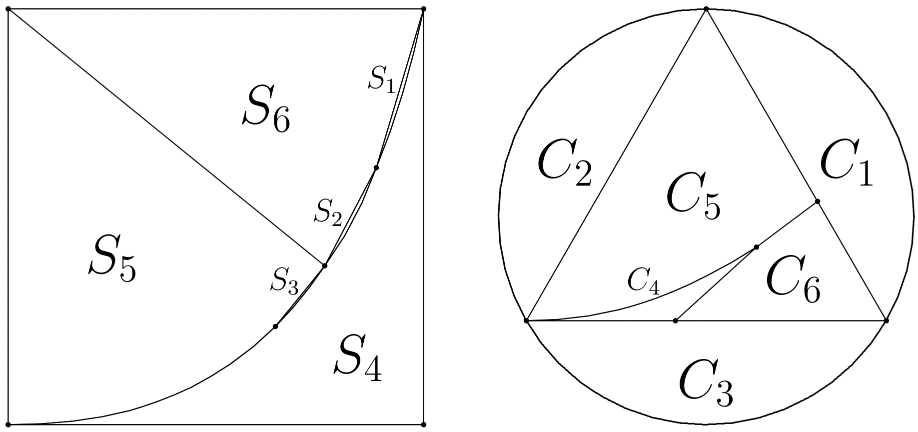

Figure 3. The constructed congruence by dissection

The square S can be represented as $S = \gamma(P)$ with $\gamma \in \mathbf{Aff}_+$. This gives the dissection

$$S = \gamma(P_1) \cup \gamma(P_2) \cup \gamma(P_3) \cup \gamma(P_4) \cup \gamma(P_5) \cup \gamma(P_6) \quad \text{with} \quad \gamma \in \mathbf{Aff}_+.$$

The circular disc C is dissected into an equilateral triangle, that can be expressed as $\delta(T)$ with $\delta \in \mathbf{Aff}_+$, and into three discs C_i, $i = 1, 2, 3$, each of which being the convex hull of a circular arc covering an angle of size $\frac{2\pi}{3}$. By Lemma 1 (i), the discs $P_i = \mathrm{conv}(\alpha^{i-1}(\Gamma))$ and the discs C_i, $i = 1, 2, 3$, are congruent with respect to $\mathbf{Aff}_+$. We pick $\eta_i \in \mathbf{Aff}_+$ such that $C_i = \eta_i(P_i)$, $i = 1, 2, 3$. Using this and the above representation of T we obtain the dissection of $C = C_1 \cup C_2 \cup C_3 \cup \delta(T)$ into

$$C = \eta_1(P_1) \cup \eta_2(P_2) \cup \eta_3(P_3) \cup \delta\alpha^3(P_4) \cup \delta(P_5) \cup \delta\beta(P_6) \quad \text{with} \quad \alpha, \beta, \delta, \eta_i \in \mathbf{Aff}_+.$$

The boundaries of the pieces P_i, $i = 1, \ldots, 6$, split into finitely many line segments, elliptic arcs, and affine images of Δ. The corollary shows that Δ is twice continuously differentiable and convex, as well as the line segments and the elliptic arcs are. This proves Theorem 1.

3. Proof of Theorem 2

Lemma 3. *For every $k \geq 1$, there exist numbers $0 < a < a_1 < a_2 < a_3 < a_4 < 1$ and $0 < b < a^k$ and a strictly increasing and infinitely differentiable function $f : [a,1] \to [b,1]$ with the following properties:*

(i) $f(a) = b$, $f(1) = 1$, $af'(a) = bf'(1)$, $f^{(l)}(a) = f^{(l)}(1) = 0$ *for all* $l \geq 2$.

(ii) *The graph Γ_0 of $f|_{[a_3,a_4]}$ and the arc $\Theta = \big\{(\cos(\varphi), \sin(\varphi)) : 0 \leq \varphi \leq \frac{2\pi}{3}\big\}$ are congruent with respect to $\mathbf{Aff}_+$.*

(iii) *The arc Γ_0 and the line segment $\Lambda_0 = \overline{(a_3, f(a_3))(a_4, f(a_4))}$ bound a disc D_0 situated below Λ_0.*

(iv) *The graph Γ_+ of $f|_{[a_4,1]}$ and the line segment $\Lambda_+ = \overline{(a_4, f(a_4))(1, f(1))}$ bound a disc D_+ situated above Λ_+.*

(v) *The graph Γ_- of $f|_{[a_2,a_3]}$ and the line segment $\Lambda_- = \overline{(a_2, f(a_2))(a_3, f(a_3))}$ bound a disc D_- situated above Λ_-.*

(vi) *The graph $\overline{\Gamma}_-$ of $f|_{[a_1,a_2]}$ and the line segment $\overline{\Lambda}_- = \overline{(a_1, f(a_1))(a_2, f(a_2))}$ bound a disc $\overline{D}_-$ situated below $\overline{\Lambda}_-$ and congruent with D_- with respect to $\mathbf{Aff}_+$.*

(vii) *The graph $\overline{\Gamma}_+$ of $f|_{[a,a_1]}$ and the line segment $\overline{\Lambda}_+ = \overline{(a, f(a))(a_1, f(a_1))}$ bound a disc $\overline{D}_+$ situated below $\overline{\Lambda}_+$ and congruent with D_+ with respect to $\mathbf{Aff}_+$.*

(viii) *The segments $\Lambda_+, \Lambda_0, \Lambda_-$, and $\overline{\Lambda}_-$ are subsets of the tangent of f at the point $(1,1)$.*

(ix) *The segment $\overline{\Lambda}_+$ is a subset of the tangent of f at the point (a,b).*

Proof. The graph of $h_0 : \big[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\big] \to \mathbb{R}$, $h(x) = \frac{1}{2} - \sqrt{1-x^2}$, can be described as $\Sigma_0 = \big\{\big(\cos(\varphi), \sin(\varphi) + \frac{1}{2}\big) : \frac{7\pi}{6} \leq \varphi \leq \frac{11\pi}{6}\big\}$ and therefore is congruent with Θ (see Figure 4).

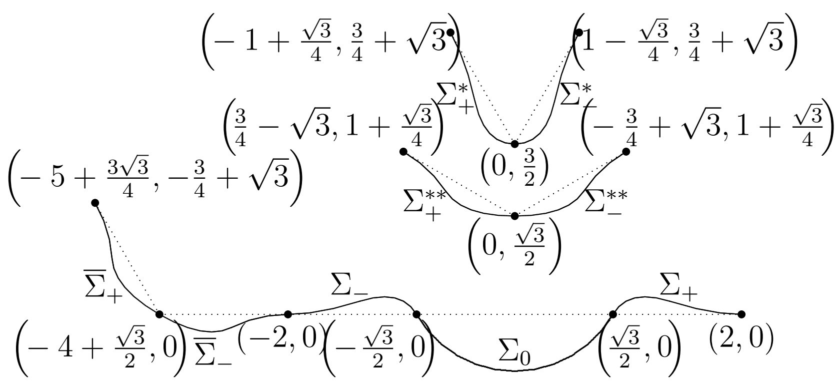

Figure 4. The arc Σ

Let $h_+ : \left[\frac{\sqrt{3}}{2}, 2\right] \to \mathbb{R}$ be a function with $h_+\left(\frac{\sqrt{3}}{2}\right) = h_+(2) = 0$, $h_+\left(\left(\frac{\sqrt{3}}{2}, 2\right)\right) \subseteq (0, \infty)$, $h'_+\left(\left[\frac{\sqrt{3}}{2}, 2\right]\right) \subseteq \left(-\frac{\sqrt{3}}{5}, \infty\right)$, and $h_+^{(l)}(2) = 0$, $l \geq 1$, such that h_0 and h_+ together form an infinitely differentiable function on $\left[-\frac{\sqrt{3}}{2}, 2\right]$. A reflection of the graph Σ_+ of h_+ with respect to the vertical axis yields the graph Σ_- of a function h_- on $\left[-2, -\frac{\sqrt{3}}{2}\right]$ such that h_-, h_0, and h_+ together are an infinitely differentiable function.

Let Σ_+^* and Σ_-^* be the images of Σ_+ and Σ_- under the rotation

$$(x, y) \mapsto \left(-\tfrac{1}{2}x - \tfrac{\sqrt{3}}{2}y + \tfrac{\sqrt{3}}{4}, \tfrac{\sqrt{3}}{2}x - \tfrac{1}{2}y + \tfrac{3}{4}\right)$$

(with center $\left(0, \frac{1}{2}\right)$ and angle $\frac{2\pi}{3}$) and its inverse

$$(x, y) \mapsto \left(-\tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y - \tfrac{\sqrt{3}}{4}, -\tfrac{\sqrt{3}}{2}x - \tfrac{1}{2}y + \tfrac{3}{4}\right),$$

respectively. Then $\Sigma_+^* \cup \Sigma_-^*$ is infinitely differentiable, because its differential behaviour at $\left(0, \frac{3}{2}\right)$ coincides with that of the unit circle centered at $\left(0, \frac{1}{2}\right)$. Next we apply the map $(x, y) \mapsto \left(\sqrt{3}x, \frac{\sqrt{3}}{3}y\right)$ to obtain $\Sigma_+^{**} \cup \Sigma_-^{**}$. Then we use

$$(x, y) \mapsto \left(\tfrac{\sqrt{3}}{2}x + \tfrac{1}{2}y - 4 + \tfrac{\sqrt{3}}{4}, -\tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y - \tfrac{3}{4}\right)$$

(a rotation mapping the right-hand end-point of Σ_-^{**} onto $(-2, 0)$ and $\left(0, \frac{\sqrt{3}}{2}\right)$ onto $\left(-4 + \frac{\sqrt{3}}{2}, 0\right)$) and arrive at $\overline{\Sigma}_+ \cup \overline{\Sigma}_-$.

The resulting arc $\Sigma = \Sigma_0 \cup \Sigma_+ \cup \Sigma_- \cup \overline{\Sigma}_+ \cup \overline{\Sigma}_-$ is infinitely differentiable, because all derivatives of the subarcs Σ_- and $\overline{\Sigma}_-$ at the common point $(-2, 0)$ vanish. Note that Σ is the graph of a function h. This is obvious for $\Sigma_0 \cup \Sigma_+ \cup \Sigma_-$. The arc $\overline{\Sigma}_+$ is obtained from Σ_+ by the map

$$(x, y) \mapsto \left(-\tfrac{1}{2}x - \tfrac{5\sqrt{3}}{6}y - 4 + \tfrac{3\sqrt{3}}{4}, \tfrac{\sqrt{3}}{2}x + \tfrac{1}{2}y - \tfrac{3}{4}\right).$$

The slope of $\overline{\Sigma}_+$ in the image of a point $(x, h_+(x)) \in \Sigma_+$ is given by the derivative $\frac{d\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}h_+(x) - \frac{3}{4}\right)}{d\left(-\frac{1}{2}x - \frac{5\sqrt{3}}{6}h_+(x) - 4 + \frac{3\sqrt{3}}{4}\right)} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}h'_+(x)}{-\frac{1}{2} - \frac{5\sqrt{3}}{6}h'_+(x)}$. This derivative has no singularities, because we have chosen $h'_+(x) > -\frac{\sqrt{3}}{5}$. Therefore $\overline{\Sigma}_+$ describes a function. Similarly, $\overline{\Sigma}_-$ is obtained from Σ_+ by

$$(x, y) \mapsto \left(x + \tfrac{2\sqrt{3}}{3}y - 4, -y\right).$$

The derivative $\frac{d(-h_+(x))}{d\left(x + \frac{2\sqrt{3}}{3}h_+(x) - 4\right)} = \frac{-h'_+(x)}{1 + \frac{2\sqrt{3}}{3}h'_+(x)}$ is not singular, for $h'_+(x) > -\frac{\sqrt{3}}{2}$.

Now we pick a map $\varrho \in \mathbf{Aff}_+$ transforming Σ into the graph of strictly increasing and infinitely differentiable function $h_\varrho : [0, 1] \to [0, 1]$ with $h_\varrho(0) = 0$, $h_\varrho(1) = 1$, and $0 < h'_\varrho(0) < h'_\varrho(1)$. (One fixes a straight line l_0 through the left-hand end-point $\left(-5 + \frac{3\sqrt{3}}{4}, -\frac{3}{4} + \sqrt{3}\right)$ of Σ whose slope is less than $\min_x h'(x)$ and a straight line l_1 through the right-hand end-point $(2, 0)$ of Σ with a slope larger than $\max_x h'(x)$. Then one defines ϱ by $\varrho\left(-5 + \frac{3\sqrt{3}}{4}, -\frac{3}{4} + \sqrt{3}\right) = (0, 0)$,

$\varrho(2,0) = (1,1)$, and $\varrho(l_0 \cap l_1) = \{(1,0)\}$. The result is illustrated in the left-hand part of Figure 5. The dotted lines are the tangents at $(0,0)$ and $(1,1)$.

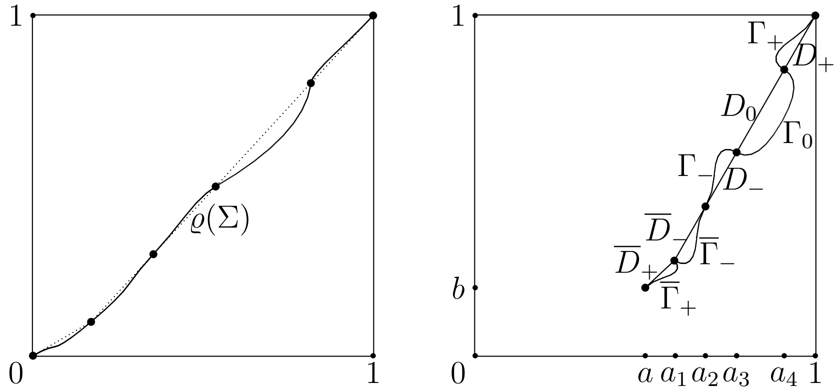

Figure 5. End of the proof of Lemma 3

Finally, we fix $0 < b < a < 1$ such that $ah'_\varrho(0) = bh'_\varrho(1)$ and $b < a^k$. Application of the map $\sigma(x,y) = ((1-a)x+a, (1-b)y+b)$ to $\varrho(\Sigma)$ yields the graph $\sigma\varrho(\Sigma)$ of the strictly increasing and infinitely differentiable function $f : [a,1] \to [b,1]$, $f(x) = (1-b)h_\varrho\big(\frac{x-a}{1-a}\big)+b$. We obtain $f(a) = (1-b)h_\varrho(0)+b = b$, $f(1) = (1-b)h_\varrho(1)+b = 1$, and $af'(a) = a\frac{1-b}{1-a}h'_\varrho(0) = b\frac{1-b}{1-a}h'_\varrho(1) = bf'(1)$. The derivatives $f^{(l)}(a)$ and $f^{(l)}(1)$ vanish for all $l \geq 2$, because in certain neighbourhoods of (a,b) and $(1,1)$ the graph of f is an affine image of the graph of h_+ close to $(2,0)$ and all derivatives $h_+^{(l)}(2)$ are zero.

The other claims of the lemma follow easily. The right-hand part of Figure 5 shows the final situation. (The illustration is highly simplified in order to improve the visibility of the curved arcs.) □

Corollary 2. *Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, $\alpha(x,y) = (ax, by)$, and let Γ be the graph of f, a, b, f being chosen as in Lemma 3. Then $\Delta = \{(0,0)\} \cup \bigcup_{i=0}^\infty \alpha^i(\Gamma)$ is the graph of a k times continuously differentiable function $g : [0,1] \to [0,1]$.*

Proof. The arcs $\alpha^i(\Gamma)$ describe the functions $g_i(x) = b^i f(a^{-i}x) : [a^{i+1}, a^i] \to [b^{i+1}, b^i]$. Consecutive functions fit continuously together, because $g_{i+1}(a^{i+1}) = b^{i+1} = g_i(a^{i+1})$. Hence Δ is the graph of a continuous function $g : [0,1] \to [0,1]$. The functions g_i are infinitely differentiable, because f is. Property (i) yields in particular

$$\begin{array}{ll} g'_i(a^{i+1}) = \frac{b^i}{a^i}f'(a) = \frac{b^{i+1}}{a^{i+1}}f'(1) = g'_{i+1}(a^{i+1}) & \text{and} \\ g_i^{(l)}(a^{i+1}) = \frac{b^i}{a^{li}}f^{(l)}(a) = 0 = \frac{b^{i+1}}{a^{l(i+1)}}f^{(l)}(1) = g_{i+1}^{(l)}(a^{i+1}) & \text{for } \ l \geq 2. \end{array}$$

Thus g is infinitely differentiable on $(0,1]$.

Since f is infinitely differentiable, there exists a bound r such that $|f^{(l)}(x)| < r$ for $x \in [a,1]$, $l \in \{1,\ldots,k\}$. Consequently, $\big|g_i^{(l)}(x)\big| = \big(\frac{b}{a^l}\big)^i \big|f^{(l)}(a^{-i}x)\big| < \big(\frac{b}{a^l}\big)^i r$ for $x \in [a^{i+1}, a^i]$, $l \in \{1,\ldots,k\}$, and in turn $g^{(l)}(0) = \lim_{x\downarrow 0} g^{(l)}(x) = 0$, $l \in \{1,\ldots,k\}$, because $0 < b < a^k \leq a^l$. This is the claimed smoothness at 0. □

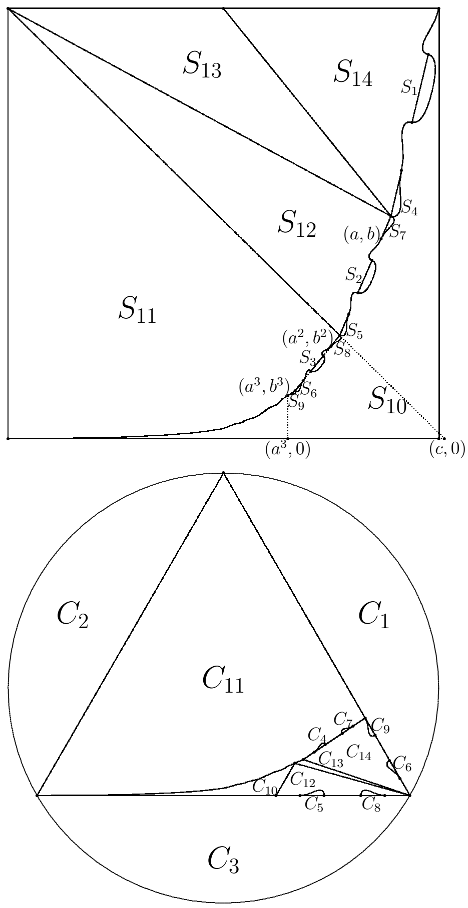

Figure 6. The dissections of a square and a circular disc

We come to the proof of Theorem 2. Given $k \geq 3$, we apply Lemma 3 and Corollary 2. Then the square S, say $S = [0,1]^2$ without loss of generality, admits the following dissection

$$S = S_1 \cup \cdots \cup S_{14}$$

(see Figure 6):

$S_i = \alpha^{i-1}(D_0)$ for $i = 1,2,3$, $S_i = \alpha^{i-4}\big(\overline{D}_-\big)$ for $i = 4,5,6$, and $S_i = \alpha^{i-7}\big(\overline{D}_+\big)$ for $i = 7,8,9$. The boundary of S_{10} consists of the polygonal arc $\overline{(0,0)(1,0)(1,1)}$ and Δ.

S_{11} is bounded by $\{(0,0)\} \cup \bigcup_{i=3}^{\infty} \alpha^i(\Gamma) = \alpha^3(\Delta)$, $\overline{(a^3,b^3)\alpha^2(a_1,f(a_1))\alpha^2(a_2,f(a_2))}$, $\alpha^2(\Gamma_-)$, $\overline{\alpha^2(a_3,f(a_3))\alpha^2(a_4,f(a_4))}$, $\alpha^2(\Gamma_+)$, and $\overline{(a^2,b^2)\alpha(a_1,f(a_1))(0,1)(0,0)}$.

The boundary of the disc S_{12} splits into the arcs $\overline{(a,b)(a_1,f(a_1))(0,1)\alpha(a_1,f(a_1))\alpha(a_2,f(a_2))}$, $\alpha(\Gamma_-)$, $\overline{\alpha(a_3,f(a_3))\alpha(a_4,f(a_4))}$, and $\alpha(\Gamma_+)$. S_{13} is the triangle $\triangle\big((0,1)(a_1,f(a_1))\big(\frac{1}{2},1\big)\big)$. The disc S_{14} is bounded by $\overline{(1,1)\big(\frac{1}{2},1\big)(a_1,f(a_1))(a_2,f(a_2))}$, Γ_-, $\overline{(a_3,f(a_3))(a_4,f(a_4))}$, and Γ_+.

Let $(c,0)$ be the intersection point of the horizontal axis with the straight line through $(0,1)$ and $\alpha(a_1,f(a_1))$. The triangle $T = \triangle((0,0)(c,0)(0,1))$ is dissected into

$$T = \alpha^3(S_{10}) \cup S_{11} \cup \alpha^2(D_+) \cup \alpha^2(D_-) \cup P \tag{1}$$

where P is the non-convex pentagon with the vertices $(a^3, 0)$, $(c, 0)$, $\alpha(a_1, f(a_1))$, $\alpha^2(a_1, f(a_1))$, and $(a^3, b^3) = \alpha^2(a, b)$ (see the dotted lines in Figure 6). The triangles $\mathrm{conv}(S_{12})$ and $\mathrm{conv}(S_{14})$ admit the dissections

$$\mathrm{conv}(S_{12}) = S_{12} \cup \alpha(D_+) \cup \alpha(D_-), \quad \mathrm{conv}(S_{14}) = S_{14} \cup D_+ \cup D_-. \tag{2}$$

We split P into three triangles $\eta_{12}(\mathrm{conv}(S_{12}))$, $\eta_{13}(S_{13})$, and $\eta_{14}(\mathrm{conv}(S_{14}))$ with $\eta_{12}, \eta_{13}, \eta_{14} \in \mathbf{Aff}_+$. Using (2) we can refine the dissection (1) to

$$\begin{aligned} T = {} & \alpha^3(S_{10}) \cup S_{11} \cup \alpha^2(D_+) \cup \alpha^2(D_-) \cup \eta_{12}(S_{12}) \cup \eta_{12}\alpha(D_+) \cup \\ & \cup \eta_{12}\alpha(D_-) \cup \eta_{13}(S_{13}) \cup \eta_{14}(S_{14}) \cup \eta_{14}(D_+) \cup \eta_{14}(D_-). \end{aligned}$$

By Lemma 3 (vi) and (vii), the discs $S_i = \alpha^{i-4}\big(\overline{D}_-\big)$, $i = 4, 5, 6$, and $S_i = \alpha^{i-7}\big(\overline{D}_+\big)$, $i = 7, 8, 9$, can be written as $S_4 = \eta_4^{-1}\alpha^2(D_-)$, $S_5 = \eta_5^{-1}\eta_{12}\alpha(D_-)$, $S_6 = \eta_6^{-1}\eta_{14}(D_-)$, $S_7 = \eta_7^{-1}\alpha^2(D_+)$, $S_8 = \eta_8^{-1}\eta_{12}\alpha(D_+)$, and $S_9 = \eta_9^{-1}\eta_{14}(D_+)$ with appropriate $\eta_4, \ldots, \eta_9 \in \mathbf{Aff}_+$. This yields

$$T = \bigcup_{i=4}^{9} \eta_i(S_i) \cup \alpha^3(S_{10}) \cup S_{11} \cup \bigcup_{i=12}^{14} \eta_i(S_i). \tag{3}$$

As in the proof of Theorem 1, we dissect the circular disc C into an inscribed equilateral triangle, that we express as $\delta(T)$ with $\delta \in \mathbf{Aff}_+$, and into three discs C_i, $i = 1, 2, 3$, each of which being the convex hull of a circular arc covering an angle of size $\frac{2\pi}{3}$. By Lemma 3 (ii), the discs C_i and the discs $S_i = \alpha^{i-1}(D_0) = \alpha^{i-1}(\mathrm{conv}(\Gamma_0))$, $i = 1, 2, 3$, are congruent with respect to $\mathbf{Aff}_+$. We pick $\eta_i \in \mathbf{Aff}_+$ such that $C_i = \eta_i(S_i)$, $i = 1, 2, 3$. Using this and (3) we obtain the final dissection of $C = C_1 \cup C_2 \cup C_3 \cup \delta(T)$ into

$$C = \bigcup_{i=1}^{3} \eta_i(S_i) \cup \bigcup_{i=4}^{9} \delta\eta_i(S_i) \cup \delta\alpha^3(S_{10}) \cup \delta(S_{11}) \cup \bigcup_{i=12}^{14} \delta\eta_i(S_i),$$

where $\alpha, \delta, \eta_i \in \mathbf{Aff}_+$. Corollary 2 implies that all pieces S_i belong to $\mathcal{D}^k$. This completes the proof.

We would like to remind that the arc Δ is infinitely differentiable in all points apart from $(0, 0)$ (see the proof of Corollary 2). So the discs $S_1, \ldots, S_9, S_{12}, S_{13}, S_{14}$ belong to $\mathcal{D}^\infty$ and S_{10}, S_{11} are "not far" from $\mathcal{D}^\infty$.

We finally remark that the congruence by dissection of S and C uses a piecewise congruence of the quadrangle $\mathrm{conv}(S_{12}) \cup S_{13} \cup \mathrm{conv}(S_{14})$ with the pentagon $P \subseteq S$ (dotted in Figure 6). This is essentially realized by the three triangles $\mathrm{conv}(S_{12})$, S_{13}, $\mathrm{conv}(S_{14})$ and by their images $\eta_{12}(\mathrm{conv}(S_{12}))$, $\eta_{13}(S_{13})$, $\eta_{14}(\mathrm{conv}(S_{14}))$, respectively. By a good use of the freedom in the construction of Δ, one could probably obtain a situation where a congruence by dissection of the quadrangle and the pentagon can be realized by the aid of two pieces. This should be the key to a congruence by dissection of S and C based on 13 pieces only.

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