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# General Embedding Problems and Two-distance Sets in Minkowski Planes

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**Abstract.** This paper describes a complete classification of 2-distance sets in 2-dimensional normed real linear spaces, also called Minkowski planes. 2-distance sets are point sets characterized by a property of the induced metric of the points in the surrounding metric space: at most two different distances are allowed between different points of a 2-distance set. Considering the problem from this metric point of view, it is a special embedding problem of finite metric spaces into suitable Minkowski spaces. The solution of the problem has both an algebraic part (analytical geometry) as well as an discrete part. The reason for this is that for each one of finitely many combinatorial candidates, characterized by the relative position of the points and the distinction between large and small distances, the problem can be transformed into a system of polynomial equations and inequalities whose unknown variables are geometric coordinates and the occurring distance. Both parts together were handled with the use of a computer program, using some evolved external mathematical libraries and systems (polymake, nauty, Core Library, CoCoA) and following the modern trend that numerical computations are based on exact arithmetics.

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## 1. Introduction

Minkowski spaces $\mathbb{M}^d$, which are in this paper finite-dimensional Banach spaces, provide a geometric model which generalizes Euclidean geometry. As in the Euclidean space, the distance between two points, which belong to some $d$-dimensional real linear space with origin $\mathbf{0}$, e.g., to $\mathbb{R}^d$, only depends on their difference vector. But the function assigning a length to each vector, called the norm $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}$, can be arbitrary as long as the following three axioms are satisfied:

- the norm is positive definite: $\|\mathbf{0}\| = 0$ and $\|\mathbf{x}\| > 0$ for all $\mathbf{x} \neq \mathbf{0}$,
- the norm is homogeneous: $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^d$ and all $\lambda \in \mathbb{R}$, and
- the triangle inequality is satisfied: $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

Geometrically, we can represent the space $\mathbb{M}^d$ by its unit ball

$$B = B(\mathbb{M}^d) = \{\, \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq 1 \,\}\,. \tag{1}$$

It turns out that $B$ can be an arbitrary convex body with $B = -B$, i.e., $B$ is convex and compact, has $\mathbf{0}$ in its interior and is symmetric with respect to $\mathbf{0}$. For the theory of Minkowski spaces see also the book of Thompson [5]. The norm is determined by $B$ in the following way:

$$\|\mathbf{x}\| = \|\mathbf{x}\|_B = \min\{\, \lambda \geq 0 : \mathbf{x} \in \lambda B \,\}\,. \tag{2}$$

Minkowski spaces are affine spaces. If we have some geometric configuration in a Minkowski space $\mathbb{M}^d$ with unit ball $B$, then the metric properties, which can be expressed only using the distances function, do not change if we apply an invertible affinely linear map $\alpha$ to the whole configuration and $\mathbb{M}^d$. The unit ball of $\alpha(\mathbb{M}^d)$ is $\alpha(B) - \alpha(\mathbf{0})$, i.e., $B$ is transformed by the linear component of $\alpha$.

In this paper we are mostly interested in 2-dimensional Minkowski spaces $\mathbb{M}^2$, called Minkowski planes, and in 2-distance sets $S$ of $\mathbb{M}^2$. This means that between all pairs of different points in $S$ there occur at most two distinct distances. We do not fix the Minkowski planes in advance, but are looking for suitable Minkowski planes where some special kind of 2-distance sets $S$ are possible. Because of that, we define the pair $(\mathbb{M}^2, S)$ to be a 2-*distance set* if the set

$$\operatorname{dist}(\mathbb{M}^2, S) := \{\, \|\mathbf{s}_1 - \mathbf{s}_2\|_{\mathbb{M}^2} : \mathbf{s}_1, \mathbf{s}_2 \in S, \mathbf{s}_1 \neq \mathbf{s}_2 \,\} \tag{3}$$

contains at most two elements and if $S$ contains at least two elements. Nevertheless we still call $S$ a 2-distance set if no confusion arises.

To obtain a classification of all possible 2-distance sets, we define some equivalence relations to identify 2-distance sets with identical position of the points and identical metric structure, see Section 4. The resulting equivalence classes are called 2-distance configuration.

For a fixed metric structure, we arrive on a special kind of an embedding problem of finite metric spaces into suitable Minkowski spaces. A *metric space* is a pair $(X, \rho)$ of a set $X$ and the distance function (also called metric) $\rho$ :

$X \times X \to \mathbb{R}$ satisfying the well known conditions symmetry ($\rho(x,y) = \rho(y,x)$ for all $x, y \in X$), positive definiteness ($\rho(x,y) > 0\ \forall x \neq y$) and the triangle inequality ($\rho(x,z) \leq \rho(x,y) + \rho(y,z)\ \forall x,y,z \in X$). We consider every Minkowski space $\mathbb{M}^d$ as the metric space $(\mathbb{R}^d, \rho_{\mathbb{M}^d})$ with $\rho_{\mathbb{M}^d}(\mathbf{x},\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\|_{B(\mathbb{M}^d)}$. Some general theory about such embedding problems is provided in Section 2. These results motivate our notion of relative position of the points, see Section 3.

Having fixed the metric structure and the relative position of the points, we can obtain the answer to classification questions from the solution of a system of polynomial equations and inequalities. Since the conditions are expressed by polynomials, the systems can be solved exactly by some algorithms, at least theoretically. The author wrote some computer programs which generated and solved all these systems, using the special structure of the systems. Further, a lot of pictures representing the classification graphically are obtained from the computational results, presented in Section 5.

The proof of correctness of the classification is very important but difficult to realize on just a couple of pages. The obvious approach is to show that the algorithms are correct, to show that they are correctly implemented, and provide the output as well as the program sources. But such a proof is not easy to realize, nor is its verification.

In analogy to classical proofs, our solution process – which is here an electronic computation done by a computer program – will not only produce the required answer to the question, but also a certificate. This certificate can be used by another, much simpler, computer program to verify that the answer is indeed correct.

With this approach, the main dilemma to trust the output of a computer program remains. But on the other hand, the critical part of the source code to be checked for correctness is much smaller. Additionally, it is possible to check at least parts of the proof without the help of a computer. This idea is followed in Section 6.

Some material presented in this article is already published in the authors PhD thesis [6].

## 2. Embedding metric spaces in Minkowski spaces

We say that a function $\phi : X \to Y$ is an *embedding* of the metric space $(X, \rho)$ into the metric space $(Y, \varrho)$, if for all $a, b \in X$ we have $\varrho(\phi(a), \phi(b)) = \rho(a, b)$. Without loss of generality we can restrict our considerations to $X = [n] := \{1, 2, \ldots, n\}$.

In this paper we will focus on the first of the following two decision problems.

**Task 1.** (General Decision Problem on Embedding $([n], \rho)$ in $\mathbb{M}^d$)
Decide whether or not there is an embedding of a given metric space $([n], \rho)$, $n \geq 2$, into a suitable Minkowski space $\mathbb{M}^d$ of given dimension $d \geq 1$.

**Task 2.** (Special Decision Problem on Embedding $([n], \rho)$ in $\mathbb{M}^d$)
Decide whether or not there is an embedding of a given metric space $([n], \rho)$, $n \geq 2$, into the given Minkowski space $\mathbb{M}^d$ with $d \geq 1$.

In connection to Task 1 we also consider the algorithmic task to determine all possible embeddings. The chosen description will be made more precise later on and is technically more involved.

**Task 3.** (General Description of $d$-Embeddings of $\rho$)
Describe all possible embeddings of a given metric space $([n], \rho)$, $n \geq 2$, into a suitable Minkowski space $\mathbb{M}^d$ of given dimension $d \geq 1$.

We will transform Task 1 and Task 3 into the more analytic task of determining admissibility and the task of describing the solution set – which is in the sequel denoted as "solving" the systems – respectively, of a finite number of finite systems of equations and inequalities in $\mathbb{R}^m$, with $m = m(d, n) = \binom{n-1}{d}$. A finite *system of equations and inequalities in* $X$, is a triple $\mathcal{S} = (E, W, S)$ of finite sets of functions $f : X \to \mathbb{R}$. All functions $f \in E \cup W \cup S$ are called *restrictions* of $\mathcal{S}$. The *union* of two systems $(E, W, S)$ and $(E', W', S')$ is $(E, W, S) \cup (E', W', S') := (E \cup E', W \cup W', S \cup W')$.

We will only consider systems in $\mathbb{R}^m$, possibly after identifying the set of functions $f : Y \to \mathbb{R}$ with $\mathbb{R}^m$ in case of $|Y| = m$.

We say that $\mathbf{x} \in \mathbb{R}^m$ is a *solution vector* of $\mathcal{S}$ if $f(\mathbf{x}) = 0$ for all $f \in E$, $f(\mathbf{x}) \geq 0$ for all $f \in W$, and $f(\mathbf{x}) > 0$ for all $f \in S$. The *solution set* $L = L(\mathcal{S})$ is the set of all solution vectors of $\mathcal{S}$, $m = \dim \mathcal{S}$ is called the *dimension* of $\mathcal{S}$. $\mathcal{S}$ is called *admissible* if $L(\mathcal{S}) \neq \emptyset$. Two systems in $X$ are called *equivalent* if they have the same solution set.

$\mathcal{S}$ is called a linear (polynomial; homogeneous) system if each restriction $f$ is an affinely linear (a polynomial, or a homogeneous – i.e., $f(\lambda\mathbf{x}) = \lambda^p f(\mathbf{x})$ for some $p = p(f) \in \mathbb{N}$) function from $\mathbb{R}^m$ to $\mathbb{R}$, respectively.

We will transform Task 1 and Task 3 to homogeneous polynomial systems $(E, W, S)$ such that $W$ and $S$ only contain linear functions and the polynomials in $E$ have total degree at most 2, see Theorem 17.

For the special case of polytopal Minkowski spaces, Task 2 and its classification form can both be transformed to the task of solving one linear system in $\mathbb{R}^{d(n-1)}$ and combinatorial evaluation of the solution set. The interested reader can find this transformation in [6].

### 2.1. Transformation of the embedding task to analytical systems

**Definition 4.** *We say that a set $U \subset \mathbb{R}^d$ is in weak convex position if $U$ is a subset of the relative boundary of its convex hull:*

$$U \subset \operatorname{rel}\operatorname{bd}(\operatorname{conv} U)\,.$$

**Theorem 5.** *The map $e : X \to \mathbb{R}^d$ is an embedding of a given metric space $(X, \rho)$ into a suitable Minkowski space $\mathbb{M}^d$ (depending on $e$) if and only if the set*

$$U := \left\{\, \rho(x,y)^{-1}(e(x) - e(y)) : x \neq y, \{x,y\} \subset X \,\right\} \tag{4}$$

*is bounded and in weak convex position.*

Note that for finite sets $X$ the set $U$ in (4) is, of course, bounded.

*Proof of Theorem* 5. We can assume that $U$ linearly spans $\mathbb{R}^d$, otherwise we consider $e$ as function into an affinely linear subspace of lower dimension. This does neither change the property of weak convex position, nor of the existence of a suitable Minkowski space. We can easily embed a $(d-1)$-dimensional Minkowski space into a hyperplane of a suitable $d$-dimensional Minkowski space by choosing a bipyramid as unit ball.

By definition, $e$ is an embedding into some Minkowski space $\mathbb{M}^d$ with unit ball $B$ if and only if

$$\|e(\mathbf{x}) - e(\mathbf{y})\|_B = \rho(\mathbf{x}, \mathbf{y}) \qquad \text{for all } \mathbf{x}, \mathbf{y} \in X. \tag{5}$$

Now $\|e(\mathbf{x}) - e(\mathbf{x})\|_B = \rho(\mathbf{x}, \mathbf{x}) = 0$ holds for all $\mathbf{x} \in X$. For $\mathbf{x} \neq \mathbf{y}$ we get that $\rho(\mathbf{x}, \mathbf{y}) > 0$. The system (5) is equivalent to $\|\rho(\mathbf{x}, \mathbf{y})^{-1}(e(\mathbf{x}) - e(\mathbf{y}))\|_B = 1$ for all $\mathbf{x}, \mathbf{y} \in X$ with $\mathbf{x} \neq \mathbf{y}$. This is equivalent to $U \subset \partial B$.

If $U$ is bounded and in weak convex position, i.e., $U \subset \partial(\operatorname{conv} U)$, we consider the set $B := \operatorname{cl} \operatorname{conv} U$. $B$ is a centered (since $U$ is centrally symmetric), compact convex body with $\partial(\operatorname{conv} U) = \partial B$. Thus $B$ is the unit ball of a Minkowski space $\mathbb{M}^d(B)$ so that $e$ is an embedding of $(X, \rho)$ into $\mathbb{M}^d(B)$.

If now $e$ is an embedding into the Minkowski space $\mathbb{M}^d$ with unit ball $B$, we have $U \subset \partial B$. Thus $U$ is bounded. Every vector $\mathbf{x} \in U$ belongs to the boundary of $B \supset \operatorname{conv} U$ and to $\operatorname{conv} U$, too. Thus we also have $\mathbf{x} \in \partial(\operatorname{conv} U)$. Consequently, $U$ is in weak convex position. □

**Theorem 6.** *Let $U \subset \mathbb{R}^d$ be a $k$-dimensional centered set, i.e., $k = \dim \operatorname{lin} U$, and let $V(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ denote the $k$-dimensional volume of the parallelepiped $P = \overline{\mathbf{0}\mathbf{x}_1} + \overline{\mathbf{0}\mathbf{x}_2} + \cdots + \overline{\mathbf{0}\mathbf{x}_k}$ spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$. Then $U$ is in weak convex position if and only if the following inequalities hold for all $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1} \in U$:*

$$\begin{aligned} V(\mathbf{x}_1, \ldots, \mathbf{x}_k) \leq V(\mathbf{x}_1, \ldots, \mathbf{x}_{k-2}, \mathbf{x}_{k-1}, \mathbf{x}_{k+1}) &+ V(\mathbf{x}_1, \ldots, \mathbf{x}_{k-2}, \mathbf{x}_k, \mathbf{x}_{k+1}) \\ &+ \cdots + V(\mathbf{x}_2, \ldots, \mathbf{x}_{k+1}). \end{aligned} \tag{6}$$

For any family of objects $o_i$, $i \in M \subset \mathbb{N}$, and a set $\{i_1, \ldots, i_m\} = I \subset M$ with $i_1 < i_2 < \cdots < i_m$ we use the notation $(o_i)_{i \in I}$ for the $m$-tuple $(o_{i_1}, o_{i_2}, \ldots, o_{i_m})$. Such $m$-tuples can be used to denote the $m$ arguments to a function: $V((\mathbf{x}_i)_{i \in I}) = V(\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_m})$. We write (6) more precisely as

$$V((\mathbf{x}_i)_{i \in [k]}) \leq \sum_{j \in [k]} V((\mathbf{x}_i)_{i \in [k+1] \setminus \{j\}}).$$

If in particular $k = d$, then we have that $V(\mathbf{x}_1, \ldots, \mathbf{x}_d) = |\det(\mathbf{x}_1, \ldots, \mathbf{x}_d)|$ is the absolute value of the determinant of the matrix whose columns are $\mathbf{x}_1, \ldots, \mathbf{x}_{d-1}$ and $\mathbf{x}_d$. For $k = d = 2$ (6) becomes

$$|\det(\mathbf{a}, \mathbf{b})| \leq |\det(\mathbf{a}, \mathbf{c})| + |\det(\mathbf{b}, \mathbf{c})| \,, \tag{7}$$

which must hold for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in U$.

For the simple case $k = 1$ the system (6) becomes $\|\overline{\mathbf{0a}}\| = \|\overline{\mathbf{0b}}\|$ for all $\mathbf{a}, \mathbf{b} \in U$, where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^d$.

*Proof of Theorem* 6. We can assume that $U$ linearly spans $\mathbb{R}^d$, i.e., that $k = d$.

First assume that $U$ is in weak convex position. We verify the inequalities (6). Take arbitrary $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \mathbf{x}_{k+1} \in U$. If $V(\mathbf{x}_1, \dots, \mathbf{x}_k) = 0$ then (6) holds trivially. Otherwise, the vectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ span $\mathbb{R}^d$, thus there are $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ with $\mathbf{x}_{k+1} = \lambda_1 \mathbf{x}_1 + \dots + \lambda_k x_k$. Since $U$ is centered, we can achieve that all $\lambda_i \geq 0$ $(i = 1, \dots, k)$ by possibly interchanging $\mathbf{x}_i$ with $-\mathbf{x}_i \in U$. This does not modify (6) since the volumes $V(\mathbf{a}, \dots, \mathbf{b}, \mathbf{x}, \mathbf{c}, \dots, \mathbf{d}) = V(\mathbf{a}, \dots, \mathbf{b}, -\mathbf{x}, \mathbf{c}, \dots, \mathbf{d})$ are invariant under inversion of a spanning vector. Consequently, $\mathbf{x}_{k+1}$ belongs to the convex cone spanned by $\mathbf{x}_1, \dots, \mathbf{x}_k$ with apex $\mathbf{0}$. Since $U$ is in weak convex position, $\mathbf{x}_{k+1}$ cannot belong to the interior of the simplex with vertices $\mathbf{0}, \mathbf{x}_1, \dots, \mathbf{x}_k$. Thus $\lambda_1 + \dots + \lambda_k \geq 1$, since $\mathbf{x}_{k+1}$ would belong to the hyperplane containing $\mathbf{x}_1, \dots, \mathbf{x}_k$ if $\lambda_1 + \dots + \lambda_k = 1$. We get that

$$\lambda_1 V(\mathbf{x}_1, \dots, \mathbf{x}_k) + \dots + \lambda_k V(\mathbf{x}_1, \dots, \mathbf{x}_k) \geq V(\mathbf{x}_1, \dots, \mathbf{x}_k),$$

which turns out to be (6), since $\lambda_1 V(\mathbf{x}_1, \dots, \mathbf{x}_k) = V(\lambda_1 \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) = V(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \dots + \lambda_k x_k, \mathbf{x}_2, \dots, \mathbf{x}_k) = V(\mathbf{x}_{k+1}, \mathbf{x}_2, \dots, \mathbf{x}_k) = V(\mathbf{x}_2, \dots, \mathbf{x}_{k+1})$ and analogously $\lambda_2 V(\mathbf{x}_1, \dots, \mathbf{x}_k) = V(\mathbf{x}_1, \mathbf{x}_{k+1}, \mathbf{x}_3, \dots, \mathbf{x}_k) = V(\mathbf{x}_1, \mathbf{x}_3, \dots, \mathbf{x}_{k+1})$, ..., and $\lambda_k V(\mathbf{x}_1, \dots, \mathbf{x}_k) = V(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{x}_{k+1})$.

For the other direction we assume that (6) holds for all $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \mathbf{x}_{k+1} \in U$ and prove by contradiction that $U$ is in weak convex position. If $U$ were not in weak convex position, $U \not\subset \partial(\operatorname{conv} U)$, there must be some $\mathbf{u} \in U$ with $\mathbf{u} \notin \partial(\operatorname{conv} U)$, thus $\mathbf{u} \in \operatorname{int}(\operatorname{conv} U)$.

If $\mathbf{u} = \mathbf{0}$, then we have the following contradiction to inequality (6): set $\mathbf{x}_{k+1} := \mathbf{u} = \mathbf{0}$ and take $\mathbf{x}_1, \dots, \mathbf{x}_k$ as $k$ linearly independent vectors of $U$ (note that $\dim U = d$). Then $V(\mathbf{x}_1, \dots, \mathbf{x}_k) > 0$ but $V(\mathbf{x}_1, \dots, \mathbf{x}_{k-2}, \mathbf{x}_{k-1}, \mathbf{x}_{k+1}) + V(\mathbf{x}_1, \dots, \mathbf{x}_{k-2}, \mathbf{x}_k, \mathbf{x}_{k+1}) + \dots + V(\mathbf{x}_2, \dots, \mathbf{x}_{k+1}) = 0$. Thus we have $\mathbf{u} \neq \mathbf{0}$.

We can additionally assume that $U$ is a finite set, otherwise replace $U$ by some centered subset $U' \subset U$ such that still $\mathbf{u} \in \operatorname{int}(\operatorname{conv} U')$ and $\mathbf{u} \in U'$, which exists by Caratheodory's Theorem.

Consider the ray $[\mathbf{0}, \mathbf{u}\rangle$ which intersects $\partial(\operatorname{conv} U)$ in some point $\mathbf{u}' = \mu \mathbf{u}$ for $\mu > 1$. Again by Caratheodory's Theorem there are affinely independent points $\mathbf{x}_1, \dots, \mathbf{x}_k \in U$ with $\mathbf{u}' \in \operatorname{conv}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$, since $\mathbf{u}'$ is contained in some facet of $\partial(\operatorname{conv} U)$ which is of dimension $k-1$. Thus there are real numbers $\lambda'_1, \dots, \lambda'_k \geq 0$ with $\mathbf{u}' = \lambda'_1 \mathbf{x}_1 + \dots + \lambda'_k \mathbf{x}_k$ and $\lambda'_1 + \dots + \lambda'_k = 1$. So we have with $\lambda_i := \lambda'_i / \mu$ that $u = \lambda_1 \mathbf{x}_1 + \dots + \lambda_k \mathbf{x}_k$ and $\lambda_1, \dots, \lambda_k \geq 0$ and $\lambda_1 + \dots + \lambda_k < 1$. Multiplying by $V(\mathbf{x}_1, \dots, \mathbf{x}_k) > 0$ yields

$$\lambda_1 V(\mathbf{x}_1, \dots, \mathbf{x}_k) + \dots + \lambda_k V(\mathbf{x}_1, \dots, \mathbf{x}_k) < V(\mathbf{x}_1, \dots, \mathbf{x}_k),$$

which contradicts (since again $\lambda_1 V(\mathbf{x}_1, \dots, \mathbf{x}_k) = V(\mathbf{x}_2, \dots, \mathbf{x}_k, \mathbf{u})$, ..., and $\lambda_k V(\mathbf{x}_1, \dots, \mathbf{x}_k) = V(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{u})$, see above) the inequality (6) for $\mathbf{x}_{k+1} = \mathbf{u}$.

This contradiction completes our proof. □

Furthermore we mention that the condition $k = \dim \operatorname{lin} U$ is important in Theorem 5. For $k > \dim U$ the inequalities (6) are trivially satisfied as $0 \le 0$, since $\dim U = \dim(U \cup \{\mathbf{0}\})$, but $U$ is in general not in weak convex position! Thus the inequalities (6) are necessary for $U$ to be in weak convex position if $k \ge \dim U$. The inequalities (6) are sufficient for $U$ to be in weak convex position if $k = \dim U$.

Since an affinely linear bijection preserves the property to be in weak convex position, to decide whether $U \subset \mathbb{R}^d$ is in weak convex position or not for instances with $\dim U < d$ we can transform it to another instance $U' \subset \mathbb{R}^k$ of full dimension $\dim U' = k$.

Before we combine the results of Theorem 5 and Theorem 6 to get systems of equations and inequalities representing the general embedding problem, we summarize our results in an algorithm.

**Algorithm 7.** Input: A function $e : [n] \to \mathbb{R}^d$, and a metric space $([n], \rho)$

Output: Yes/No, whether or not $e$ is an embedding of $([n], \rho)$ into a suitable Minkowski space $\mathbb{M}^d$

1. We first construct the set $U \in \mathbb{R}^d$ by (4).
2. Then we determine the dimension $k = \dim U = \dim(\operatorname{lin}(e([n])))$ as the rank of the matrix $[e(1)e(2) \cdots e(n)]$.
3. To calculate all the values of the volumes in (6) we construct some linear function $a$ which projects $\operatorname{lin} U$ injectively onto $\mathbb{R}^{\dim U}$. For some constant $c > 0$ we then have

$$V((\mathbf{x}_i)_{i\in[k]}) = c \left|\det((a(\mathbf{x}_i))_{i\in[k]})\right| \tag{8}$$

   for all $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \operatorname{lin} U$. It is not necessary to compute $c$.
4. So we can check whether or not (6) holds for all $(k+1)$-tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_{k+1})$ of vectors from $U$ by using (8).

This algorithm is the starting point to transform the general embedding task into the admissibility or solution task of an analytical system of equations and inequalities. For each $k = 1, 2, \ldots, d$ we get a system $\operatorname{SysEm}(\rho, k)$ of equations and inequalities in $\mathbb{R}^{nk}$ for the case that there is a $k$-dimensional embedding $e$ of $([n], \rho)$ into $\mathbb{R}^k$. Note that we identify the set of functions $e : [n] \to \mathbb{R}^k$, with $\mathbb{R}^{nk}$. All restrictions $f$ of these systems are positively homogeneous functions of degree $k$, i.e., $f(\lambda\mathbf{x}) = |\lambda|^k f(\mathbf{x})$, as they are linear combinations of absolute values of homogeneous polynomials. Using the notation $\mathbb{P}_n := [n]^2 \setminus \{\,(i,i) : i \in [n]\}$, and $I = (I_1, \ldots, I_{k+1}) = ((I_{i,1}, I_{i,2}))_{i\in[k+1]} \in (\mathbb{P}_n)^{k+1}$, $e : [n] \to \mathbb{R}^k$ we get the following:

$$\begin{aligned} w^I_{\rho,k}(e) &:= -\frac{\left|\det((e(I_{i,1})-e(I_{i,2}))_{i\in[k]})\right|}{\Pi_{i\in[k]}\rho(I_i)} \\ &\quad + \sum_{j\in[k]} \frac{\left|\det((e(I_{i,1})-e(I_{i,2}))_{i\in[k+1]\setminus\{j\}})\right|}{\Pi_{i\in[k+1]\setminus\{j\}}\rho(I_i)} \geq 0\,, \qquad (9) \\ s_{n,k}(e) &:= \sum_{J\in(\mathbb{P}_n)^k} \left|\det((e(J_{i,1})-e(J_{i,2}))_{i\in[k]})\right| > 0\,, \qquad (10) \\ \mathrm{SysEm}(\rho,k) &:= \left(\emptyset, \{\, w^I_{\rho,k} : I \in (\mathbb{P}_n)^{k+1}\,\}, \{s_{n,k}\}\right)\,. \qquad (11) \end{aligned}$$

Note that (9) is equivalent to (6) for $U$ defined by (4) in view of (8). (10) is just one possibility to ensure that $\dim U = k$.

**Corollary 8.** *The metric space $([n],\rho)$, $n \geq 2$, can be embedded into a suitable Minkowski space $\mathbb{M}^d$ of fixed dimension $d \geq 1$ if and only if for at least one $k \in [d]$ the system* $\mathrm{SysEm}(\rho,k)$ *is admissible. The set of all d-dimensional embeddings $e : [n] \to \mathbb{R}^d$, i.e., satisfying* $\dim \operatorname{aff} e([n]) = d$, *is exactly the solution set of* $\mathrm{SysEm}(\rho,d)$.

The first part of Corollary 8 can be strengthened a little bit since we can ignore lower-dimensional embeddings.

**Proposition 9.** *The metric space $([n],\rho)$, $n \geq d+1$, can be embedded into a suitable Minkowski space $\mathbb{M}^d$ of dimension $d \geq 1$ if and only if the system* $\mathrm{SysEm}(\rho,d)$ *is admissible.*

*Proof of Proposition* 9. We show that if there is a lower dimensional embedding, then also a full dimensional embedding can be constructed. Assume that $e : ([n],\rho) \to \mathbb{M}^d_o$, $i \mapsto \mathbf{s}_i$ is an embedding where $L := \operatorname{aff}\{\mathbf{s}_1,\dots,\mathbf{s}_n\}$ has dimension $\dim L < d$. Without loss of generality we can assume that $L$ is a linear subspace, i.e., $\mathbf{0} \in L$. Otherwise we can consider the translation $e' := e - e(1)$ instead of $e$, which is an embedding of $([n],\rho)$ into $\mathbb{M}^d_o$, too. Next we consider an inclusion maximal affinely independent set $S \subset \{\mathbf{s}_1,\dots,\mathbf{s}_n\}$. $S$ has exactly $\dim L + 1 < n$ elements. Thus there is some $k \in [n]$ with $\mathbf{s}_k \notin S$. Note that $\mathbf{s}_i \neq \mathbf{s}_j$ for all $i \neq j$ since $\rho(i,j) > 0$.

We will extend the unit ball $\tilde{B} := B(\mathbb{M}^d_o) \cap L$ of the linear subspace $L$ of $\mathbb{M}^d_o$ to a unit ball in some linear subspace $L'$ of $\mathbb{R}^d$ with dimension $\dim L' = \dim L + 1$. For this we fix any direction $\mathbf{x} \in \mathbb{R}^d \setminus L$ and define $B := \tilde{B} + \overline{(-\mathbf{x})\mathbf{x}}$ as prism over $\tilde{B}$, and $L' := \operatorname{lin} B$. Now we shift $\mathbf{s}_k$ a little bit in direction $\mathbf{x}$ to $\mathbf{s}'_k := \mathbf{s}_k + \epsilon\mathbf{x}$. If $\epsilon > 0$ is small enough, all lengths stay the same, $\|\mathbf{s}_k - \mathbf{s}_j\|_{\tilde{B}} = \|\mathbf{s}_k - \mathbf{s}_j\|_B = \|\mathbf{s}'_k - \mathbf{s}_j\|_B$ for all $j \in [n] \setminus \{k\}$, but $\dim \operatorname{aff}\{\mathbf{s}_1,\dots,\mathbf{s}'_k,\dots,\mathbf{s}_n\} = \dim L + 1$.

If $\dim L' = \dim L + 1 < d$, then we repeat this procedure $d - 1 - \dim L$ times. We obtain an embedding with full dimension $d$, and thus a solution of $\mathrm{SysEm}(\rho,d)$. □

### 2.2. Transformation to polynomial systems

We will transform the system $\mathrm{SysEm}(\rho, d)$ into other systems

- which have only homogeneous polynomial restrictions of maximal degree two,
- which have only linear inequality restrictions, i.e., non-linear restrictions are only allowed as equations, and
- whose solutions represent equivalence classes of affinely equivalent embeddings up to scaling. More precisely, if we apply a reversible affine transformation to $e$, then both embeddings correspond to the same solution of the analytical system up to scalar multiplication by a nonzero real number.

#### 2.2.1. Omitting absolute values of sub-expressions

For the first point, we replace all terms $|T|$ by the term $\mathrm{fsign}_T \cdot T$. If we can assure that $\mathrm{fsign}_T = \operatorname{sign} T$, then $|T| = \mathrm{fsign}_T T$ and the restrictions stay the same. The corresponding numbers $\mathrm{fsign}_T \in \{-1, 0, 1\}$ are introduced as parameter. The condition $\mathrm{fsign}_T = \operatorname{sign} T$ is equivalent to $\mathrm{fsign}_T T > 0$ if $\mathrm{fsign}_T \neq 0$, and to $T = 0$ if $\mathrm{fsign}_T = 0$. For the function set $F$ we denote by $F_{A \to B}$ the set of functions defined by expressions which are modified expressions[1] of functions in $F$ by replacing each occurrence of $A$ by $B$.

**Lemma 10.** *The system $\mathcal{S} = (E, W, S)$ is admissible if and only if at least one of the three systems $\mathcal{S}_{T+} := (E_{|T| \to T}, W_{|T| \to T}, S_{|T| \to T} \cup \{T\})$, $\mathcal{S}_{T-} := (E_{|T| \to -T}, W_{|T| \to -T}, S_{|T| \to -T} \cup \{-T\})$, and $\mathcal{S}_{T0} := (E_{|T| \to 0} \cup \{T\}, W_{|T| \to 0}, S_{|T| \to 0})$ is admissible. Note that we identify the expression $T$ with the function evaluating this expression. The solution set is the union of the three pairwise disjoint solution sets of the replaced systems, $L(\mathcal{S}) = L(\mathcal{S}_{T+}) \dot\cup L(\mathcal{S}_{T-}) \dot\cup L(\mathcal{S}_{T0})$.*

For $\mathrm{SysEm}(\rho, d)$ we have to apply Lemma 10 several times where $T = \det((e(J_{i,1}) - e(J_{i,2}))_{i \in [d]})$ for $J \in (\mathbb{P}_n)^d$. This yields finitely many systems with only homogeneous polynomial restrictions.

#### 2.2.2. Using Plücker coordinates

Now we introduce new variables to get systems which are linear in its inequality restrictions. The new variables $\underline{b}_I$ are indexed by $I := (I_1, \ldots, I_d) \in \mathbb{N}^d$, satisfying $1 \leq I_1 < I_2 < \cdots < I_d < n$, with the intended meaning $\underline{b}_I = \underline{b}_I^e$ for some $e : [n] \to \mathbb{R}^d$, where

$$\underline{b}_I^e := \det((e(I_i) - e(n))_{i \in [d]}) = \det(e(I_1) - e(n), \ldots, e(I_d) - e(n)) \,. \tag{12}$$

So the set of indices is $\mathrm{Seq}_{d,n} := \{\, (I_1, \ldots, I_d) \in \mathbb{N}^d : 1 \leq I_1 < I_2 < \cdots < I_d < n \,\}$ and we can consider $\underline{b} \in \mathbb{R}^{\mathrm{Seq}_{d,n}}$ which is identified with $\mathbb{R}^{m(d,n)}$ since $\left|\mathrm{Seq}_{d,n}\right| = \binom{n-1}{d} =: m(d, n)$.

[1]The considered expression for a function must be fixed and will be clear from the context.

The vector $\underline{b}^e$ is invariant under bijective affine transformations of $e$ up to a scalar multiple $c \in \mathbb{R} \setminus \{0\}$: for the affine map $a : \mathbb{R}^d \to \mathbb{R}^d$ we have $\underline{b}^{a \circ e} = c\underline{b}^e$.

The variables $\underline{b}_I$ can be considered as the Plücker coordinates of an appropriately chosen element $h$ of the Grassmann variety $G_d(\mathbb{R}^{n-1})$. The $d$-dimensional linear subspace $h$ of $\mathbb{R}^{n-1}$ can be identified with an equivalence class of all labeled $n$-point sets in $\mathbb{R}^d$ with respect to affine transformations. For $e$ and $\{e(1), \dots, e(n)\}$ we first consider a translation by $-e(n)$, form column-wise the matrix $M$ of all non-trivial obtained vectors $\big[e(1) - e(n)| \dots |e(n-1) - e(n)\big]$. The linear span of all rows of $M$ is $h \in G_d(\mathbb{R}^{n-1})$. The usual way to parameterize $G_d(\mathbb{R}^{n-1})$ in the $(\binom{n-1}{d} - 1)$-dimensional real projective space is by using all $d \times d$-minors of an $(n-1) \times d$-matrix whose columns form a basis of $h$, such as the matrix $M$.

From the structure of $\mathrm{SysEm}(\rho, d)$ it follows that we can express all its restriction by using $\underline{b}^e_I$, $I \in \mathrm{Seq}_{d,n}$, only instead of $e$ itself.

To see this, we define in two steps for each $J \in ([n]^2)^d$ a linear functional $b(J) : \mathbb{R}^{\mathrm{Seq}_{d,n}} \to \mathbb{R}$ mapping $\underline{b}$ to the value of the corresponding determinant, i.e., with

$$b(J)(\underline{b}^e) = \det((e(J_{i,1}) - e(J_{i,2}))_{i \in [d]}) \,. \tag{13}$$

This generalizes the variables via $\underline{b}_{(I_1,\dots,I_d)} = b((I_1, n), (I_2, n), \dots, (I_d, n))(\underline{b})$.

**Definition 11.** *For $I = (I_1, \dots, I_d) \in [n]^d$ and $J := (I_i, n)_{i \in [d]} \in ([n]^2)^d$ we define*

1. *$b(J) := 0$ if $|\{I_1, I_2, \dots, I_d, n\}| < d + 1$, i.e., if $I_i = n$ for some $1 \le i \le d$, or if $I_i = I_j$ for some $1 \le i < j \le n$, and otherwise*
2. *$b(J)(\underline{b}) := \operatorname{sign}\delta \cdot \underline{b}_{\delta(I)}$ if $\delta : [d] \to [d]$ denotes the unique permutation which sorts $I$ strictly monotone increasing, i.e., with $\delta(I_i) < \delta(I_{i+1})$ for all $1 \le i < d$. Thus $\delta(I) := (\delta(I_i))_{i \in [d]} \in \mathrm{Seq}_{d,n}$. As usual, the sign $\operatorname{sign}\delta$ of $\delta$ is $+1$ $(-1)$ if there is an even (odd, respectively) number of transpositions (permutation interchanging exactly two elements) whose composition is $\delta$.*

Note that case 2 trivially includes $b(J)(\underline{b}) := \underline{b}_I$ if $I \in \mathrm{Seq}_{d,n}$ using the identity for $\delta$.

Now we have defined $b(J)$ for all $J \in ([n] \times \{n\})^d$ as linear functionals on $\mathbb{R}^{\mathrm{Seq}_{d,n}}$, which we already identified with $\mathbb{R}^{m(d,n)}$.

Now for arbitrary $J = ((J_{1,1}, J_{1,2}), (J_{2,1}, J_{2,2}), \dots, (J_{d,1}, J_{d,2})) \in ([n]^2)^d$ we define

$$b(J) := \sum_{K \in \{1,2\}^d} (-1)^{d + \sum K} b(((J_{i,K_i}, n))_{i \in [d]}) \,, \tag{14}$$

where $\sum K = \sum_{i=1}^d K_i$ stands for the sum of the components of $K \in \{1,2\}^d$.

Note that (14) "redefines" $b(J)$ for $J_{1,2} = J_{2,2} = \cdots = J_{d,2} = n$ as the sum of $b(J)$ itself (for $K = (1, \dots, 1)$) and of $2^d - 1$ zeros for all other $K$.

**Proposition 12.** *For all $J \in ([n]^2)^d$ the linear functional $b(J)$ defined by Definition 11 and by (14) has the property (13) regarding its value for $\underline{b}^e$ which was constructed from $n$ labeled points in $\mathbb{R}^d$, $e : [n] \to \mathbb{R}^d$.*

*Additionally, $b(J)$ is antisymmetric and "linear" in each pair $J_i$ of $J$:*

$$b(\delta(J)) = \operatorname{sign}\delta \cdot b(J) \qquad \forall J \in ([n]^2)^d,\ \delta : [d] \to [d] \text{ bijective} \tag{15}$$

$$b((j,i),I) = -b((i,j),I) \qquad \forall i,j \in [n],\ I \in ([n]^2)^{d-1} \tag{16}$$

$$b((i,k),I) = b((i,j),I) + b((j,k),I) \qquad \forall i,j,k \in [n],\ I \in ([n]^2)^{d-1}. \tag{17}$$

Conversely, for every vector $\underline{b} \in \mathbb{R}^{\mathrm{Seq}_{d,n}} \equiv \mathbb{R}^{m(d,n)}$ there is a function $e : [n] \to \mathbb{R}^d$ with $\underline{b} = \underline{b}^e$ provided $\underline{b}$ satisfies the well known Grassmann-Plücker relations, see e.g. [7]. For $I \in ([n]^2)^{d+1}$, $J \in ([n]^2)^{d-1}$ we obtain

$$0 = \mathrm{GPR}_{I,J}(\underline{b}) := \sum_{j=1}^{d+1} (-1)^j \cdot b((I_i)_{i\in[d+1]\setminus\{j\}})(\underline{b}) \cdot b(I_j, J)(\underline{b})\,, \tag{18}$$

$$E_{\det}(n,d) := \{\, \mathrm{GPR}_{I,J} : I \in ([n]^2)^{d+1}, J \in ([n]^2)^{d-1} \,\}\,, \tag{19}$$

$$\mathrm{SysDet}(n,d) := (E_{\det}(n,d), \emptyset, \emptyset)\,. \tag{20}$$

**Lemma 13.** *The solution set of* $\mathrm{SysDet}(n,d)$ *is exactly the set of all determinants*

$$L(\mathrm{SysDet}(n,d)) = \{\, \underline{b}^e | e : [n] \to \mathbb{R}^d \,\}\,.$$

**Remark 14.** Instead of a complete proof we will only construct a function $e : [n] \to \mathbb{R}^d$ from a given $\underline{b} \in L(\mathrm{SysDet}(n,d))$, $\underline{b} \neq 0$, such that $\underline{b} = \underline{b}^e$. For our construction we choose some $o \in [n]$ with $e(o) = \mathbf{0}$ and for $i = 1,\dots,d$ some $a_i \in [n]$ which is mapped to the $i$-th unit vector scaled by $c \neq 0$.

Since $\underline{b} \neq 0$ we can find $o, a_1, \dots, a_d \in [n]$ with $c := b(C)(\underline{b}) \neq 0$ for $C := (a_i, o)_{i\in[d]}$.

We first construct $\tilde{e}^{C,\underline{b}} : [n] \to \mathbb{R}^d$ via

$$\begin{aligned}\tilde{e}^{C,\underline{b}}(x) := (&b((x,o),(a_2,o),\dots,(a_d,o))(\underline{b}),\\ &b((a_1,o),(x,o),(a_3,o),\dots,(a_d,o))(\underline{b}),\\ &\vdots\\ &b((a_1,o),\dots,(a_{d-1},o),(x,d))(\underline{b}))\,.\end{aligned}$$

The $i$-th coordinate of $\tilde{e}^{C,\underline{b}}(x)$ is $b(C^{(x,o),i})(\underline{b})$, where the sequence $C^{(x,y),i} \in ([n]^2)^d$ is almost $C$, except for the $i$-th pair which is $(x,y)$, $C_i^{(x,y),i} := (x,y)$ and $C_l^{(x,y),i} := C_l = (a_l, o)$ for $l \in [d] \setminus \{i\}$. This results in $\underline{b}^{\tilde{e}^{C,\underline{b}}} = c^{d-1}\underline{b}$.

To correct this scalar factor we define the function $e$ for $x \in [n]$ as

$$e(x) := \left( \tilde{e}^{C,\underline{b}}(x)_1, \frac{\tilde{e}^{C,\underline{b}}(x)_2}{c}, \dots, \frac{\tilde{e}^{C,\underline{b}}(x)_d}{c} \right)\,. \tag{21}$$

The first coordinate of $e(x)$ is the same as of $\tilde{e}^{C,\underline{b}}(x)$, and the remaining coordinates of $e(x)$ are the ones of $\tilde{e}^{C,\underline{b}}(x)$ divided by $c$. This gives $\underline{b}^e = \underline{b}$.

**Remark 15.** Note that due to our definition the condition $0 = \mathrm{GPR}_{I,J}(\underline{b})$ in (18) is invariant under permutations of the pairs $I_1, I_2, \dots, I_{d+1}$ and also under permutations of the pairs $J_1, J_2, \dots, J_{d-1}$. More precisely, for two permutations $\alpha : [d+1] \to [d+1]$, $\beta : [d-1] \to [d-1]$ and $I \in ([n]^2)^{d+1}$, $J \in ([n]^2)^{d-1}$ we have

$$\mathrm{GPR}_{(I_{\alpha(i)})_{i\in[d+1]},(J_{\beta(i)})_{i\in[d-1]}} = \operatorname{sign}\alpha \cdot \operatorname{sign}\beta \cdot \mathrm{GPR}_{I,J} \ .$$

Additionally, replacing a pair $(i,j)$ by $(j,i)$ of $I$ or of $J$ again yields the same condition since $\mathrm{GPR}_{((I_{1,2},I_{1,1}),I_2,\dots,I_{d+1}),J} = \mathrm{GPR}_{I,((J_{1,2},J_{1,1}),J_2,\dots,J_{d-1})} = -\mathrm{GPR}_{I,J}$.

For the particular case of $d = 2$ the pairs in $I$ may also be exchanged with the pair in $J$:

$$\mathrm{GPR}_{(I_1,I_2,I_3),(J_1)} = -\mathrm{GPR}_{(J_1,I_2,I_3),(I_1)} \quad (I_1, I_2, I_3, J_1 \in [n]^2)\,.$$

For $d \geq 3$ such a statement is not true in general.

From Remark 15 and the linearity of the Grassmann-Plücker relations with respect to the components of $I$ and $J$ we obtain the following simplification.

**Corollary 16.** $\mathrm{SysDet}(n,d)$ *is equivalent to* $\mathrm{SysDetRed}(n,d) := (E_{\mathrm{det,red}}(n,d), \emptyset, \emptyset)$ *with*

$$\begin{aligned} E_{\mathrm{det,red}}(n,d) &:= \big\{ \mathrm{GPR}_{I,J} : I \in ([n-1]\times\{n\})^{d+1}, J \in ([n-1]\times\{n\})^{d-1}, \\ &\qquad I_{1,1} < I_{2,1} < \dots < I_{d+1,1}, J_{1,1} < J_{2,1} < \dots < J_{d-1,1} \big\} \text{ if } d \geq 3, \quad (22) \\ E_{\mathrm{det,red}}(n,2) &:= \big\{ \mathrm{GPR}_{((a,n),(b,n),(c,n)),(d,n)} : 1 \leq a < b < c < d < n \big\}\,. \quad (23) \end{aligned}$$

Note that we can further reduce the number of equations in $E_{\mathrm{det,red}}(n,d)$ provided that $\underline{b}(1,2,\dots,d) \neq 0$, e.g., $L(E_{\mathrm{det}}(n,2), \emptyset, \{\underline{b}(1,2)\}) = L(\{\mathrm{GPR}_{((1,n),(2,n),(c,n)),(d,n)} : 3 \leq c < d < n\}, \emptyset, \{\underline{b}(1,2)\})$.

### 2.2.3. Equivalent embedding system

Following the ideas of 2.2.1, we use the notation $\mathrm{fsign}(J) := \mathrm{fsign}_{b(J)}$ for $J \in ([n]^2)^d$. So fsign is a function $([n]^2)^d \to \{-1,0,1\}$. Together with the transformation and restrictions obtained in 2.2.2 and the standard way to eliminate fractions we obtain from the system $\mathrm{SysEm}(\rho,d)$ the following family of equations and inequalities in $\mathbb{R}^{m(d,n)}$ whose restrictions are polynomials of degree at most two for equations and at most one for the inequalities. For $I \in (\mathbb{P}_n)^{d+1}$ and $J \in (\mathbb{P}_n)^d$ we get and define the following:

$$\begin{aligned} 0 \leq w^I_{\rho,d,\mathrm{fsign}}(\underline{b}) &:= -\rho(I_{d+1}) \cdot \mathrm{fsign}((I_i)_{i\in[d]}) \cdot b((I_i)_{i\in[d]})(\underline{b}) + \\ &\quad + \sum_{j\in[d]} \rho(I_j) \cdot \mathrm{fsign}((I_i)_{i\in[d+1]\setminus\{j\}}) \cdot b((I_i)_{i\in[d+1]\setminus\{j\}})(\underline{b}), \quad (24) \\ 0 < s^J_{n,d,\mathrm{fsign}}(\underline{b}) &:= \mathrm{fsign}(J) \cdot b(J)(\underline{b}) \qquad \text{if } \mathrm{fsign}(J) \neq 0, \quad (25) \end{aligned}$$

$$\begin{aligned}
0 = e^J_{n,d} &:= b(J) \qquad \text{if } \operatorname{fsign}(J) = 0, && (26)\\
E_{\text{sign}}(d, \operatorname{fsign}) &:= \{e^J_{n,d} : J \in (\mathbb{P}_n)^d, \operatorname{fsign}(J) = 0\}, && (27)\\
W_{\text{conv}}(\rho, d, \operatorname{fsign}) &:= \{\, w^I_{\rho,d,\text{fsign}} : I \in (\mathbb{P}_n)^{d+1}\}, && (28)\\
S_{\text{sign}}(d, \operatorname{fsign}) &:= \{\, s^J_{n,d,\text{fsign}} : J \in (\mathbb{P}_n)^d, \operatorname{fsign}(J) \neq 0\}, && (29)\\
\operatorname{SysEmD}(\rho, d, \operatorname{fsign}) &:= (E_{\text{sign}}(d, \operatorname{fsign}) \cup E_{\text{det,red}}(n, d), \\
&\qquad W_{\text{conv}}(d, \operatorname{fsign}, \rho), S_{\text{sign}}(d, \operatorname{fsign}))\,. && (30)
\end{aligned}$$

Note that $w^I_{\rho,d}(e) = \frac{w^I_{\rho,d,\text{fsign}^*}(\underline{b}^e)}{\Pi_{i\in[d+1]}\rho(I_i)}$ if we choose $\operatorname{fsign}^* : J \mapsto \operatorname{sign}(b(J)(\underline{b}^e))$. The condition $s_{n,d}(e) > 0$ is implicitly represented by $0 < s^J_{n,d,\text{fsign}^*}(\underline{b}^e)$ if for at least one $J \in (\mathbb{P}_n)^d$ we have that $\operatorname{fsign}^*(J) \neq 0$, i.e., if we assure that $\operatorname{fsign}^* \not\equiv 0$.

So we get that

$$\underline{b}^{\cdot}(L(\operatorname{SysEm}(\rho, d))) = \bigcup_{\operatorname{fsign}:(\mathbb{P}_n)^d\to\{-1,0,1\},\operatorname{fsign}\not\equiv 0} L(\operatorname{SysEmD}(\rho, d, \operatorname{fsign})) \tag{31}$$

where $\underline{b}^{\cdot} : \mathbb{R}^{nd} \to \mathbb{R}^{m(d,n)}, e \mapsto \underline{b}^e$ denoted the mapping of labeled points to its Plücker coordinates.

Two functions $f, g : [n] \to \mathbb{R}^d$ are called *affinely equivalent* if there is some affinely linear bijection $a : \mathbb{R}^d \to \mathbb{R}^d$ with $f = a \circ g$, i.e., $f(i) = a(g(i))$ for all $i \in [n]$. This describes an equivalence relation in the set $\left(\mathbb{R}^d\right)^{[n]}$. We identify $\left(\mathbb{R}^d\right)^{[n]}$ with $\mathbb{R}^{dn}$.

Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ are called *positive equivalent* (*direction equivalent*) if there is some $\lambda > 0$ ($\lambda \neq 0$, respectively) with $\mathbf{x} = \lambda\mathbf{y}$. This describes two equivalence relations in the set $\mathbb{R}^m$. We summarize our results with the following theorem.

**Theorem 17.** *The metric space* $([n], \rho)$ *with* $n \geq d + 1$ *can be embedded into a suitable Minkowski space* $\mathbb{M}^d$ *if and only if there is a nontrivial (i.e., not equal to zero) function* $\operatorname{fsign} : (\mathbb{P}_n)^d \to \{-1, 0, 1\}$ *(called formal sign function) such that the homogeneous polynomial system* $\operatorname{SysEmD}(\rho, d, \operatorname{fsign})$ *of equations and inequalities in* $\mathbb{R}^{m(d,n)}$, $m(d, n) = \binom{n-1}{d}$, *is admissible.*

*Then equation* (31) *holds true and there is a one-to-one correspondence between*

1. *all affine equivalence classes (i.e., equivalence classes with respect to affine equivalence) of embeddings* $e : [n] \to \mathbb{R}^d$ *of* $([n], \rho)$ *into a suitable Minkowski space* $\mathbb{M}^d$ *which are full dimensional (i.e.,* $\dim \operatorname{aff} e([n]) = d$*), and*
2. *all direction equivalence classes (i.e., equivalence classes with respect to direction equivalence) in the union of* $L(\operatorname{SysEmD}(\rho, d, \operatorname{fsign}))$ *for all* $\operatorname{fsign} \in \{-1, 0, 1\}^{(\mathbb{P}_n)^d}$ *with* $\operatorname{fsign} \not\equiv 0$.

We note that a direction equivalence class $C$ in the union of $L(\operatorname{SysEmD}(\rho, d, \operatorname{fsign}))$, $\operatorname{fsign} \not\equiv 0$, has the form $C = [\underline{b}]_> \cup [-\underline{b}]_>$. Here $[\underline{b}]_>$ denotes the equivalence class of $\underline{b} \in \mathbb{R}^{m(d,n)}$ with respect to positive equivalence. So $C$ is connected with two systems $\operatorname{SysEmD}(\rho, d, \operatorname{fsign})$ and $\operatorname{SysEmD}(\rho, d, -\operatorname{fsign})$, where $\operatorname{fsign} = \operatorname{sign} \underline{b}$ and

$\underline{b}$ is a solution of $\mathrm{SysEmD}(\rho, d, \mathrm{fsign})$. Thus, the affine equivalence classes of full dimensional embeddings are in one-to-one relation to all positive equivalence classes of the union of $L(\mathrm{SysEmD}(\rho, d, \mathrm{fsign}))$, where fsign traverses a subset of all non-trivial formal sign functions $\mathrm{fsign} : (\mathbb{P}_n)^d \to \{0, \pm 1\}$ which contains of all pairs fsign, $-$ fsign exactly one representative.

Using standard combinatorial calculations we obtain the following corollary.

**Corollary 18.** *The $m = m(d,n) = \binom{n-1}{d}$-dimensional system $\mathrm{SysEmD}(\rho, d, \mathrm{fsign})$ has $|E_{\mathrm{sign}}(d, \mathrm{fsign})| + |S_{\mathrm{sign}}(d, \mathrm{fsign})| + |W_{\mathrm{conv}}(d, \mathrm{fsign}, \rho)| \leq \binom{n}{2}^d + \binom{n}{2}^{d+1} \in O(n^{2d+2})$ linear restrictions and additional at most*

$$|E_{\mathrm{det,red}}(n,d)| \leq \binom{n-1}{d+1}\binom{n-1}{d-1} \in O(n^{2d})$$

*polynomial restrictions of degree at most 2 as equations. For $d = 2$ we get the stronger bound*

$$|E_{\mathrm{det,red}}(n,2)| \leq \binom{n-1}{4}.$$

*Especially for $n < d+3$, the system $\mathrm{SysEmD}(\rho, d, \mathrm{fsign})$ is in fact a homogeneous linear system.*

**Remark 19.** Note that by Wolfe [8] for each metric $\rho$ with $n \leq d+2$, there is at least one fsign such that the system $\mathrm{SysEmD}(\rho, d, \mathrm{fsign})$ is admissible since all metric spaces with $d+2$ points can be embedded into the $d$-dimensional $\ell_\infty$-space.

For $d = 2$ and $n = 5$ we get that $|E_{\mathrm{det,red}}(5,2)| \leq 1$. This allows to decide admissibility of $\mathrm{SysEmD}(\rho, d, \mathrm{fsign})$ using the well developed technique of quadratic programming.

**Remark 20.** We call $(E', W', S')$ a subsystem of the system $(E, W, S)$ if $E' \subset E$, $W' \subset W$ and $S' \subset S$. Obviously, for the solution set $L'$ and $L$ the converse relation $L \subset L'$ holds.

For the embedding systems $\mathrm{SysEmD}(\rho, d, \mathrm{fsign})$ there are "natural" subsystems, namely the embedding systems $\mathrm{SysEmD}(\rho|_X, d, \mathrm{fsign}|_X)$ of metric subspaces $(X, \rho|_X)$ of $([n], \rho)$, $X \subsetneq [n]$. These subsystems are smaller in both the number of restrictions and also the dimension.

For the classification of 2-distance configurations it was very important to use the information obtained from the solution of the subsystem to simplify the system $\mathrm{SysEmD}(\rho, d, \mathrm{fsign})$ and also to reduce the number of formal sign functions fsign to be considered. From admissible systems $\mathrm{SysEmD}(\rho|_X, d, \mathrm{fsign}|_X)$ we can obtain information about redundant restrictions, reducing the total number of restrictions, and about implicit equations, reducing the dimension of the system to be solved.

### 2.3. Algorithmic solvability

From Theorem 17 and the theory of real closed fields, see e.g. [9], we get

**Corollary 21.** *The general decision problem on embedding* $([n], \rho)$ *in* $\mathbb{M}^d$ *(Task 1) is algorithmically decidable if* $\rho$ *is given by rational functions on some parameters belonging to a given semi-algebraic set* $P$*. This includes the case where* $\rho$ *is explicitly given by* $\binom{n}{2}$ *algebraic numbers which can be achieved with* $|P| = 1$*. If* $\rho$ *is given by* $\binom{n}{2}$ *rational numbers than this is of course possible without any parameters. For* $|P| > 1$ *the answer to the decision problem is in general the partition of* $P = P_1 \dot{\cup} P_2$ *into two semi-algebraic sets* $P_1$ *and* $P_2$ *such that for* $p \in P_1$ *there is an embedding of* $([n], \rho(p))$ *into a suitable Minkowski space and for* $p \in P_2$ *there are no such embeddings.*

Note that for arbitrary instances of this embedding problem we cannot expect to get an answer to this decision problem by computers from today, due to the enormous costs (time and memory) of this approach.

**Corollary 22.** *The general description problem on embedding* $([n], \rho)$ *in* $\mathbb{M}^d$ *(Task 3) is algorithmically solvable if* $\rho$ *is given by rational functions on a* $k$*-dimension real parameter vector* $(p_1, \ldots, p_k)$ *belonging to a given semi-algebraic set* $P \subset \mathbb{R}^k$*. In general, the description is based on a finite set* $C$ *of semi-algebraic cells (i.e., semi-algebraic subsets) of* $\mathbb{R}^{K+k} = \mathbb{R}^K \times \mathbb{R}^k$ *for some integer* $K \geq 0$*. For example, it could be given as a subset of all cells of a cylindrical decomposition (CAD) of* $\mathbb{R}^{K+k}$ *with* $K = m(d, n)$*. Then each point* $(x, p)$ *within a cell* $c$ *of* $C$ *corresponds to an affine equivalence class* $[e]$ *of embeddings* $e$ *of* $([n], \rho(p))$ *into a suitable Minkowski space* $\mathbb{M}^d$*. A suitable* $e = e(c, x, p)$ *can be constructed from* $(x, p)$ *using polynomials over* $\mathbb{Z}$ *as coordinates. The unit ball of a corresponding suitable* $\mathbb{M}^d = \mathbb{M}^d(c, x, p)$ *can be constructed from* $(x, p)$ *as the convex hull of finitely many points whose coordinates are polynomials over* $\mathbb{Z}$ *in* $(x, p)$*. All these polynomials depend only on the cell* $c$ *but not on* $(x, p)$ *itself.*

*For each* $p \in P$*, each affine equivalence class of embeddings* $e$ *of* $([n], \rho(p))$ *into a suitable Minkowski space* $\mathbb{M}^d$ *is represented exactly once by some* $c \in C$ *and* $x \in \mathbb{R}^k$ *with* $(x, p) \in c$ *as* $[e(c, x, p)]$*.*

Note that similar statements also hold for the special decision problem on embedding $([n], \rho)$ in $\mathbb{M}^d(B)$, Task 2, and its description version. For these problems the corresponding Minkowski space must be given by a semi-algebraic description of the boundary of its unit ball. Note that for rational $\rho$ and polytopal unit balls $B$ with rational description these problems can be solved without quantifier elimination methods.

## 3. Relative position of points

There are two important needs to formalize the concept of the "relative position" which a configuration of $n$ labeled points can have.

Our first motivation is to obtain a classification of 2-distance sets for which the relative position of the points is beneath the induced metric of the points one criterion to distinguish "different" 2-distance configurations.

The second motivation originates from Section 2.2.1 resulting in Theorem 17. It turns out that the formal sign function fsign : $(\mathbb{P}_n)^d \to \{-1, 0, 1\}$ there is in fact an analytical description of the relative position of the points $e(1), \dots, e(n) \in \mathbb{R}^d$. Thus the general embedding problem can be transformed into one polynomial system for each fixed relative position.

**Definition 23.** *The oriented relative position of the $n$ labeled points $e(1), \dots, e(n)$ with $e : [n] \to \mathbb{R}^d$ is the function* fsign : $(\mathbb{P}_n)^d \to \{0, \pm 1\}$*,* $J \mapsto \mathrm{sign}(b(J)(\underline{b}^e))$*, see* (13)*. The relative position of the $n$ labeled points is the unordered pair* $\{\mathrm{fsign}, -\mathrm{fsign}\}$ *of oriented relative position $e$ and the reorientation* $-\mathrm{fsign}$ *of* fsign*.*

**Remark 24.** Note that the previous definition implicitly assumes that the points $e(1), \dots, e(n)$ affinely span $\mathbb{R}^d$, i.e., that the oriented relative position is not identical to zero. If $k := \dim \mathrm{aff}\, e([n]) < d$, then there is no natural definition of an oriented relative position. But the relative position can be easily defined using an arbitrary bijective affinely linear transformation from $\mathrm{aff}\, e([n])$ to $\mathbb{R}^k$.

**Remark 25.** For $d = 2$ and the full-dimensional case we have that $\mathrm{fsign}((x, y), (z, w)) = +1$ if and only if there are no parallel lines containing $e(x), e(y)$ and $e(z), e(w)$, respectively, and the oriented angle between the vectors $e(x) - e(y)$ and $e(z) - e(w)$ is between 0 and $\pi$.

**Remark 26.** There is another basic concepts for the relative position of $n$ points in $\mathbb{R}^d$: the acyclic oriented matroid of affine dependencies of a point configuration. The *chirotope*, also called basis orientation, of the point configuration $(e(1), \dots, e(n))$ in $\mathbb{R}^d$ is the antisymmetric function

$$\chi(i_1, i_2, \dots, i_{d+1}) := \ \mathrm{sign} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ e(i_1) & e(i_2) & \cdots & e(i_{d+1}) \end{pmatrix} \in \{0, \pm 1\}, \qquad i_1, i_2, \dots, i_{d+1} \in [n]\,.$$

Then we can represent the oriented matroid of the point configuration $(e(1), \dots, e(n))$ by the pair $\{\chi, -\chi\}$, see also [7]. Definition 23 introduces a concept of relative position which allows a finer distinction between labeled sets of points since $\chi(i_1, i_2, \dots, i_{d+1}) = \mathrm{fsign}((i_2, i_1), (i_3, i_1), \dots, (i_{d+1}, i_1))$. Roughly speaking, for $d = 2$ this concept adds to the chirotope the information whether or not two lines – each defined by containing two points of the configuration – intersect. If they intersect, this concept also describes the ordering of the intersection point and the two defining points along each of the two lines. But also the relative position by Definition 23 can be described by another oriented matroid: the "big oriented matroid" of the points as described in [7].

### 3.1. Relative position in the plane

The oriented relative position of $n$ distinct labeled points in the plane can also be represented by a circular sequence of permutations, also called *allowable sequence*, see [10]. We will use a modification of this idea by omitting the requirement that there is one linear projection $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $\phi(e(1)) < \phi(e(2)) < \cdots < \phi(e(n))$. The advantages over the function $e$ are the amount of stored data, a simple human-readable representation (of course, pictures are even better) and a natural way to represented partially known relative positions. Note that we assume for this representation that $e$ is injective, which was not necessary for the representation by $\pm\,\mathrm{fsign}$.

**Definition 27.** *The direction set of $e : [n] \to \mathbb{R}^2$ with respect to the direction $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^2$ is*

$$\mathrm{dirs}(e, \mathbf{v}) := \{\, (i, j) \in \mathbb{P}_n : \exists \lambda > 0 : e(i) - e(j) = \lambda \mathbf{v} \,\} .$$

*The oriented position sequence of $e : [n] \to \mathbb{R}^2$ is the cyclic sequence of non-empty direction sets $\mathrm{dirs}(e, \mathbf{v})$ as $\mathbf{v}$ is rotated anti-clockwise around the origin. An oriented position list of $e : [n] \to \mathbb{R}^2$ is any complete sub-sequence $(\mathrm{dirs}(e, \mathbf{v}_1), \mathrm{dirs}(e, \mathbf{v}_2), \ldots, \mathrm{dirs}(e, \mathbf{v}_k))$ of the oriented position sequence where the direction vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ "cover" a linear half-plane of $\mathbb{R}^2$, i.e., if $\mathrm{dirs}(e, \mathbf{v}_1), \mathrm{dirs}(e, \mathbf{v}_2), \ldots, \mathrm{dirs}(e, \mathbf{v}_k), \mathrm{dirs}(e, -\mathbf{v}_1), \mathrm{dirs}(e, -\mathbf{v}_2), \ldots, \mathrm{dirs}(e, -\mathbf{v}_k)$ is one complete cycle of the oriented position sequence.*

We generalize these notion without knowing the points.

**Definition 28.** *An abstract direction set of order $n$ is any subset of $\mathbb{P}_n$. An (abstract) oriented position sequence of order $n$ is a cyclic sequence of non-empty abstract direction sets of order $n$ with finite cycle length. An (abstract) oriented position list $l$ of order $n$ is a finite sequence of non-empty abstract direction sets of order $n$. The oriented position sequence of $l = (l_1, \ldots, l_k)$ is the cyclic sequence $\mathrm{CycSeq}(l) := \ldots, l_1, l_2, \ldots, l_k, \mathrm{opp}(l_1), \mathrm{opp}(l_2), \ldots, \mathrm{opp}(l_k), l_1, \ldots,$ where $\mathrm{opp} : \mathbb{N}^2 \to \mathbb{N}^2$ is defined by $\mathrm{opp}(i, j) := (j, i)$. $l$ and $\mathrm{CycSeq}\, l$ are called complete oriented position list, or sequence, respectively, of order $n$, if each $(i, j) \in \mathbb{P}_n$ occurs exactly once in the sets $l_1, l_2, \ldots, l_k, \mathrm{opp}(l_1), \mathrm{opp}(l_2), \ldots, \mathrm{opp}(l_k)$.*

**Definition 29.** *To define for $s \in \mathbb{Z}$ the shifting-operation on abstract oriented position lists $l = (l_1, \ldots, l_k)$ we first introduce indices into $\mathrm{CycSeq}(l) = \ldots, l_{-1}, l_0, l_1, \ldots$ with $l_{2kz+i} := l_i$ and $l_{2kz+i+k} := \mathrm{opp}(l_i)$ for all $z \in \mathbb{Z}$ and $i \in [k]$. Then*

$$\mathrm{Shift}(l, s) := (l_{1+s}, l_{2+s}, \ldots, l_{k+s}) . \tag{32}$$

*The reorientation operation on abstract oriented positions lists is defined as*

$$\mathrm{Mirror}((l_1, \ldots, l_k)) := (l_k, l_{k-1}, \ldots, l_1) . \tag{33}$$

**Proposition 30.** *Assume that we are given two labeled sets $e_i : [n] \to \mathbb{R}^2$ of distinct points in the plane affinely spanning $\mathbb{R}^2$, $i = 1, 2$, and that $l^i$ are oriented position lists of $e_i$.*

1. *$e_1, e_2$ have the same oriented relative position if and only if they have the same oriented position sequence.*

2. *$l^1, l^2$ are complete oriented position lists of order $n$, each one consisting of more than one direction set.*
3. *$e_1, e_2$ have the same oriented relative position if and only if there is some $s \in \mathbb{Z}$ with $l^1 = \mathrm{Shift}(l^2, s)$.*
4. *$e_1, e_2$ have the same relative position if and only if there is some $s \in \mathbb{Z}$ with $l^1 = \mathrm{Shift}(l^2, s)$ or $l^1 = \mathrm{Shift}(\mathrm{Mirror}(l^2), s)$.*

Note that the description of relative position by equivalence classes of oriented position lists also works for collinear points in $\mathbb{R}^2$ and for $\mathbb{R}^1$ as well but in this case we loose the distinction of the two orientations.

**Proposition 31.** *Assume that we are given two labeled sets $e_i : [n] \to \mathbb{R}^d$ of distinct collinear points, $i = 1, 2$, $d \in \{1, 2\}$, $n > 1$. Assume that for $i = 1, 2$ the relative position of $e_i$ is $\{\pm\,\mathrm{fsign}^i\}$, and that $l^i$ are oriented position lists of $e_i$. In case $d = 1$ this means that $l^i = (\mathrm{dirs}(e_i, 1))$ or that $l^i = (\mathrm{dirs}(e_i, -1))$.*

1. *$e_1, e_2$ have the same relative position if and only if they have the same oriented position sequence.*
2. *$l^1, l^2$ are complete oriented position lists of order $n$, each one consisting of exactly one direction set.*
3. *$e_1, e_2$ have the same relative position if and only if there is some $s \in \{0, 1\}$ with $l^1 = \mathrm{Shift}(l^2, s)$.*
4. *$l^i = \mathrm{Mirror}(l^i)$ for $i = 1, 2$.*

**Proposition 32.** *The oriented relative position of $e : [n] \to \mathbb{R}^2$, $n > 1$, can be computed as $\mathrm{fsign}^l$ from any oriented position list $l = (l_1, \dots, l_k)$ of $e$. Denote $\mathrm{CycSeq}(l) = \dots, l_{-1}, l_0, l_1, \dots$ . Then if $k > 1$ the points are not all collinear, the affine dimension of $e(1), \dots, e(n)$ equals $d(l) := 2$, and for each $J \in (\mathbb{P}_n)^2$ there are integers $i, j$ with $0 \le j < 2k$ and $I_1 \in l_i$ and $I_2 \in l_{i+j}$. Then it holds*

$$\mathrm{sign}(b(J)(\underline{b}^e)) = \mathrm{fsign}^l(J) := \begin{cases} 0 & \text{if } j \in \{0, k\}\,, \\ 1 & \text{if } 0 < j < k\,, \\ -1 & \text{if } k < j < 2k\,. \end{cases}$$

*If $k = 1$, then the affine dimension of $e(1), \dots, e(n)$ equals $d(l) := 1$. We define $\mathrm{fsign}^l : \mathbb{P}_n \to \{0, \pm 1\}$ by $\mathrm{fsign}^l(J) := 1$ if $J \in l_1$ and $\mathrm{fsign}^l(J) := -1$ for all other $J \in \mathbb{P}_n$. Then $\{\pm\,\mathrm{fsign}^l\}$ is the relative position of $e$.*

Thus the embedding system corresponding to a metric $\rho$ on $[n]$ together with a complete oriented position list $l$ of order $n$ is $\mathrm{SysEmD}(\rho, d(l), \mathrm{fsign}^l)$.

**Lemma 33.** *The system $\mathrm{SysEmD}(\rho, d(l), \mathrm{fsign}^l)$ for $d(l) = 2$, where again $l = (l_1, \dots, l_k)$ with $l_i \subset \mathbb{P}_n$ for all $i \in [k]$, has the following implicit linear equations for $i \in [k]$, $A, B \in l_i$, $C \in \mathbb{P}_n$; $\underline{b} \in L(\mathrm{SysEmD}(\rho, d(l), \mathrm{fsign}^l))$:*

$$\begin{aligned} 0 = e_{\rho,l}^{A,B,C}(\underline{b}) \;&:=\; \rho(A) \cdot \mathrm{fsign}^l(B, C) \cdot b(B, C)(\underline{b}) \\ &\qquad -\rho(B) \cdot \mathrm{fsign}^l(A, C) \cdot b(A, C)(\underline{b}) \qquad (34) \\ &= \; w_{\rho,2,\mathrm{fsign}^l}^{(A,C,B)}(\underline{b}) = -w_{\rho,2,\mathrm{fsign}^l}^{(B,C,A)}(\underline{b})\,. \qquad (35) \end{aligned}$$

*In case of* $d(l) = 1$, *all triangle inequalities are in fact implicit equations:* $(A, B \in \mathbb{P}_n)$

$$\begin{aligned} 0 = \rho(A) \cdot \mathrm{fsign}^l(B) \cdot b(B)(\underline{b}) - \rho(B) \cdot \mathrm{fsign}^l(A) \cdot b(A)(\underline{b}) &= w^{(A,B)}_{\rho,1,\mathrm{fsign}^l}(\underline{b}) \\ &= -w^{(B,A)}_{\rho,1,\mathrm{fsign}^l}(\underline{b}) \,. \end{aligned}$$

Besides the consequences of Lemma 33 there is one geometrically intuitive way to reduce the number of restrictions in $\mathrm{SysEmD}(\rho, 2 = d(l), \mathrm{fsign}^l)$: a planar $n$-gon is convex if and only if all the inner angles on vertices are at most $\pi$, i.e., if it is locally convex at every vertex.

Assume that we have fixed for a given $l = (l_1, \dots, l_k)$ one $L_i \in l_i$ for each $i \in [k]$, and set $L_{i+k} := L_i$ for $i = 1, 2$.

$$\begin{aligned} E_{\mathrm{SameDir}}(2, l) &:= \{\, e^{L_i,B,C}_{\rho,l} : i \in [k], B \in l_i \setminus \{L_i\}, C \in \mathbb{P}_n, C_1 < C_2 \,\}\,, \\ E_{\mathrm{SameDir}}(1, l) &:= \{\, w^{(A,B)}_{\rho,1,\mathrm{fsign}^l} : A, B \in L_1 \,\}\,, \\ W_{\mathrm{conv,red}}(\rho, 2, l) &:= \{\, w^{(L_i,L_{i+2},L_{i+1})}_{\rho,2,\mathrm{fsign}^l} : i \in [k] \,\}\,, \\ W_{\mathrm{conv,red}}(\rho, 1, l) &:= \emptyset\,, \\ \mathrm{SysEmPredLin}(\rho, l) &:= \big(E_{\mathrm{sign}}(d(l), l) \cup E_{\mathrm{SameDir}}(d(l), \mathrm{fsign}^l), \\ &\qquad W_{\mathrm{conv,red}}(\rho, d(l), l), S_{\mathrm{sign}}(d(l), \mathrm{fsign}^l)\big)\,, \\ \mathrm{SysEmPred}(\rho, l) &:= \mathrm{SysEmPredLin}(\rho, l) \cup \mathrm{SysDetRed}(n, d(l)) \\ \mathrm{SysEmPLin}(\rho, l) &:= \mathrm{SysEmPredLin}(\rho, l) \cup \big(\emptyset, W_{\mathrm{conv}}(\rho, d(l), \mathrm{fsign}^l), \emptyset\big)\,, \\ \mathrm{SysEmP}(\rho, l) &:= \mathrm{SysEmPLin}(\rho, l) \cup \mathrm{SysDetRed}(n, d(l)), \\ &= \mathrm{SysEmD}(\rho, d(l), \mathrm{fsign}^l) \cup (E_{\mathrm{SameDir}}(d(l), l), \emptyset, \emptyset)\,. \end{aligned}$$

**Lemma 34.** *For every complete position list* $l$ *of order* $n$ *and any metric* $\rho$ *on* $[n]$ *the three systems* $\mathrm{SysEmD}(\rho, d(l), \mathrm{fsign}^l)$, $\mathrm{SysEmP}(\rho, l)$ *and* $\mathrm{SysEmPred}(\rho, l)$ *are equivalent.*

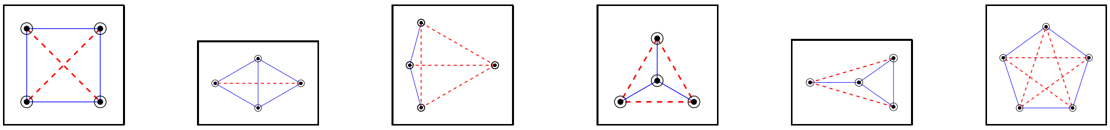
Having in mind the main strategy for the classification of 2-distance sets, solving the linear part of the embedding systems, i.e., the subsystems of restrictions which were known to be linear in advance, the system $\mathrm{SysEmPred}(\rho, l)$ is not as useful as $\mathrm{SysEmP}(\rho, l)$ since in general the solution set of its linearization $\mathrm{SysEmPredLin}(\rho, l)$ is larger than the solution set of the linearization SysEmPLin $(\rho, l)$ of $\mathrm{SysEmP}(\rho, l)$.

## 4. Equivalent 2-distance sets in Minkowski planes

Recall from the introduction that we call the pair $(\mathbb{M}^2, S)$ consisting of a Minkowski plane $\mathbb{M}^2$ and a subset $S \subset \mathbb{R}^2$ a 2-*distance set* if $S$ contains at least two elements and the set $\mathrm{dist}(\mathbb{M}^2, S)$ of distances within $S$, see (3), contains at most two elements. We denote the set of all 2-distance sets by $\mathfrak{C}_2$. If even $|\mathrm{dist}(\mathbb{M}^2, S)| = 1$ then $(\mathbb{M}^2, S)$ is usually called an *equilateral set.*

It is known that every 2-distance set $S$ contains at most 9 points, see [11, Theorem 3] for more general $k$-distance sets in Minkowski spaces. Thus each 2-distance set $S$ is finite, and the set $\operatorname{dist}(\mathbb{M}^2, S)$ contains a positive minimum.

**Example 35.** In the Euclidean plane $\mathbb{E}^2$ there are the following 2-distance sets:



This list is complete in the sense of maximal $\mathbb{E}^2$-2-distance configurations, and of course already known, see [12]. We have $\operatorname{dist}(\mathbb{E}^2, S) = \{1, \sqrt{2}\}$, $\{1, \sqrt{3}\}$, $\{1, \sqrt{2+\sqrt{3}}\}$, $\{1, \sqrt{3}\}$, $\{1, \sqrt{2+\sqrt{3}}\}$, and $\{1, \frac{1}{2}(1+\sqrt{5})\}$, respectively.

There are many 2-distance sets which are almost the same concerning geometric intuition.

Note that we do not regard the points in $S$ as labeled points for the purpose of classification. Thus permuting the points in some representation which uses a numbering of points will result in an representation of the same 2-distance set or equivalence class for any of the following equivalence relations in $\mathfrak{C}_2$.

### 4.1. Strong equivalence

We will silently identify two 2-distance sets, if one is the image of the other with respect to some isometry and scaling. Any bijective embedding of the metric spaces $(X, \rho)$ into the metric spaces $(\tilde{X}, \tilde{\rho})$ is called an *isometry*.

**Definition 36.** *We call two 2-distance sets $(\mathbb{M}^2, S)$ and $(\tilde{\mathbb{M}}^2, \tilde{S})$ strongly equivalent, $(\mathbb{M}^2, S) \equiv_s (\tilde{\mathbb{M}}^2, \tilde{S})$, if and only if there is an isometry $\phi : \mathbb{M}^2 \to \tilde{\mathbb{M}}^2$ and a positive real number $\lambda$ such that $\phi(S) = \lambda\tilde{S}$.*

Note that by the Mazur-Ulam theorem, $\phi$ can only be an affinely linear function, see Thompson [5].

Obviously, the relation $\equiv_s$ of strong equivalence is an equivalence relation in the set $\mathfrak{C}_2$.

### 4.2. Affine equivalence

**Definition 37.** *We call two 2-distance sets $(\mathbb{M}^2, S)$ and $(\tilde{\mathbb{M}}^2, \tilde{S})$ affinely equivalent, $(\mathbb{M}^2, S) \equiv_a (\tilde{\mathbb{M}}^2, \tilde{S})$, if and only if there is an affinely linear function $A : \mathbb{M}^2 \to \tilde{\mathbb{M}}^2$ and a positive real number $\lambda$ such that $A(S) = \lambda\tilde{S}$ and $\|A\mathbf{x} - A\mathbf{y}\|_{\tilde{\mathbb{M}}^2} = \|\mathbf{x} - \mathbf{y}\|_{\mathbb{M}^2}$ for all $\mathbf{x}, \mathbf{y} \in S$.*

Thus, $(\mathbb{M}^2, S) \equiv_a (\tilde{\mathbb{M}}^2, \tilde{S})$ if there is an affinely linear function $A$ whose restricting to $S$ is an isometry between the metric spaces $(S, \rho_{\mathbb{M}^2}|_{S\times S})$ and $(A(S), \rho_{\tilde{\mathbb{M}}^2}|_{A(S)\times A(S)})$, and if $A(S)$ is a homothetic copy of $\tilde{S}$.

The relation $\equiv_a$ of affine equivalence is an equivalence relation in $\mathfrak{C}_2$, too.

### 4.3. Full equivalence

Since we consider homothetic 2-distance sets $S$ as equivalent, we introduce the following notion for the metric induced by $S$ in $\mathbb{M}^2$.

**Definition 38.** *The normalized metric induced by a labeled set $e : [n] \to X$ in a metric space $(X, \rho)$ is the function $\rho|_e : [n]^2 \to \mathbb{R}$, defined by $(a, b) \mapsto \frac{1}{\min\{\rho(e(i),e(j)):(i,j)\in\mathbb{P}_n, e(i)\neq e(j)\}}\rho(e(a), e(b))$.*

**Definition 39.** *We call two 2-distance sets $(\mathbb{M}^2, S)$ and $(\tilde{\mathbb{M}}^2, \tilde{S})$ fully equivalent, $(\mathbb{M}^2, S) \equiv_f (\tilde{\mathbb{M}}^2, \tilde{S})$, if and only if there is a bijection $\varphi : S \to \tilde{S}$ and a labeling $e : [n] \to S$ such that the labeling $\tilde{e} := \phi \circ e : [n] \to \tilde{S}$ has the same relative position (see Definition 23) as $e$ and if the normalized metric (see Definition 38) induced by $\tilde{e}$ in $\tilde{\mathbb{M}}^2$ coincides with the normalized metric induced by $e$ in $\mathbb{M}^2$.*

Thus we can describe every equivalence class of full equivalence in $\mathfrak{C}_2$, called *full 2-distance configuration*, with representative $(\mathbb{M}^2, \{e(1), \dots, e(n)\})$ by an complete abstract oriented position list $l$ of order $n$, together with a metric $\rho$ on $[n]$ with $\rho(\mathbb{P}_n) = \{1, r\}$, $r \geq 1$.

**Remark 40.** Two such representations $(l^i, \rho_i)$ with $d(l^i) = 2$ represent the same full 2-distance configuration (provided each of them represents one), if and only if there is a permutation $\phi$ of $[n]$ and some $s \in \mathbb{Z}$ with $\phi(l^1) = \mathrm{Shift}(l^2, s)$ or $\phi(l^1) = \mathrm{Shift}(\mathrm{Mirror}(l^2), s)$ and $\phi(\rho_1) = \rho_2$. Note that we write for $(i, j) \in [n]^2$: $\phi((i, j)) := (\phi(i), \phi(j))$, $\phi(\{p_1, \dots, p_m\}) := \{\phi(p_1), \dots, \phi(p_m)\}$ for $p_1, \dots, p_m \in \mathbb{P}_n$ and for $l_1, \dots, l_k \subset \mathbb{P}_n$ finally $\phi((l_1, \dots, l_k)) := (\phi(l_1), \phi(l_2), \dots, \phi(l_k))$. Additionally, $\phi(\rho_1) : [n]^2 \to \mathbb{R}$ with $(\phi(a), \phi(b)) \mapsto \rho_1(a, b)$, i.e., $\phi(\rho_1)(x, y) := \rho_1(\phi^{-1}(x), \phi^{-1}(y))$.

We can describe the metric $\rho$ on $[n]$ with $\rho(\mathbb{P}_n) = \{1, r\}$, by the value $r \in [1, \infty)$ (the continuous part of $\rho$) together with the undirected simple graph $([n], L)$ of "large" distances, $L := \{\, \{i, j\} \subset [n] : \rho(i, j) > 1 \,\}$ (the combinatorial part of $\rho$).

Recall the structure of an complete abstract oriented position list as a sequence of abstract directions sets. For each pair $(i, j) \in \mathbb{P}_n$ exactly one of $(i, j)$ and $(j, i)$ is contained in (exactly) one of these direction sets. Now we can append the symbolic information whether or not $\{i, j\} \in L$, i.e., a symbolic representation of the distance $\rho(i, j)$ to be embedded, to all the pairs $(i, j)$, and obtain a new data structure called *position-metric-list*. So let $G := \{S, B\}$ denote the set of symbolic distances, where $S$ represents 1 and $B$ stands for $r$. Triples $(i, j, \varrho)$ with $(i, j) \in \mathbb{P}_n$ and $\varrho \in G$ are called *abstract direction.* A position-metric-list of order $n$ is a finite sequence of nonempty sets of abstract directions. Its *associated oriented position list* $l(w)$ is obtained in the obvious way be omitting its distance components. A *complete position-metric-list* of order $n$ is a position-metric-list $w$ such that $l(w)$ is a complete oriented position list of order $n$.

Summarizing we can represent any full 2-distance configuration by the pair $(w, r)$ consisting of a complete position-metric-list $w$ of order $n$ and the real number $r$ which is the ratio of maximal and minimal positive distances among the considered points.

### 4.4. Similar 2-distance sets

Having the representation $(w, r)$ of full 2-distance configurations in mind, we get another equivalence relation by ignoring the value of $r$.

**Definition 41.** *We call two* 2*-distance sets* $(\mathbb{M}^2, S)$ *and* $(\tilde{\mathbb{M}}^2, \tilde{S})$ *similar,* $(\mathbb{M}^2, S) \sim (\tilde{\mathbb{M}}^2, \tilde{S})$*, if and only if there is a bijection* $\varphi : S \to \tilde{S}$ *and a labeling* $e : [n] \to S$ *such that the labeling* $\tilde{e} := \phi \circ e : [n] \to \tilde{S}$ *has the same relative position (see Definition* 23*) as* $e$ *and if the sets of large distances coincide:* $L := \{ \{i, j\} \subset [n] : \rho_{B(\mathbb{M}^2)}(e(i), e(j)) > \min \rho_{B(\mathbb{M}^2)}(e(\mathbb{P}_n)) \} = \tilde{L} := \{ \{i, j\} \subset [n] : \rho_{B(\tilde{\mathbb{M}}^2)}(\tilde{e}(i), \tilde{e}(j)) > \min \rho_{B(\tilde{\mathbb{M}}^2)}(\tilde{e}(\mathbb{P}_n)) \}$*.*

Thus, $(\mathbb{M}^2, S) \sim (\tilde{\mathbb{M}}^2, \tilde{S})$, if and only if they both can produce the same representation as a position-metric-list $w$ as described above.

### 4.5. Summary of the equivalence relations

We have introduced four different equivalence relations in $\mathfrak{C}_2$: strong equivalence $\equiv_s$, affine equivalence $\equiv_a$, full equivalence $\equiv_f$, and similarity $\sim$. Of course, more are possible and maybe useful, for example consider the mentioned graph of large distances or the concept of relative position using the oriented matroid of the point configuration.

**Definition 42.** *We call each equivalence class* $T = [C] := \{ C' \in \mathfrak{C}_2 : C' \sim C \}$ *of similarity of a* 2*-distance set* $C \in \mathfrak{C}_2$ *a* 2*-distance configuration. In the same way we call the equivalence class* $[C]_s$ *of strong equivalence of* $C$ *strong* 2*-distance configuration, the equivalence class* $[C]_a$ *of affine equivalence of* $C$ *affine* 2*-distance configuration, and the equivalence class* $[C]_f$ *of full equivalence of* $C$ *(as already mentioned) full* 2*-distance configuration.*

**Proposition 43.** *In the sequence* $\equiv_s$, $\equiv_a$, $\equiv_f$, $\sim$, *of the introduced relations, each following relation is more general than the preceding one: for all* $C \in \mathfrak{C}_2$ *we have*

$$[C]_s \subset [C]_a \subset [C]_f \subset [C] .$$

Assume that $w$ is a position-metric-list of $C = (\mathbb{M}^2, S) \in \mathfrak{C}_2$ for a labeling $e : [n] \to S \subset \mathbb{R}^2$ and $\operatorname{dist}(\mathbb{M}^2, S) = \{1, r\}$. Then we define $T(w) := [C]$, $D([C], r) := [C]_f$ and $A(w, r, \underline{b}^e) := [C]_a$.

Note that there are 2-distance sets $C$ with $[C]_s \neq [C]_a \neq [C]_f \neq [C]$, for example we can take any $C \in D(T^5_{30}, 1.73)$, see the following section. But also $[C]_s = [C]_a = [C]_f = [C]$ is possible for some 2-distance sets $C$.

## 5. Classification results

We will present five enumerations to classify all 2-distance configurations, all full 2-distance configurations, all affine 2-distance configurations and some further information regarding the quantity of strong 2-distance configurations belonging to each of the affine 2-distance configurations.

The first enumeration in Section 5.1 is a complete list of pictures of all 94 different 2-distance configurations in Minkowski planes. It also contains all information about the uncountably many full 2-distance configurations.

In Section 5.2 we summarize the classification of 2-distance configurations by enumerating all 11 maximal 2-distance configurations. Their combinatorial representations are not contained as sub structures in other ones. We show a enumeration of the pictures and another one of corresponding position-metric-lists.

The forth enumeration shown in Section 5.3 gives in conjunction with the first enumeration an overview of all affine 2-distance configurations. For each full 2-distance configuration consisting of more than one affine 2-distance configuration there is a picture illustrating all corresponding affine 2-distance configurations using a parameter which is a point belonging to, e.g., a triangle without its boundary. In general, one such picture can represent some class of full 2-distance configurations with corresponding affine 2-distance configurations with different values of $r$ at once.

Finally, the last enumeration in Section 5.4 is a partition of all affine 2-distance configurations into three categories. Each category represents some information about the quantity of corresponding strong 2-distance configurations. There are pictures representing classes of affine 2-distance configurations as in the forth enumeration, possibly split into several classes with smaller parameter set.

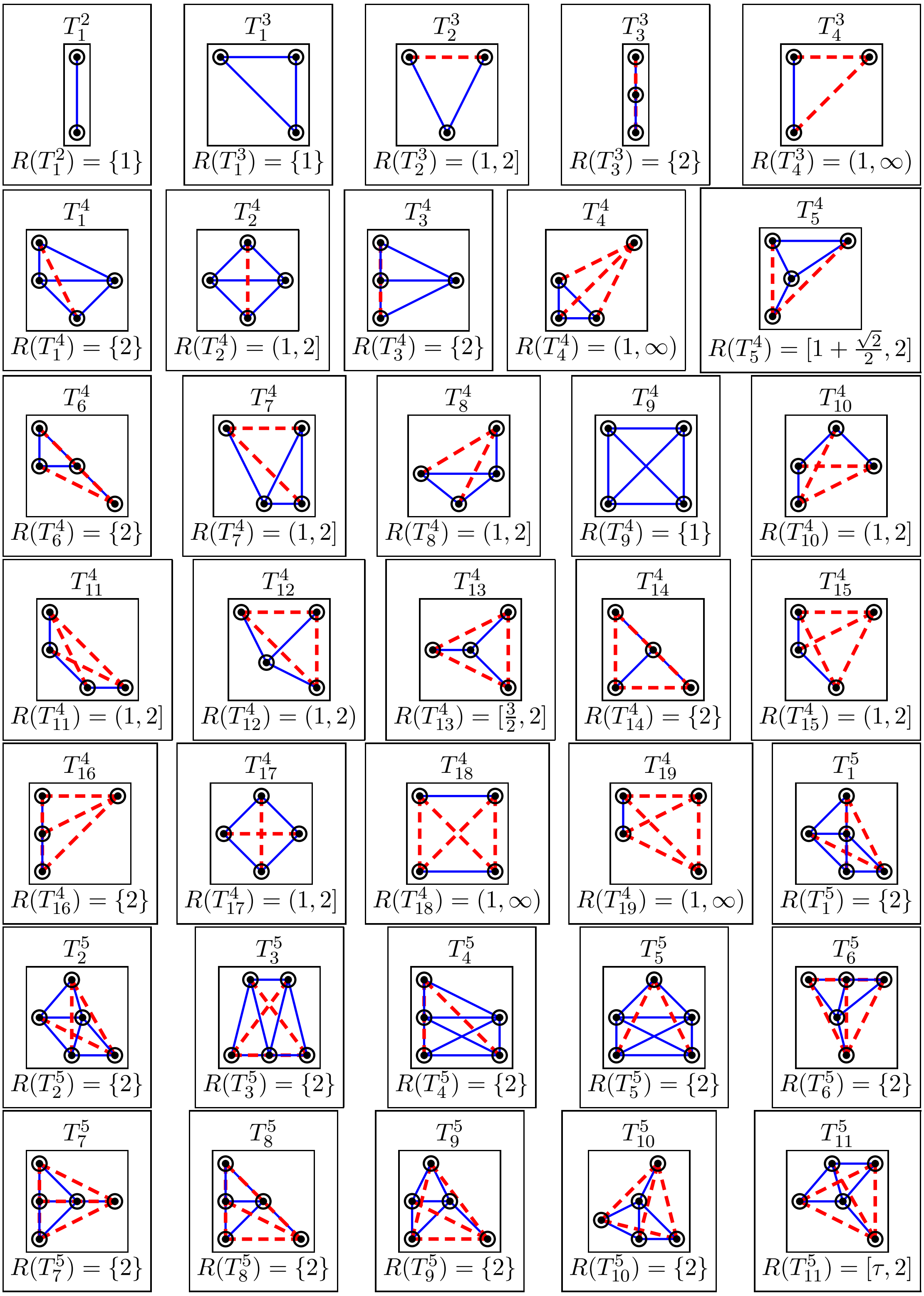
For formal descriptions of the represented data please see [13].

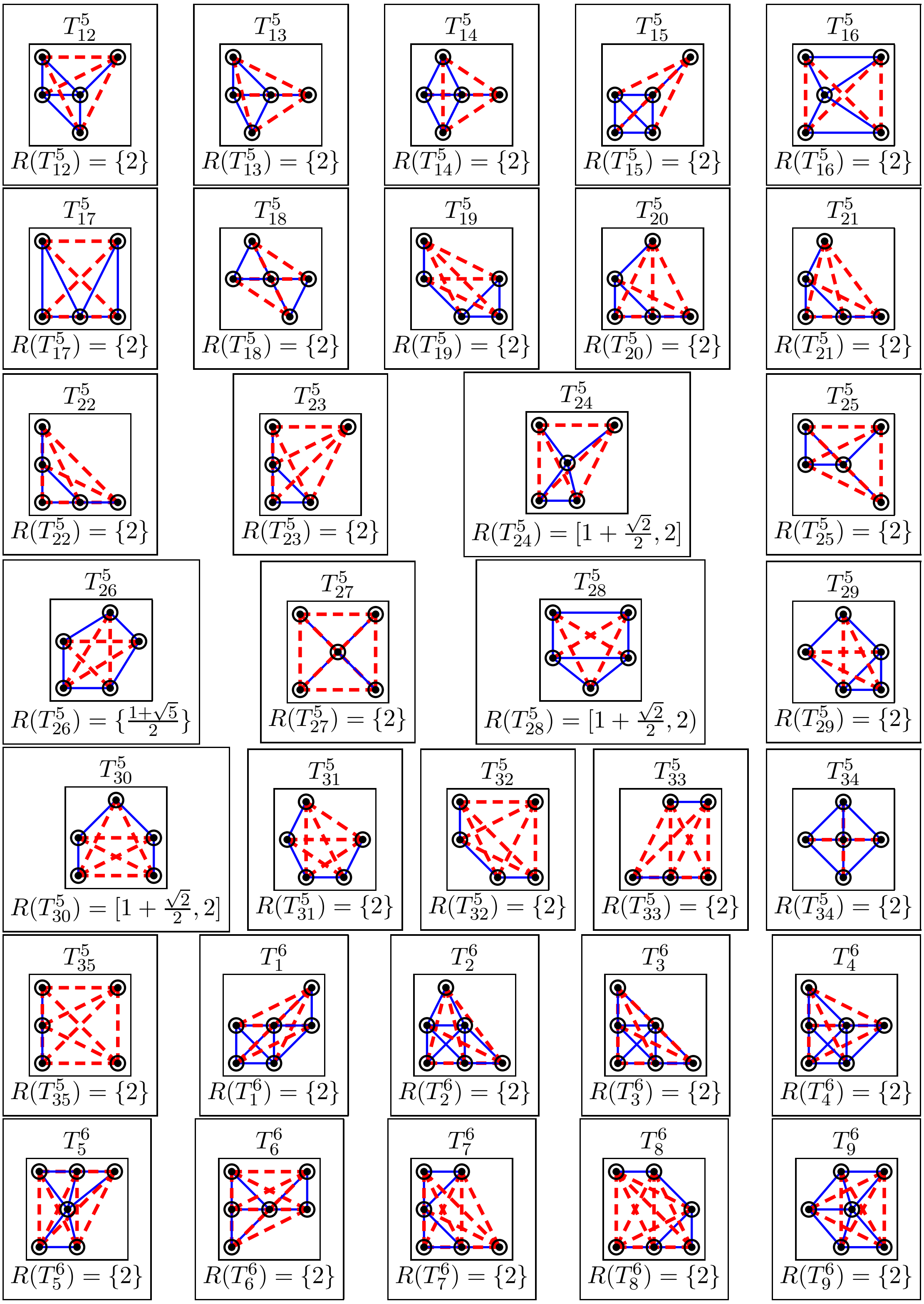
### 5.1. Full 2-distance configurations

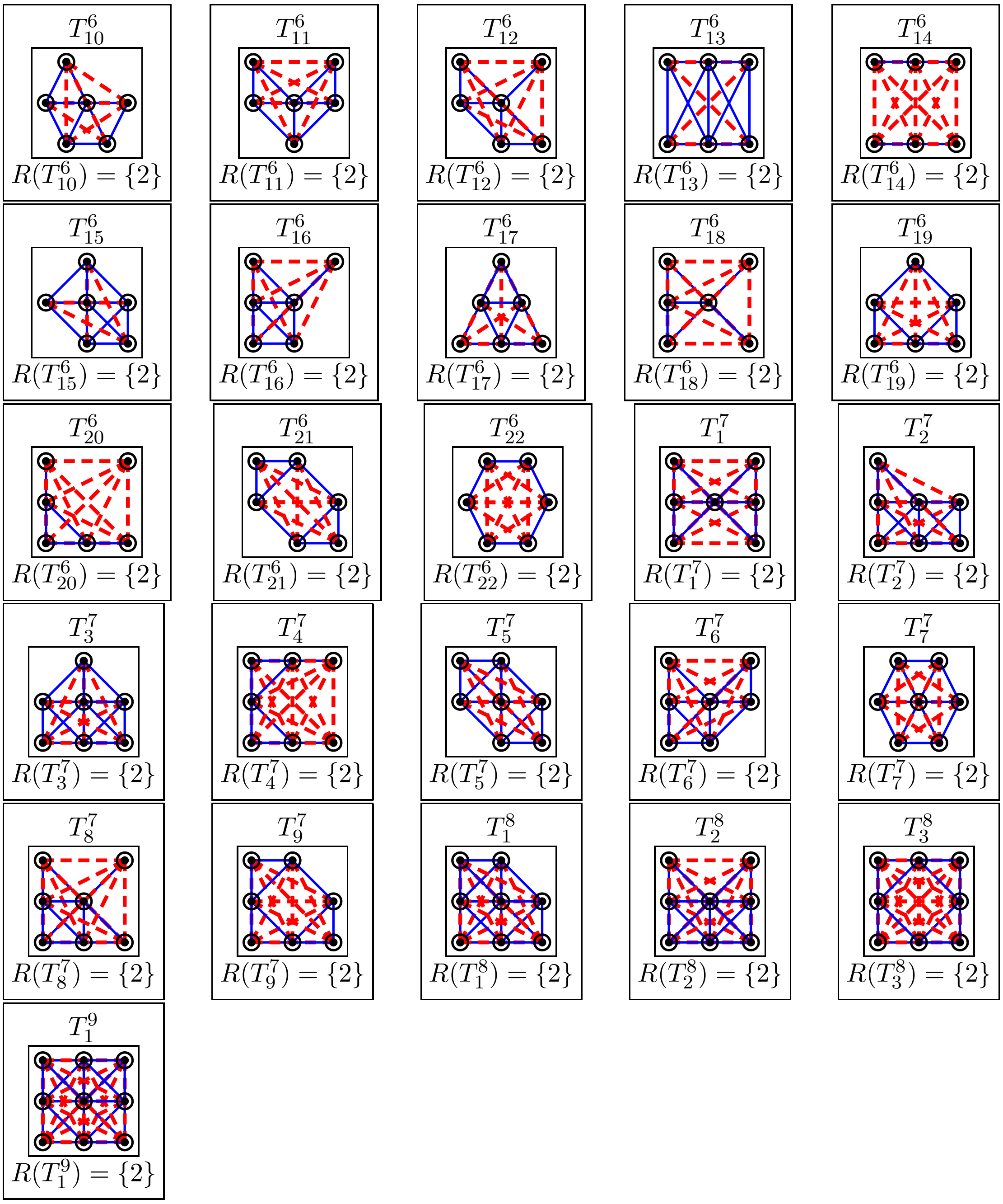
Now we present a list of all 94 different 2-distance configurations $T = T_k^n$. Here $n$ is again the number of points and $k$ is some number to yield unique symbols for the equivalence classes. $T_k^n$ can be described by a complete position-metric-list $w$ of order $n$, see Section 5.2.

We will visualize $T = [(\mathbb{M}^2, S)]$ by drawing all points of the set $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ as small "double"-balls ⊙. $(\mathbb{M}^2, S)$ was chosen so that $\operatorname{dist}(\mathbb{M}^2, S) = \{1, r\}$ for some $r \geq 1$, including the case $r = 1$ for equilateral sets. We connect $\mathbf{s}_i, \mathbf{s}_j$ by a straight line if $\|\mathbf{s}_i - \mathbf{s}_j\| = 1$. Otherwise, i.e., if $\|\mathbf{s}_i - \mathbf{s}_j\| = r > 1$, we connect them by a dashed line. We do not visualize the corresponding Minkowski plane $\mathbb{M}^2$, because one suitable $\mathbb{M}^2$ can always be constructed from this picture together with the value of $r$. The relative position should be clear from the first glance at these pictures for all but two exceptions, $T_8^4$and $T_{10}^4$. But since all 2-distance configurations which are not maximal are drawn as they occur in the picture of a larger 2-distance configuration, the relative position becomes clear from the picture of $T_{28}^5$ and $T_1^9$, respectively, in Section 5.2.

All different full 2-distance configurations are obtained as $D = D(T, r^*)$, where $T$ is a 2-distance configuration and $r^* \in R(T)$. The sets $R(T_k^n) \subset [1, \infty)$ are shown in this list as well, giving a complete classification of full 2-distance configurations. For this we denote by $\tau$ the real root of the polynomial $x^3 - 2x^2 + x - 1$, $\tau = \operatorname{RootOf}(1, X^3 - 2X^2 + X - 1) \approx 1.754877$.

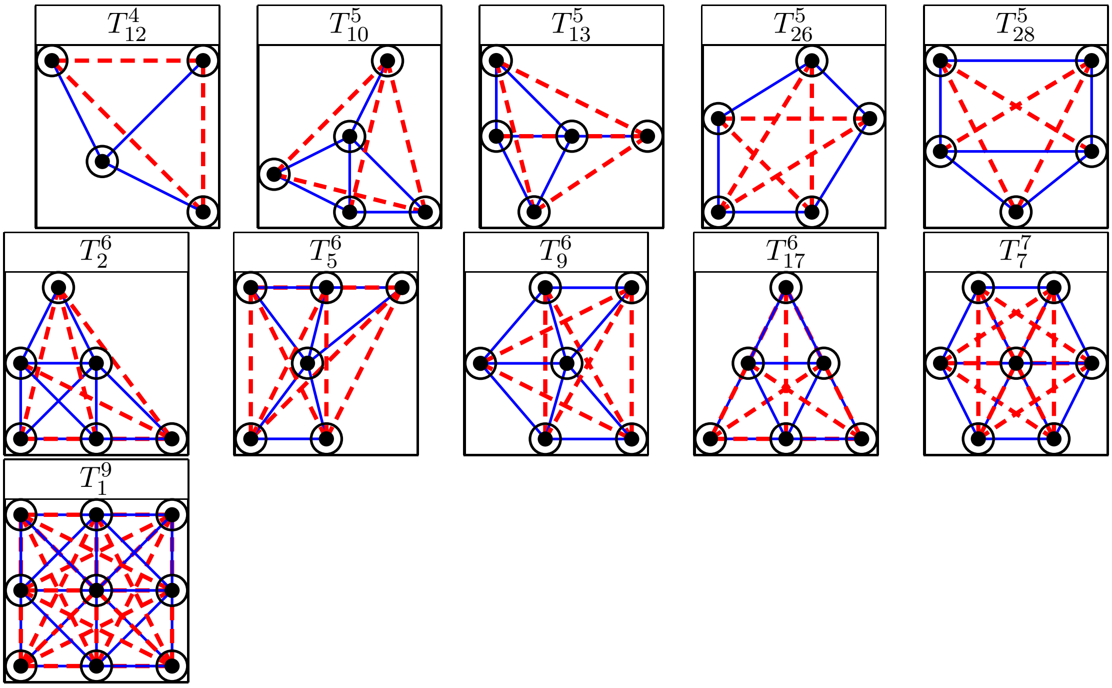

$T_1^2$
$R(T_1^2) = \{1\}$
$T_1^3$
$R(T_1^3) = \{1\}$
$T_2^3$
$R(T_2^3) = (1, 2]$
$T_3^3$
$R(T_3^3) = \{2\}$
$T_4^3$
$R(T_4^3) = (1, \infty)$
$T_1^4$
$R(T_1^4) = \{2\}$
$T_2^4$
$R(T_2^4) = (1, 2]$
$T_3^4$
$R(T_3^4) = \{2\}$
$T_4^4$
$R(T_4^4) = (1, \infty)$
$T_5^4$
$R(T_5^4) = [1 + \frac{\sqrt{2}}{2}, 2]$
$T_6^4$
$R(T_6^4) = \{2\}$
$T_7^4$
$R(T_7^4) = (1, 2]$
$T_8^4$
$R(T_8^4) = (1, 2]$
$T_9^4$
$R(T_9^4) = \{1\}$
$T_{10}^4$
$R(T_{10}^4) = (1, 2]$
$T_{11}^4$
$R(T_{11}^4) = (1, 2]$
$T_{12}^4$
$R(T_{12}^4) = (1, 2)$
$T_{13}^4$
$R(T_{13}^4) = [\frac{3}{2}, 2]$
$T_{14}^4$
$R(T_{14}^4) = \{2\}$
$T_{15}^4$
$R(T_{15}^4) = (1, 2]$
$T_{16}^4$
$R(T_{16}^4) = \{2\}$
$T_{17}^4$
$R(T_{17}^4) = (1, 2]$
$T_{18}^4$
$R(T_{18}^4) = (1, \infty)$
$T_{19}^4$
$R(T_{19}^4) = (1, \infty)$
$T_1^5$
$R(T_1^5) = \{2\}$
$T_2^5$
$R(T_2^5) = \{2\}$
$T_3^5$
$R(T_3^5) = \{2\}$
$T_4^5$
$R(T_4^5) = \{2\}$
$T_5^5$
$R(T_5^5) = \{2\}$
$T_6^5$
$R(T_6^5) = \{2\}$
$T_7^5$
$R(T_7^5) = \{2\}$
$T_8^5$
$R(T_8^5) = \{2\}$
$T_9^5$
$R(T_9^5) = \{2\}$
$T_{10}^5$
$R(T_{10}^5) = \{2\}$
$T_{11}^5$
$R(T_{11}^5) = [\tau, 2]$


$T^5_{12}$ $R(T^5_{12}) = \{2\}$
$T^5_{13}$ $R(T^5_{13}) = \{2\}$
$T^5_{14}$ $R(T^5_{14}) = \{2\}$
$T^5_{15}$ $R(T^5_{15}) = \{2\}$
$T^5_{16}$ $R(T^5_{16}) = \{2\}$
$T^5_{17}$ $R(T^5_{17}) = \{2\}$
$T^5_{18}$ $R(T^5_{18}) = \{2\}$
$T^5_{19}$ $R(T^5_{19}) = \{2\}$
$T^5_{20}$ $R(T^5_{20}) = \{2\}$
$T^5_{21}$ $R(T^5_{21}) = \{2\}$
$T^5_{22}$ $R(T^5_{22}) = \{2\}$
$T^5_{23}$ $R(T^5_{23}) = \{2\}$
$T^5_{24}$ $R(T^5_{24}) = [1 + \frac{\sqrt{2}}{2}, 2]$
$T^5_{25}$ $R(T^5_{25}) = \{2\}$
$T^5_{26}$ $R(T^5_{26}) = \{\frac{1+\sqrt{5}}{2}\}$
$T^5_{27}$ $R(T^5_{27}) = \{2\}$
$T^5_{28}$ $R(T^5_{28}) = [1 + \frac{\sqrt{2}}{2}, 2)$
$T^5_{29}$ $R(T^5_{29}) = \{2\}$
$T^5_{30}$ $R(T^5_{30}) = [1 + \frac{\sqrt{2}}{2}, 2]$
$T^5_{31}$ $R(T^5_{31}) = \{2\}$
$T^5_{32}$ $R(T^5_{32}) = \{2\}$
$T^5_{33}$ $R(T^5_{33}) = \{2\}$
$T^5_{34}$ $R(T^5_{34}) = \{2\}$
$T^5_{35}$ $R(T^5_{35}) = \{2\}$
$T^6_{1}$ $R(T^6_{1}) = \{2\}$
$T^6_{2}$ $R(T^6_{2}) = \{2\}$
$T^6_{3}$ $R(T^6_{3}) = \{2\}$
$T^6_{4}$ $R(T^6_{4}) = \{2\}$
$T^6_{5}$ $R(T^6_{5}) = \{2\}$
$T^6_{6}$ $R(T^6_{6}) = \{2\}$
$T^6_{7}$ $R(T^6_{7}) = \{2\}$
$T^6_{8}$ $R(T^6_{8}) = \{2\}$
$T^6_{9}$ $R(T^6_{9}) = \{2\}$




### 5.2. Maximal 2-distance configurations

We can compress the classification of 2-distance configurations by listing the 11 maximal 2-distance configurations. Then each of the 83 remaining 2-distance configurations is contained as sub-configuration in at least one of the following.




The maximal 2-distance configurations can be represented by the following position-metric-lists. We omitted redundant parentheses, thus instead of $(1, 2, B)$ we only write $12B$:

$T^4_{12}$ : $\{12S\}$, $\{13B\}$, $\{23S\}$, $\{14B\}$, $\{24S\}$, $\{34B\}$
$T^5_{10}$ : $\{12S\}$, $\{34S\}$, $\{14B\}$, $\{13S\}$, $\{24S\}$, $\{54B\}$, $\{23S\}$, $\{53B\}$, $\{52S\}$, $\{51B\}$
$T^5_{13}$ : $\{12S\}$, $\{34S\}$, $\{14B\}$, $\{13S\}$, $\{24S\}$, $\{54B\}$, $\{23S, 52S, 53B\}$, $\{51B\}$
$T^5_{26}$ : $\{12S, 34B\}$, $\{35S, 14B\}$, $\{24S, 15B\}$, $\{13S, 25B\}$, $\{45S, 23B\}$
$T^5_{28}$ : $\{12S\}$, $\{13B\}$, $\{23S, 45S\}$, $\{15B\}$, $\{14S\}$, $\{25B\}$, $\{24S, 35S\}$, $\{34B\}$
$T^6_2$ : $\{12S\}$, $\{13S, 34S, 52S, 14B\}$, $\{54B\}$, $\{24S, 53S\}$, $\{64B\}$, $\{62S\}$, $\{63B\}$, $\{23S, 51S\}$, $\{61B\}$, $\{65S\}$
$T^6_5$ : $\{12S\}$, $\{13S, 34S, 56S, 14B\}$, $\{24S\}$, $\{54B\}$, $\{52S\}$, $\{53B, 64B\}$, $\{23S\}$, $\{51B, 63B\}$, $\{62S\}$, $\{61B\}$
$T^6_9$ : $\{12S\}$, $\{13B\}$, $\{23S, 45S\}$, $\{43B\}$, $\{16S, 42S, 53S\}$, $\{46B\}$, $\{26S, 41S\}$, $\{56B\}$, $\{52S\}$, $\{36B, 51B\}$
$T^6_{17}$ : $\{12S, 23S, 45S, 13B\}$, $\{15B\}$, $\{14S, 25S, 46S, 16B\}$, $\{26B\}$, $\{24S, 35S, 56S, 36B\}$, $\{34B\}$
$T^7_7$ : $\{12S, 23S, 45S, 67S, 13B\}$, $\{15B, 63B\}$, $\{14S, 25S, 62S, 73S, 65B\}$, $\{64B, 75B\}$, $\{24S, 35S, 61S, 72S, 74B\}$, $\{34B, 71B\}$
$T^9_1$ : $\{12S, 23S, 45S, 56S, 78S, 89S, 13B, 46B, 79B\}$, $\{16B, 49B\}$, $\{15S, 26S, 48S, 59S, 19B\}$, $\{18B, 29B\}$, $\{14S, 25S, 36S, 47S, 58S, 69S, 17B, 28B, 39B\}$, $\{27B, 38B\}$, $\{24S, 35S, 57S, 68S, 37B\}$, $\{34B, 67B\}$

We will skip similar lists of maximal full or maximal affine 2-distance configurations because they do not represent the information much easier. The position-metric-lists of the remaining 2-distance configurations can be easily extracted from the above information.

### 5.3. Affine 2-distance configurations

Each affine 2-distance configuration $A$ is represented exactly once in the following way. Take a full 2-distance configuration $D = D(T, r)$, where $r \in R(T)$ and $T = T_k^n$. Sometimes, $D$ is itself an affine 2-distance configuration; in this case $A = D$. To simplify notations, we introduce in this case a purely formal parameter $\mathbf{0} = p \in \mathbb{R}^0 =: P(T, r)$ in the $d(T, r) := 0$-dimensional real vector space. Otherwise, we introduce a real parameter vector $p$ to describe affine 2-distance configurations contained in $D(T_k^n, r)$. The parameter $p$ must belong to some set $P(T_k^n, r) \subset \mathbb{R}^d$ of a simple geometric shape and $d(T, r) := d \leq 2$. Then each affine 2-distance configuration $A$ is represented exactly once as $A = [(\mathbb{M}^2(T_k^n, r, p), S(T_k^n, r, p))]_a$ with $r \in R(T_k^n)$ and $p \in P(T_k^n, r)$. Again, the suitable Minkowski plane $\mathbb{M}^2(T, r, p)$ can be constructed from $S$ and the metric in a canonical way and is not visualized.

The coordinates of the points in $S(T, r, p)$ are piecewise polynomials in $\mathbb{Z}[r]$. More precisely, the set $R(T)$ will occasionally be split into a few number of subsets, $R(T) = \dot{\bigcup}_{i=1,\ldots,k(T)} R(T, i)$.

For some $i$ the case $A = D$ may occur. In this case there are $2n$ mono-variate polynomials, $S_{T,i} \in (\mathbb{Z}[X]^2)^n$, which define $S(T, r, p) := \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ by evaluating the polynomials at $X = r \in R(T, i)$: $(\mathbf{s}_1, \ldots, \mathbf{s}_n) := S_{T,i}(r)$.

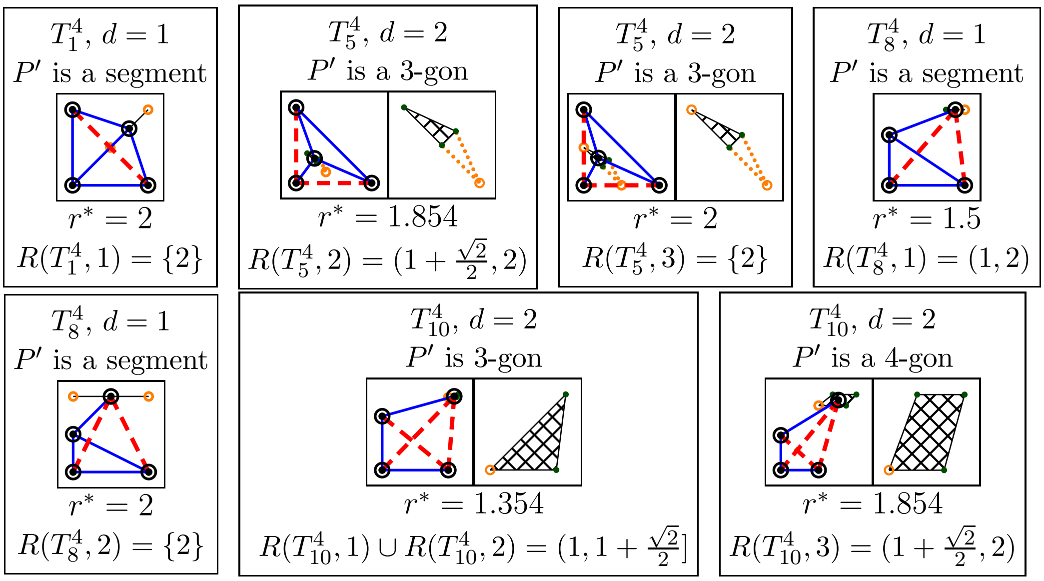
Otherwise one or more points of $S$ may be allowed to move within some bounds, so that $[(\mathbb{M}^2, S)]_f$ stays the same but $[(\mathbb{M}^2, S)]_a$ changes, for suitable $\mathbb{M}^2$. It turned out that there are at most two degrees of freedom, and that the influence of the parameter can be chosen to be affinely linear. The $2n$ coordinates of $S(T, r, p)$ are polynomials in $d + 1$ real variables $r$ and $p_1, \ldots, p_d$, summarized as $S_{T,i} \in (\mathbb{Z}[X, p_1, \ldots, p_d]^2)^n$. Fixing the first parameter $X = r$ in $S_{T,i}$, we get an affinely linear function $S_{T,i}(r) : \mathbb{R}^d \to (\mathbb{R}^2)^n$. In addition, the parameters $p \in \mathbb{R}^d$ are restricted to belong to a polyhedron, or, more precisely, to the union $P(T, i, r) \subset \mathbb{R}^d$ of interiors of some well defined faces of a polyhedron $P'(T, i, r)$. Note that we have chosen each parameter $p_j$ so that at least one coordinate of one of the points of $S$ coincides with $p_j$, possibly after multiplication with a real factor in $\mathbb{Z}[r]$. Note that in almost all cases with $d = 2$ we could take both coordinates of just one point $\mathbf{s}$ from $S$ as parameters. In all these cases the parameter range $P(T, i, r)$ coincides, up to scaling, with the range where $\mathbf{s}$ can move. The vertices of $P'(T, i, r)$ are again defined by mono-variate polynomial coordinates. The combinatorial structure of $P'(T, i, r)$ is identical for all $r \in R(T, i)$, and $P$ always consists of the same faces of $P'$ with respect to the labeled vertices.

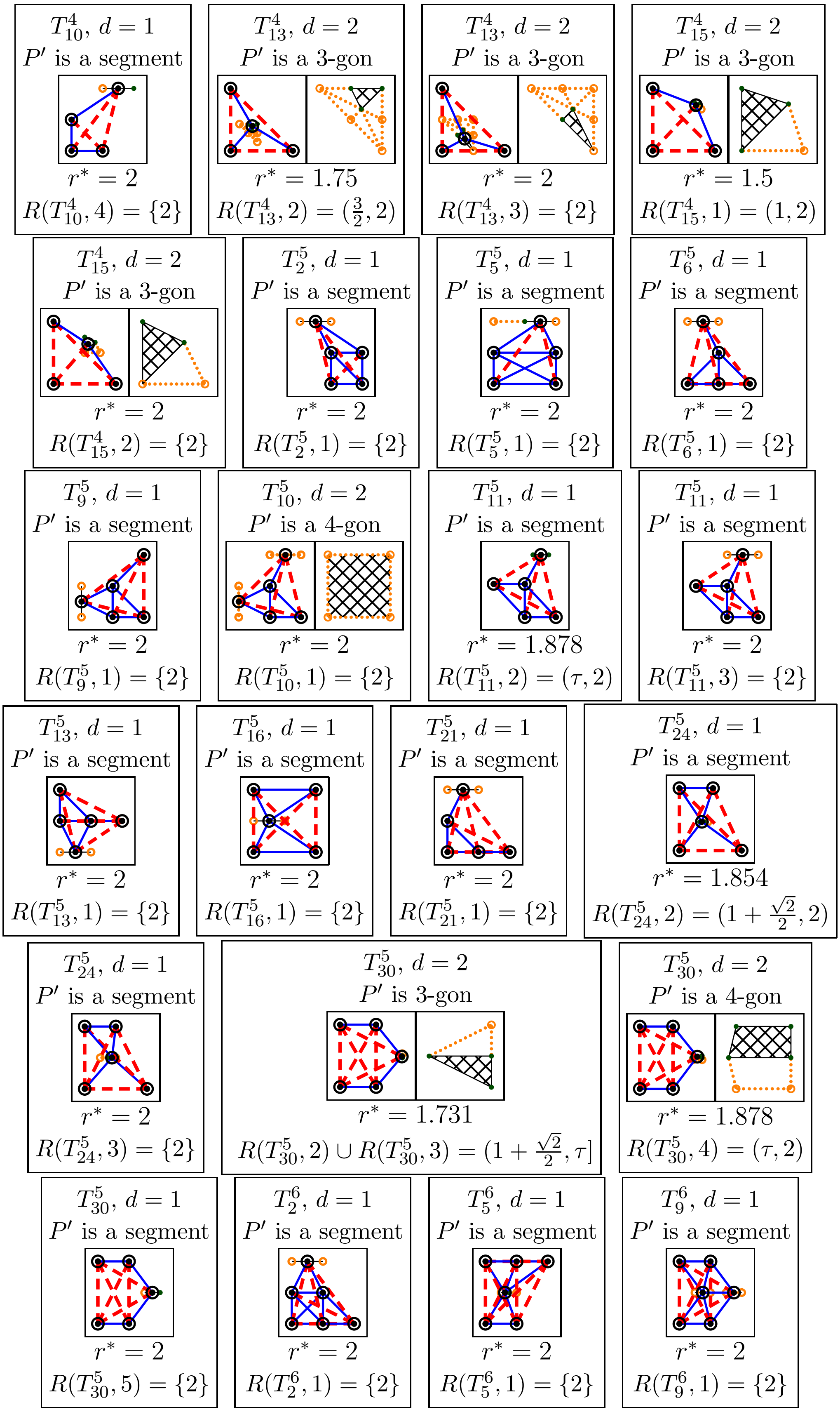
Summarizing, we have for each 2-distance configuration $T$ a refinement of $R(T)$, $R(T) = \dot{\bigcup}_{i=1,\ldots,k(T)} R(T, i)$. For each $i \in [k(T)]$ there is a construction $S = S(T, i, r, p)$ of $n$ points in the plane which is polynomial in $r \in R(T, i)$ and linear in some additional parameter $p \in \mathbb{R}^d$, with $d \in \{0, 1, 2\}$. The parameter $p$ can be chosen from a well defined parameter range $P(T, i, r) \subset \mathbb{R}^d$. Now each affine 2-distance configuration $A$ is uniquely determined by one $T$, $i \in [k_T]$, $r \in R(T,i)$ and $p \in P(T,i,r)$ as $A = [(\mathbb{M}^2, S(T,i,r,p))]_a$, where $\mathbb{M}^2$ is constructed appropriately from $S$ and the metric.

We will show nothing for each $T$ and $i \in [k(T)]$ where the full 2-distance configuration is affinely unique. Otherwise, we will produce a picture to represent all affine 2-distance configurations belonging to one of $D(T,r)$ for $r \in R(T,i)$. Below the picture we write $R(T,i)$ exactly and a floating point approximation of some fixed $r^* \in R(T,i)$. For the parameter $p^* \in P(T,i,r^*)$, which is the mean of all vertices of $P'(T,i,r^*)$, the picture contains $S(T,i,r^*,p^*)$ drawn as in the previous pictures. Additionally, we illustrate for each point of $S(T,i,r^*,p)$ its domain as $p \in P(T,i,r^*)$ varies in the same picture. Note that for some examples this illustration is to small to show any details. Because of that, another picture illustrating $P(T,i,r^*)$ is shown in case of $d = 2$ to the right hand side. Each face (vertex, edge or the interior area) of $P'$ is drawn in a way indicating whether or not it belongs to $P$. Vertices belong to $P$ if they are shown as small dark filled balls, otherwise a little larger circle line is drawn around the point. Edges belong to $P$ if they are drawn as black line and not as dotted gray line. Polygons belong to $P$ if they are crosswise shaded. The domain of the individual points in $S(T,i,r^*,p)$ on the left hand side are shown analogously.

In fact, the pictures are a little bit more complicated. Instead of just drawing $P$ and $P'$ we have drawn sometimes a larger area $P_a$, but in a style showing that it does not belong to $P$. This area is the set of all parameters $p$ such that $S(T,i,r^*,p)$ is a 2-distance set belonging to the same full 2-distance configuration. But due to symmetry within the combinatorial structure of $T$, for all $p \in P_a \setminus P$ there is some $p' \in P$ such that $S(T,i,r^*,p)$ is an affine image of $S(T,i,r^*,p')$, or more precisely, $(\mathbb{M}^2(T,i,r^*,p), S(T,i,r^*,p)) \equiv_a (\mathbb{M}^2(T,i,r^*,p'), S(T,i,r^*,p'))$.

The list of all full 2-distance configurations which contain more than one affine 2-distance configuration follows.




$T^4_{10}$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^4_{10}, 4) = \{2\}$
$T^4_{13}$, $d=2$
$P'$ is a 3-gon
$r^* = 1.75$
$R(T^4_{13}, 2) = (\frac{3}{2}, 2)$
$T^4_{13}$, $d=2$
$P'$ is a 3-gon
$r^* = 2$
$R(T^4_{13}, 3) = \{2\}$
$T^4_{15}$, $d=2$
$P'$ is a 3-gon
$r^* = 1.5$
$R(T^4_{15}, 1) = (1, 2)$
$T^4_{15}$, $d=2$
$P'$ is a 3-gon
$r^* = 2$
$R(T^4_{15}, 2) = \{2\}$
$T^5_2$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_2, 1) = \{2\}$
$T^5_5$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_5, 1) = \{2\}$
$T^5_6$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_6, 1) = \{2\}$
$T^5_9$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_9, 1) = \{2\}$
$T^5_{10}$, $d=2$
$P'$ is a 4-gon
$r^* = 2$
$R(T^5_{10}, 1) = \{2\}$
$T^5_{11}$, $d=1$
$P'$ is a segment
$r^* = 1.878$
$R(T^5_{11}, 2) = (\tau, 2)$
$T^5_{11}$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_{11}, 3) = \{2\}$
$T^5_{13}$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_{13}, 1) = \{2\}$
$T^5_{16}$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_{16}, 1) = \{2\}$
$T^5_{21}$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_{21}, 1) = \{2\}$
$T^5_{24}$, $d=1$
$P'$ is a segment
$r^* = 1.854$
$R(T^5_{24}, 2) = (1 + \frac{\sqrt{2}}{2}, 2)$
$T^5_{24}$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_{24}, 3) = \{2\}$
$T^5_{30}$, $d=2$
$P'$ is 3-gon
$r^* = 1.731$
$R(T^5_{30}, 2) \cup R(T^5_{30}, 3) = (1 + \frac{\sqrt{2}}{2}, \tau]$
$T^5_{30}$, $d=2$
$P'$ is a 4-gon
$r^* = 1.878$
$R(T^5_{30}, 4) = (\tau, 2)$
$T^5_{30}$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^5_{30}, 5) = \{2\}$
$T^6_2$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^6_2, 1) = \{2\}$
$T^6_5$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^6_5, 1) = \{2\}$
$T^6_9$, $d=1$
$P'$ is a segment
$r^* = 2$
$R(T^6_9, 1) = \{2\}$

### 5.4. Strong 2-distance configurations

For strong 2-distance configurations there is no such classification using just finite dimensional parameters. But for each affine 2-distance configuration $A$ we can give precise conditions for the unit ball of $\mathbb{M}^2$ such that $(\mathbb{M}_2, S(T,r,p)) \in A$: there must be a scaled copy of some set $F := F(T,r,p)$ contained in the unit circle.

We will not discover a complete classification of strong 2-distance configurations. It is not possible to parameterize all strong 2-distance configurations by parameters $p \in \mathbb{R}^d$ of a finite dimensional real vector space, i.e., with $d < \infty$. On the other hand, some affine 2-distance configurations $A = [(\mathbb{M}^2(T,i,r^*,p), S(T,i,r^*,p))]_a$ are in fact also strong 2-distance configurations! In these cases, the unit ball $B$ of $\mathbb{M}^2(T,i,r^*,p)$ is a polygon which is uniquely determined by $S(T,i,r^*,p)$. In general, each 2-distance set $S = S(T,i,r^*,p)$ (together with the metric determined by $L$ and $r$) determines a set $F = F(T,r^*,p)$ which must be part of the unit circle $F(T,r^*,p) \subset \partial B$ of $\mathbb{M}^2$. This condition $F(T,r^*,p) \subset \partial B$ is necessary and also sufficient for $\mathbb{M}^2$ such that $S$ induces the correct metric in $\mathbb{M}^2$ with distance values $\{1, r^*\}$.

If $F$ is in strong convex position, i.e., if each point of $F$ is a vertex of $\operatorname{conv} F$, then there is a smooth and strictly convex unit ball $B$ with $(\mathbb{M}^2(B), S) \in A$. Otherwise there are three collinear points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in F$ with $\mathbf{b} \in \operatorname{rel\,int} \overline{\mathbf{ac}}$. By convexity we get $\overline{\mathbf{ac}} \subset \partial B$. Consequently, there is no strictly convex unit ball $B$ with $(\mathbb{M}^2(B), S) \in A$. If there is another such segment $\overline{\mathbf{ce}}$ in $\partial B$, due to $\mathbf{d}, \mathbf{e} \in F$ and $\mathbf{d} \in \operatorname{rel\,int} \overline{\mathbf{ce}}$, and if $\mathbf{e}$ is not collinear with $\overline{\mathbf{ac}}$, then $\mathbf{c}$ must be a vertex of $B$, and $\partial B$ cannot be smooth. Finally, if every point of $F$ is a vertex of $B$ or belongs to the interior of a segment in $\partial B$ due to the stated reasons, then $B$ is uniquely determined by $F$.

For fixed $T$ and $i$ this combinatorial structure of $F(T,r,p)$ is not the same for all $r \in R(T,i)$ and all $p \in P(T,i,r)$. But this structure is the same for all $r \in R(T,i)$ and all $p$ belonging to the same relatively open face of the polytope $P'(T,i,r)$. So we know the combinatorial structure of the set $F(T,r,p)$ as well as the isometry group of $S(T,i,r,p)$ for each face of each set $P'(T,i,r)$, which is independent from $r \in R(T,i)$.

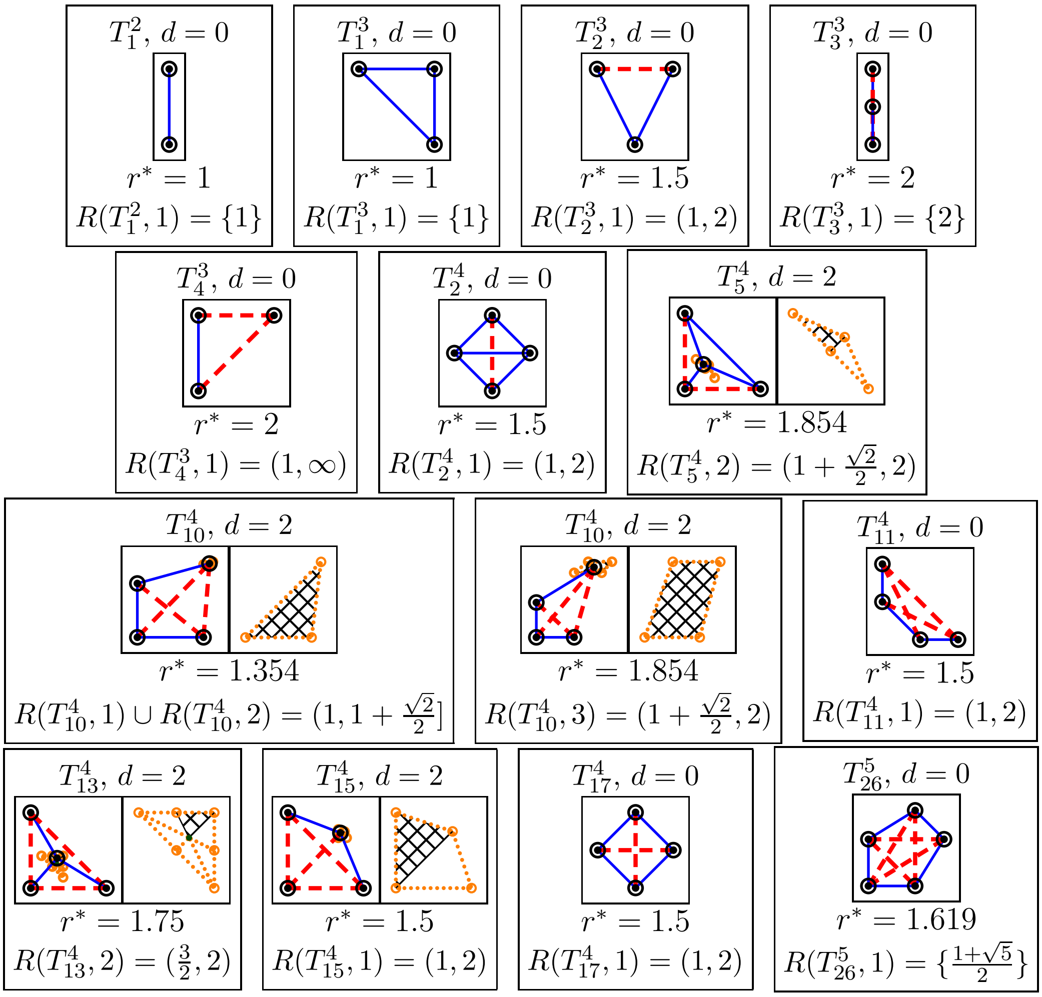
Instead of presenting the complete combinatorial structure of $F$, we restrict ourselves to distinguish three types of affine 2-distance configurations $A$.

1. There is some strictly convex Minkowski plane $\mathbb{M}^2$ with $(\mathbb{M}^2, S) \in A$.
2. There is no strictly convex Minkowski plane $\mathbb{M}^2$ with $(\mathbb{M}^2, S) \in A$, but there are planes $\mathbb{M}^2$ whose unit ball is not a polygon with $(\mathbb{M}^2, S) \in A$, i.e., $\mathbb{M}^2$ can be "partially" strictly convex.
3. There is (up to scaling) a uniquely defined plane $\mathbb{M}^2$ with $(\mathbb{M}^2, S) \in A$. This plane $\mathbb{M}^2$ has a unit ball which is a $2k$-gon ($k \in \{2,3,4\}$).

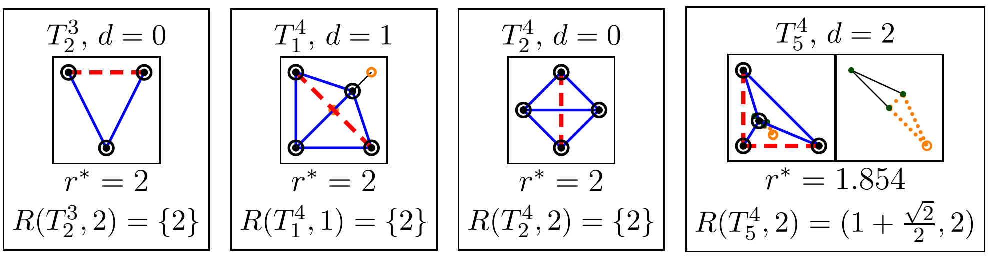
The following pictures are similar to the visualization of affine 2-distance configurations. For each affine 2-distance configuration $A$, there is exactly one picture in the corresponding section, representing for some $T = T_k^n$, $i \in [k_T]$, all $A = [(\mathbb{M}^2(T,i,r,p), S(T,i,r,p))]_a$ with $r \in R(T,i)$ and $p \in P'' \subset P(T,i,r)$. Thus

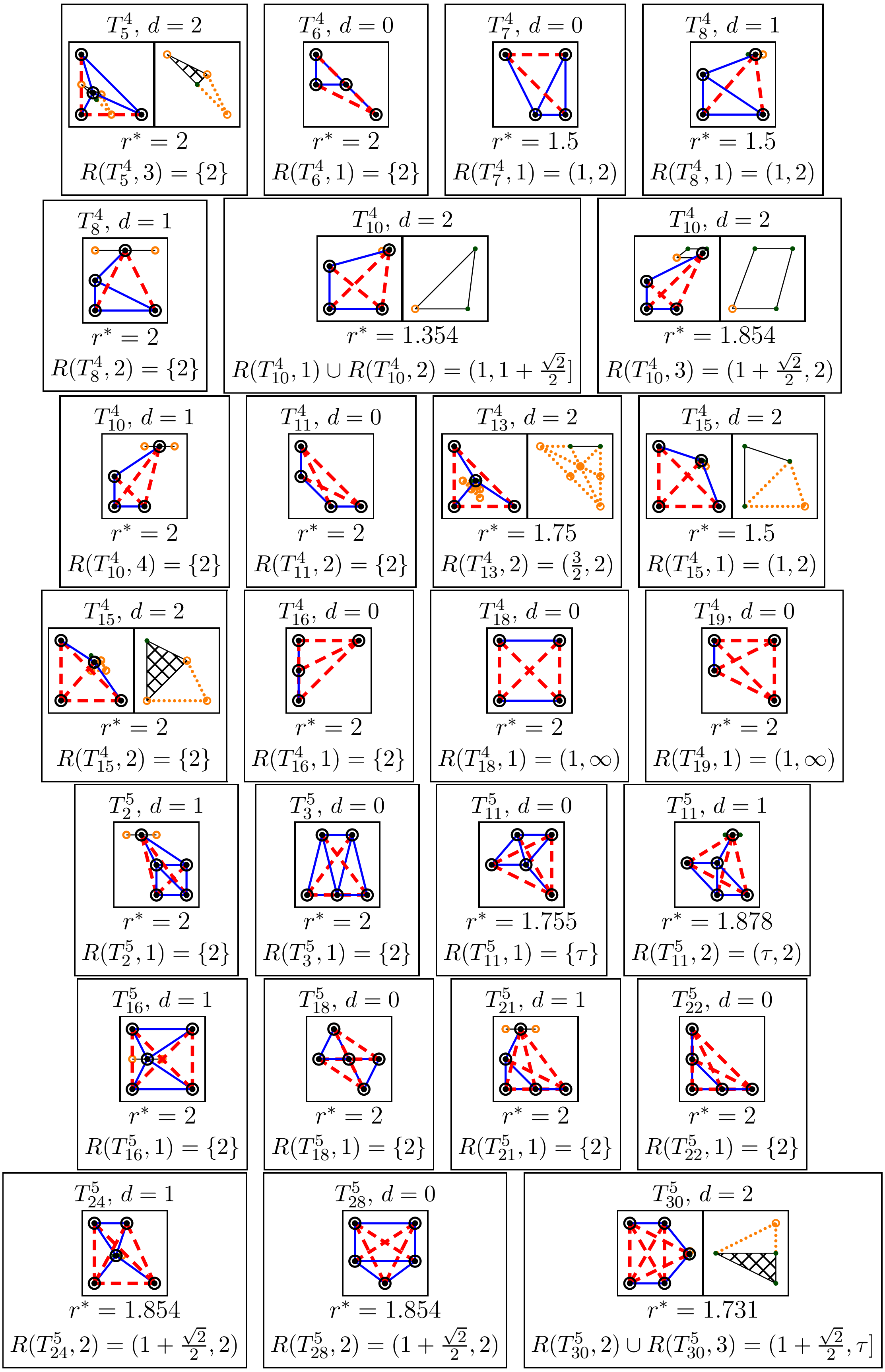
$P''$ contains the new information provided with this classification. $P''$ is the union of some faces of $P'$, drawn with the same conventions as previously used to define $P$.

### 5.4.1. Strictly convex planes possible

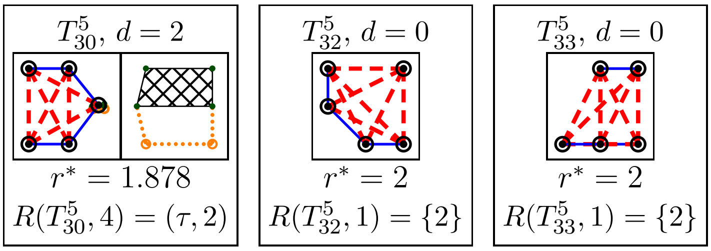



### 5.4.2. "Partially" strictly convex planes possible

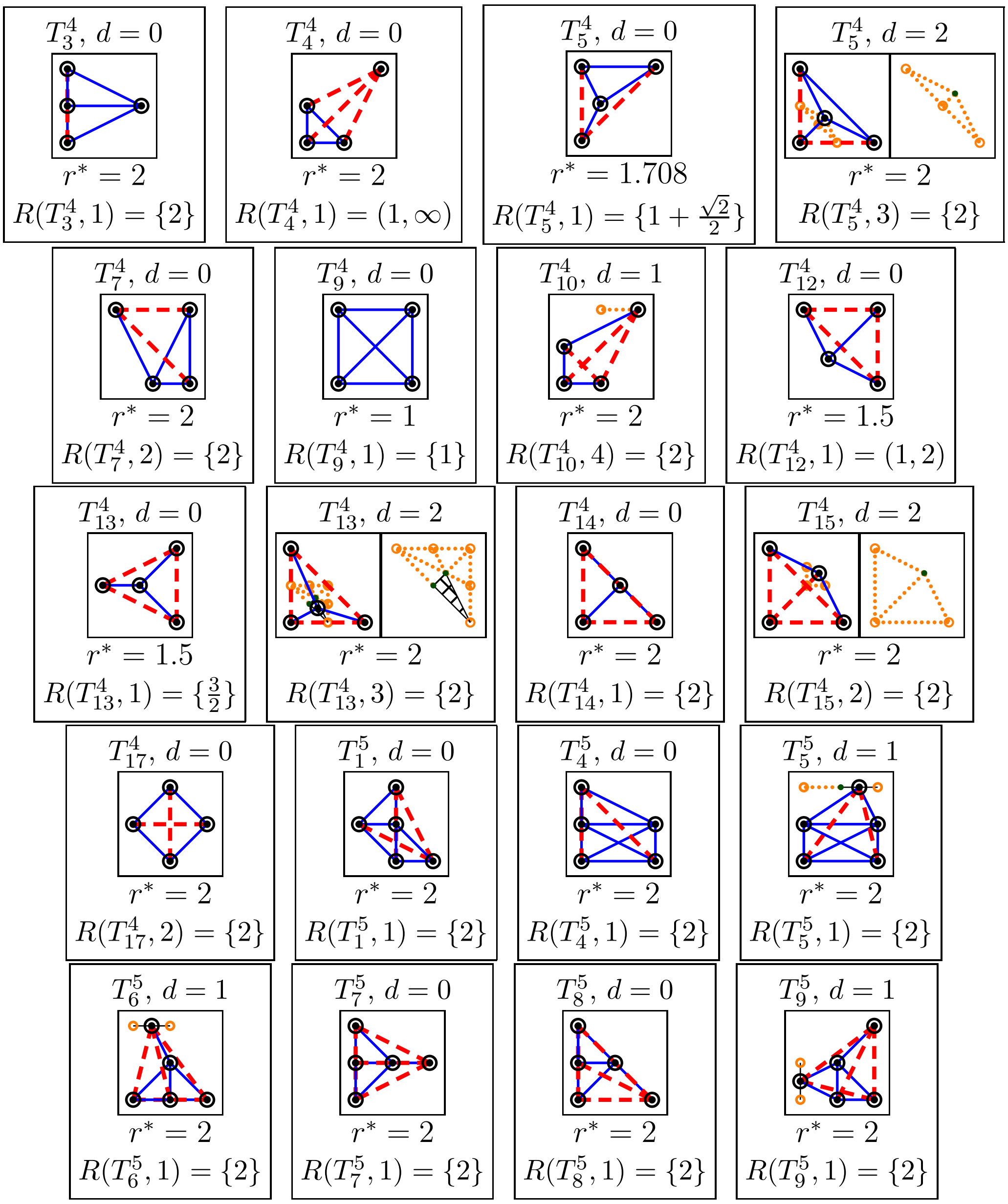


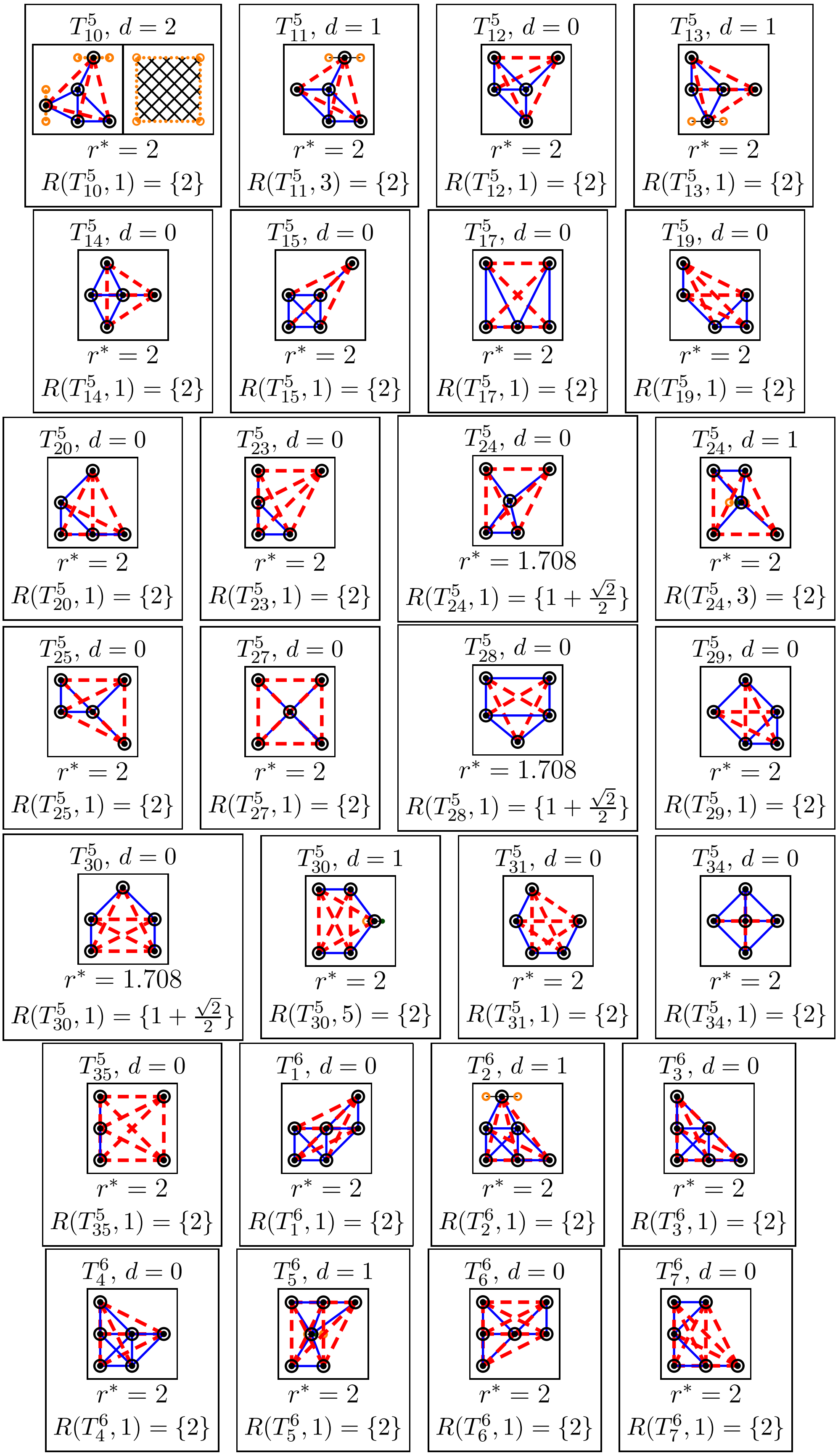


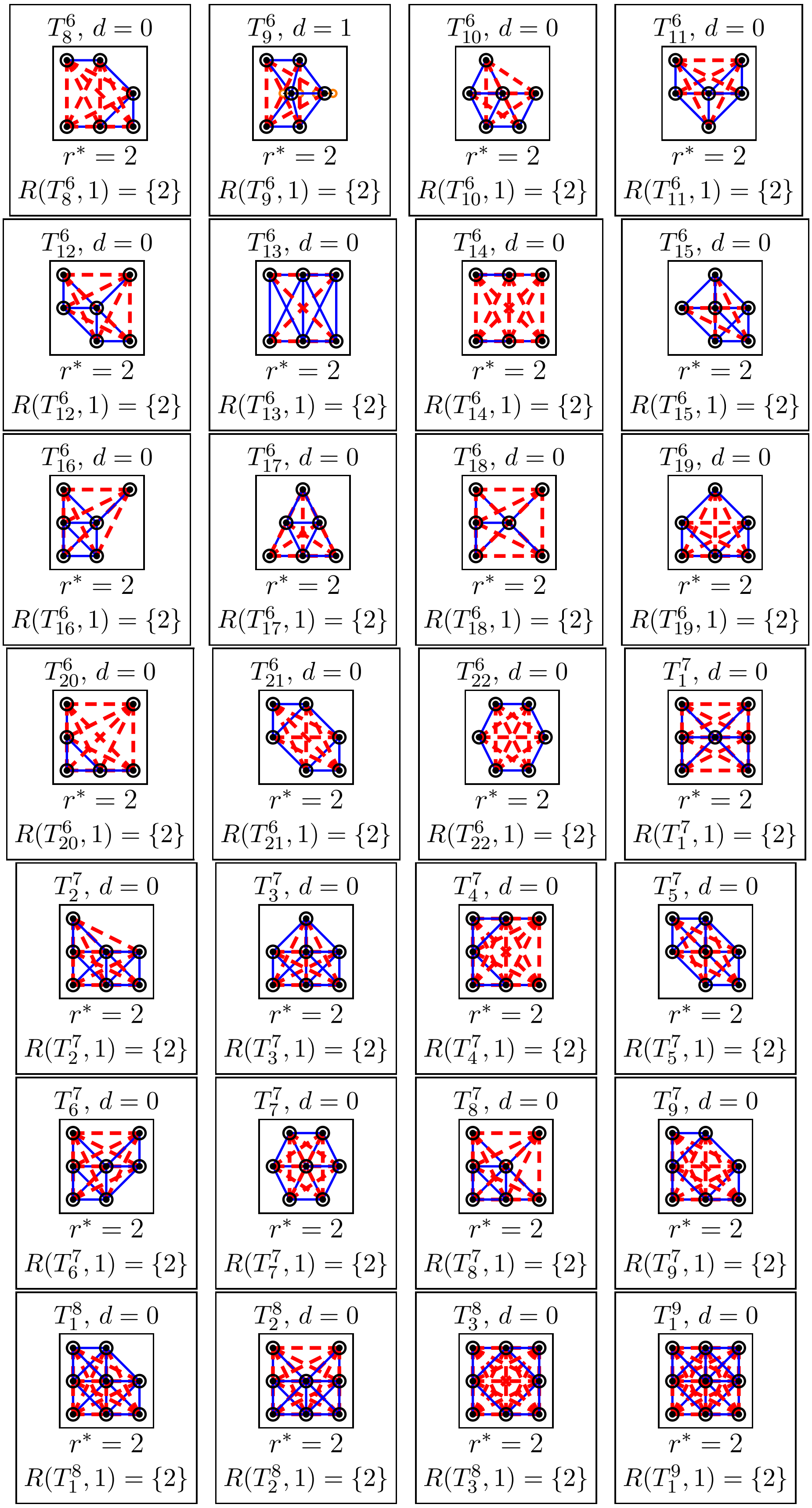
$T_5^4$, $d = 2$
$r^* = 2$
$R(T_5^4, 3) = \{2\}$
$T_6^4$, $d = 0$
$r^* = 2$
$R(T_6^4, 1) = \{2\}$
$T_7^4$, $d = 0$
$r^* = 1.5$
$R(T_7^4, 1) = (1, 2)$
$T_8^4$, $d = 1$
$r^* = 1.5$
$R(T_8^4, 1) = (1, 2)$
$T_8^4$, $d = 1$
$r^* = 2$
$R(T_8^4, 2) = \{2\}$
$T_{10}^4$, $d = 2$
$r^* = 1.354$
$R(T_{10}^4, 1) \cup R(T_{10}^4, 2) = (1, 1 + \frac{\sqrt{2}}{2}]$
$T_{10}^4$, $d = 2$
$r^* = 1.854$
$R(T_{10}^4, 3) = (1 + \frac{\sqrt{2}}{2}, 2)$
$T_{10}^4$, $d = 1$
$r^* = 2$
$R(T_{10}^4, 4) = \{2\}$
$T_{11}^4$, $d = 0$
$r^* = 2$
$R(T_{11}^4, 2) = \{2\}$
$T_{13}^4$, $d = 2$
$r^* = 1.75$
$R(T_{13}^4, 2) = (\frac{3}{2}, 2)$
$T_{15}^4$, $d = 2$
$r^* = 1.5$
$R(T_{15}^4, 1) = (1, 2)$
$T_{15}^4$, $d = 2$
$r^* = 2$
$R(T_{15}^4, 2) = \{2\}$
$T_{16}^4$, $d = 0$
$r^* = 2$
$R(T_{16}^4, 1) = \{2\}$
$T_{18}^4$, $d = 0$
$r^* = 2$
$R(T_{18}^4, 1) = (1, \infty)$
$T_{19}^4$, $d = 0$
$r^* = 2$
$R(T_{19}^4, 1) = (1, \infty)$
$T_2^5$, $d = 1$
$r^* = 2$
$R(T_2^5, 1) = \{2\}$
$T_3^5$, $d = 0$
$r^* = 2$
$R(T_3^5, 1) = \{2\}$
$T_{11}^5$, $d = 0$
$r^* = 1.755$
$R(T_{11}^5, 1) = \{\tau\}$
$T_{11}^5$, $d = 1$
$r^* = 1.878$
$R(T_{11}^5, 2) = (\tau, 2)$
$T_{16}^5$, $d = 1$
$r^* = 2$
$R(T_{16}^5, 1) = \{2\}$
$T_{18}^5$, $d = 0$
$r^* = 2$
$R(T_{18}^5, 1) = \{2\}$
$T_{21}^5$, $d = 1$
$r^* = 2$
$R(T_{21}^5, 1) = \{2\}$
$T_{22}^5$, $d = 0$
$r^* = 2$
$R(T_{22}^5, 1) = \{2\}$
$T_{24}^5$, $d = 1$
$r^* = 1.854$
$R(T_{24}^5, 2) = (1 + \frac{\sqrt{2}}{2}, 2)$
$T_{28}^5$, $d = 0$
$r^* = 1.854$
$R(T_{28}^5, 2) = (1 + \frac{\sqrt{2}}{2}, 2)$
$T_{30}^5$, $d = 2$
$r^* = 1.731$
$R(T_{30}^5, 2) \cup R(T_{30}^5, 3) = (1 + \frac{\sqrt{2}}{2}, \tau]$




### 5.4.3. Unique polygonal unit ball




$T^5_{10}$, $d = 2$
$r^* = 2$
$R(T^5_{10}, 1) = \{2\}$
$T^5_{11}$, $d = 1$
$r^* = 2$
$R(T^5_{11}, 3) = \{2\}$
$T^5_{12}$, $d = 0$
$r^* = 2$
$R(T^5_{12}, 1) = \{2\}$
$T^5_{13}$, $d = 1$
$r^* = 2$
$R(T^5_{13}, 1) = \{2\}$
$T^5_{14}$, $d = 0$
$r^* = 2$
$R(T^5_{14}, 1) = \{2\}$
$T^5_{15}$, $d = 0$
$r^* = 2$
$R(T^5_{15}, 1) = \{2\}$
$T^5_{17}$, $d = 0$
$r^* = 2$
$R(T^5_{17}, 1) = \{2\}$
$T^5_{19}$, $d = 0$
$r^* = 2$
$R(T^5_{19}, 1) = \{2\}$
$T^5_{20}$, $d = 0$
$r^* = 2$
$R(T^5_{20}, 1) = \{2\}$
$T^5_{23}$, $d = 0$
$r^* = 2$
$R(T^5_{23}, 1) = \{2\}$
$T^5_{24}$, $d = 0$
$r^* = 1.708$
$R(T^5_{24}, 1) = \{1 + \frac{\sqrt{2}}{2}\}$
$T^5_{24}$, $d = 1$
$r^* = 2$
$R(T^5_{24}, 3) = \{2\}$
$T^5_{25}$, $d = 0$
$r^* = 2$
$R(T^5_{25}, 1) = \{2\}$
$T^5_{27}$, $d = 0$
$r^* = 2$
$R(T^5_{27}, 1) = \{2\}$
$T^5_{28}$, $d = 0$
$r^* = 1.708$
$R(T^5_{28}, 1) = \{1 + \frac{\sqrt{2}}{2}\}$
$T^5_{29}$, $d = 0$
$r^* = 2$
$R(T^5_{29}, 1) = \{2\}$
$T^5_{30}$, $d = 0$
$r^* = 1.708$
$R(T^5_{30}, 1) = \{1 + \frac{\sqrt{2}}{2}\}$
$T^5_{30}$, $d = 1$
$r^* = 2$
$R(T^5_{30}, 5) = \{2\}$
$T^5_{31}$, $d = 0$
$r^* = 2$
$R(T^5_{31}, 1) = \{2\}$
$T^5_{34}$, $d = 0$
$r^* = 2$
$R(T^5_{34}, 1) = \{2\}$
$T^5_{35}$, $d = 0$
$r^* = 2$
$R(T^5_{35}, 1) = \{2\}$
$T^6_{1}$, $d = 0$
$r^* = 2$
$R(T^6_{1}, 1) = \{2\}$
$T^6_{2}$, $d = 1$
$r^* = 2$
$R(T^6_{2}, 1) = \{2\}$
$T^6_{3}$, $d = 0$
$r^* = 2$
$R(T^6_{3}, 1) = \{2\}$
$T^6_{4}$, $d = 0$
$r^* = 2$
$R(T^6_{4}, 1) = \{2\}$
$T^6_{5}$, $d = 1$
$r^* = 2$
$R(T^6_{5}, 1) = \{2\}$
$T^6_{6}$, $d = 0$
$r^* = 2$
$R(T^6_{6}, 1) = \{2\}$
$T^6_{7}$, $d = 0$
$r^* = 2$
$R(T^6_{7}, 1) = \{2\}$


$T^6_8$, $d = 0$
$r^* = 2$
$R(T^6_8, 1) = \{2\}$
$T^6_9$, $d = 1$
$r^* = 2$
$R(T^6_9, 1) = \{2\}$
$T^6_{10}$, $d = 0$
$r^* = 2$
$R(T^6_{10}, 1) = \{2\}$
$T^6_{11}$, $d = 0$
$r^* = 2$
$R(T^6_{11}, 1) = \{2\}$
$T^6_{12}$, $d = 0$
$r^* = 2$
$R(T^6_{12}, 1) = \{2\}$
$T^6_{13}$, $d = 0$
$r^* = 2$
$R(T^6_{13}, 1) = \{2\}$
$T^6_{14}$, $d = 0$
$r^* = 2$
$R(T^6_{14}, 1) = \{2\}$
$T^6_{15}$, $d = 0$
$r^* = 2$
$R(T^6_{15}, 1) = \{2\}$
$T^6_{16}$, $d = 0$
$r^* = 2$
$R(T^6_{16}, 1) = \{2\}$
$T^6_{17}$, $d = 0$
$r^* = 2$
$R(T^6_{17}, 1) = \{2\}$
$T^6_{18}$, $d = 0$
$r^* = 2$
$R(T^6_{18}, 1) = \{2\}$
$T^6_{19}$, $d = 0$
$r^* = 2$
$R(T^6_{19}, 1) = \{2\}$
$T^6_{20}$, $d = 0$
$r^* = 2$
$R(T^6_{20}, 1) = \{2\}$
$T^6_{21}$, $d = 0$
$r^* = 2$
$R(T^6_{21}, 1) = \{2\}$
$T^6_{22}$, $d = 0$
$r^* = 2$
$R(T^6_{22}, 1) = \{2\}$
$T^7_1$, $d = 0$
$r^* = 2$
$R(T^7_1, 1) = \{2\}$
$T^7_2$, $d = 0$
$r^* = 2$
$R(T^7_2, 1) = \{2\}$
$T^7_3$, $d = 0$
$r^* = 2$
$R(T^7_3, 1) = \{2\}$
$T^7_4$, $d = 0$
$r^* = 2$
$R(T^7_4, 1) = \{2\}$
$T^7_5$, $d = 0$
$r^* = 2$
$R(T^7_5, 1) = \{2\}$
$T^7_6$, $d = 0$
$r^* = 2$
$R(T^7_6, 1) = \{2\}$
$T^7_7$, $d = 0$
$r^* = 2$
$R(T^7_7, 1) = \{2\}$
$T^7_8$, $d = 0$
$r^* = 2$
$R(T^7_8, 1) = \{2\}$
$T^7_9$, $d = 0$
$r^* = 2$
$R(T^7_9, 1) = \{2\}$
$T^8_1$, $d = 0$
$r^* = 2$
$R(T^8_1, 1) = \{2\}$
$T^8_2$, $d = 0$
$r^* = 2$
$R(T^8_2, 1) = \{2\}$
$T^8_3$, $d = 0$
$r^* = 2$
$R(T^8_3, 1) = \{2\}$
$T^9_1$, $d = 0$
$r^* = 2$
$R(T^9_1, 1) = \{2\}$

## 6. Verification of the classification

Obviously, the stated classification of 2-distance configurations is correct if this holds for the classification of full 2-distance configurations.

The main challenge is to prove that this classification is complete, see Sections 6.3 and 6.5. It is not difficult to see that all the stated descriptions in fact represent some 2-distance sets, and that their equivalence classes are really distinct. The first part was assured additionally by concrete examples which were checked automatically. The second part can be certified with suitable invariants.

We will also discuss how the classification of affine 2-distance configuration can be verified in Section 6.6. Of course, this process operates on the analytical descriptions which were not shown. The presented pictures were produced automatically from this data.

The presented categories regarding strong 2-distance configurations, as well as complete descriptions of the combinatorics of the sets $F(T, r, p)$ which were omitted, were directly constructed from the classification of affine 2-distance configurations and its certificates, see Section 6.8.

### 6.1. General proof methods and certificates

Instead of publishing the complete software for generating the classification and proving its correctness, we generated additional information, called certificates. Certificates can be considered as some specific information to obtain from a general proof method for some generic assertion a specific proof for a concrete assertion.

To verify that the certificate fits the general proof method for the given assertion, a lot of mathematical conditions have to be checked. These tests can either be checked by simple calculations or there are further certificates contained which must be checked recursively by another general proof method.

Although theoretically this verification can be done by a human being, the large number of required steps for the classification of 2-distance configurations makes a complete manual verification impossible. But nevertheless we can check manually some samples.

Because of that, we have to use a computer program to verify our assertions by some general proof methods using the given certificates. Note that the requirements of the formal representation of statements and certificates are different for usage by a human or by a computer program which is designed to be easily understood by other mathematicians. We have chosen to use a representation suitable for a computer program. Some conversion tools are provided to get a human readable representation as well.

In the following we introduce generic assertions, the corresponding certificates and sketch general proof methods. The concrete assertions and certificates are provided by the author on his web page [13], together with source code and documentation for the implemented verification computer programs. For more details, the chosen representation and how to use these programs please see there.

### 6.2. Number representation and basic certificates

As usual, we represent integer numbers by the optional sign "-" and a sequence of decimal digits. Rational numbers $r \in \mathbb{Q}$ can be represented as pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$, $r = \frac{p}{q}$. This representation is unique if $p, q$ are relatively prime. Mono-variate polynomials $p \in R[X]$ over some ring $R$ (e.g., $R = \mathbb{Z}$) can be represented by the finite sequence $(p_0, \ldots, p_d)$ in $R$ for $p(X) = p_0 + p_1 X + \cdots + p_d X^d$. Intervals in $\mathbb{R}$ whose boundary belongs to the set $R \subset \mathbb{R}$ can easily be represented by symbolic classification of the lower and upper boundaries as "open" or "closed" and the corresponding bounds in $R$ with added symbolic representation of $\pm\infty$. All algebraic numbers $a$ can be represented by $(i, p) \in \mathbb{N} \times \mathbb{Z}[X]$ if $a \in \mathcal{A}$ is the $i$-th smallest root of $p$, $a = \mathrm{RootOf}(i, p)$. This representation is unique if $p$ is irreducible and has a positive leading coefficient.

Now the relations $a < b$, $a = b$ as well as simple arithmetic $a = b + c$, $a = b \cdot c$ is easily checked for $a, b \in \mathbb{Q}$ given in the above representations without a need for certificates. What about algebraic numbers $a, b \in \mathcal{A}$? Using the Core Library [3], we can check these relations without certificates, too. Without using the Core Library we can verify the assertion $a < b$ for $a := \mathrm{RootOf}(i, p)$ and $b \in \mathbb{Q}$ or vice versa by Sturm sequences, see [9], without certificates. Thus to prove $a := \mathrm{RootOf}(i_a, p_a) < b := \mathrm{RootOf}(i_b, p_b)$ we can use any $r \in \mathbb{Q}$ with $a < r < b$ as certificate. $a := \mathrm{RootOf}(i_a, p_a) = b := \mathrm{RootOf}(i_b, p_b)$ is for irreducible $p_a, p_b$ and positive leading coefficients only possible for $(i_a, p_a) = (i_b, p_b)$. Otherwise we can prove the equation using a factorization of $p_a$ and $p_b$ together with a rational interval isolating $a = b$ from all remaining roots of $p_a$ or $p_b$ as certificate.

These comparisons are sufficient to verify the element relation $a \in S$, where $a := \mathrm{RootOf}(i, p) \in \mathcal{A}$ and $S$ is a semi-algebraic subset of $\mathbb{R}$, represented as finite sequence of intervals with boundaries in $\mathcal{A}$.

Finally, for a semi-algebraic subset $S$ of $\mathbb{R}$ in the above representation and any $p \in \mathbb{Z}[X]$ we can construct certificates to verify that $p(s) = 0$, $p(s) > 0$, or $p(s) \geq 0$, respectively, for all $s \in S$.

### 6.3. Complete list of candidates of full 2-distance configurations

In a first combinatorial step, we generated a list of complete position-metric-lists $w$ which might be representations of 2-distance configurations. We call them *candidates of* 2*-distance configurations.* For $r \in \mathbb{R}$, the pair $(w, r)$ is a *candidate of a full* 2*-distance configuration.* More precisely we generated a list of pairs $(w_i, P_i)$ which has the property, that every full 2-distance configuration can be represented as $(w_i, r)$ with $r \in P_i$ for some $(w_i, P_i)$ occurring in the produced list. We call any pair $(w, P)$ consisting of a (not necessarily complete) position-metric-list $w$ of order $n$ together with a set $P \subset \mathbb{R}$ a *parameter-position-metric-list* of order $n$.

In Section 6.5 we will distinguish in the second, more analytical step, these candidates $(w_i, r)$ which really represent a full 2-distance configuration – called *realizable candidate* – from the unrealizable candidates.

Note that we repeated these two steps several times for different numbers $n$, since the complete list without unrealizable candidates for $n - 1$ points provide the input for producing the list of candidates with $n$ points.

We call any position-metric-list $w'$ a *sublist* of the position-metric-list $w = (w_1, \ldots, w_k)$, if there is a sequence $(w'_1, \ldots, w'_k)$ of sets of abstract directions which is derived from $w'$ by inserting some empty sets, and with $w'_i \subset w_i$ for all $i \in [k]$.

Two position-metric-list $w, w'$ of order $n$ are called *isomorphic*, if there is a permutation $\phi$ of $[n]$ and some $s \in \mathbb{Z}$ with $\phi(w) = \mathrm{Shift}(w', s)$ or $\phi(w) = \mathrm{Shift}(\mathrm{Mirror}(w'), s)$, where the operations of permutation, Shift and Mirror are defined for position-metric-lists the same way as for abstract oriented position lists in Remark 40, by (32) and by (33). There is one exception concerning equilateral sets, since these can be represented twice with all symbolic distances equal to $S$ or all equal to $B$. So $w$ and $w'$ are isomorphic if we can transform $w'$ in a sequence of possibly reorientation, possibly switching all distances of an equilateral set, and a shift to $w$. Provided that at least one of $w$ and $w'$ represents a 2-distance configuration, $w$ and $w'$ are isomorphic if and only if they represent the same 2-distance configuration.

Assume that $I$ is a finite sequences of parameter-position-metric-lists. We say that the candidate $(w, r)$ *is isomorphically contained in* $I$ if there is some $(w^*, P^*)$ in $I$ with $r \in P^*$ and some sublist of $w$ is isomorphic to $w^*$.

### 6.3.1. Generic assertions

**Definition 44.** *For two finite sequences $Q$ and $I$ of parameter-position-metric-lists of order $n$ the assertion* $\mathrm{AssFTDCCompleteN}(n, Q, I)$ *means the following: For each complete position-metric-list $w$ of order $n' \geq n$, and $r \in \mathbb{R}$, such that $(w, r)$ is isomorphically contained in $Q$, then $(w, r)$ is also isomorphically contained in $I$.*

We will use for each $n = 3, \ldots, 9$ one main instance of $\mathrm{AssFTDCCompleteN}(n, Q, I)$. $Q$ represents exactly all full 2-distance configurations with $n - 1$ points. $I = (U, C)$ contains all candidates $U_i$ of at most $n - 1$ points which turned out (via step 2) to be unrealizable together with all candidates $C_i$ of full 2-distance configurations with exactly $n$ points not containing isomorphic copies of the unrealizable candidates with less than $n$ points, but without isomorphic duplicates. This second part $C$ is exactly the result of the current first step, while $U$ and $Q$ represent results of both steps for smaller $n$.

Additionally, a lot (about 32,000,000) of instances $\mathrm{AssFTDCCompleteN}(n, Q', I)$ where $Q'$ contains just one parameter-position-metric-list will serve as lemmas to verify recursively $\mathrm{AssFTDCCompleteN}(n, Q, I)$.

**Definition 45.** *For a finite sequences $I$ of parameter-position-metric-lists the assertion* $\mathrm{AssFTDCComplete}(I)$ *means that any full 2-distance configuration in a Minkowski plane has a representation as $(w, r)$ such that there is some $P \subset \mathbb{R}$ with $r \in P$ and $(w, P)$ is contained in $I$. For $r = 1$ (equilateral sets) this holds true with $P = [1, \infty)$.*

Please note that also the converse will be true but is not claimed by AssCompleteFTDC($I$): each $(w, P)$ in $I$ represents either equilateral sets with $P = [1, \infty)$

or $w$ is not equilateral and for all $r \in P$ the candidate $(w, r)$ is realizable by a full 2-distance configuration, which is represented this way only once.

In Section 6.5 we will see how to verify the following assertions.

**Definition 46.** *For a parameter-position-metric-lists* $(w, P)$ *the assertion* AssNonRealizableFTDC$(w, P)$ *means that for all* $r \in P$ *the candidate* $(w, r)$ *does not represent a full* 2*-distance configuration in any Minkowski plane.*

### 6.3.2. General proof methods and generic certificates

We will prove AssFTDCComplete$(I)$ by a sequence of verified statements AssFTD CCompleteN$(3, Q^2, I^3)$, AssFTDCCompleteN$(4, Q^3, I^4)$, ..., AssFTDCComplete N$(9, Q^8, I^9)$ with $I^3 = C^3$, $I^4 = (U^3, C^4)$, $I^5 = (U^3, U^4, C^5)$, ..., $I^9 = (U^3, U^4, \ldots, U^8, C^9)$, $Q^9 := C^9$ and $I = (Q^2, Q^3, \ldots, Q^9)$. Additionally we need for all $U_i$ from the sequence $U = (U^3, U^4, \ldots, U^8)$ verified statements AssFTDCUnrealizable$(U_i)$, and some mappings of integers representing the relations between the components of $I, Q^2, \ldots, Q^8, I^3, \ldots, I^9$. By induction on $n$ we can show that $Q^n$ is a list containing representations for all full 2-distance configurations of exactly $n$ points, provided that for all $(w, P)$ in $C^n$ the position-metrics-lists $w$ are complete of order $n$, and all these $(w, P)$ occur in $U^n$ or $Q^n$ or there are $(w, P_U) = U^n_i$ and $(w, P_Q) = Q^n_j$ with $P = P_U \cup P_Q$.

Please note that $U^6$, ..., $U^8$ (and $U^9$) were in fact empty since all candidates in $C^6$, ..., $C^9$ were realizable as full 2-distance configurations.

We can prove AssFTDCCompleteN$(n, Q, I)$ by a sequence of verified assertions AssFTDCCompleteN$(n, (Q_i), I)$. For one-element lists $Q'$ we use five different certificates for AssFTDCCompleteN$(n, Q', I)$. The simplest certificate only contains the index $i$ if $(w', P') := Q'_1 = I_i =: (w_i, P_i)$, or at least $w' = w_i$ and $P' \subset P_i$. Another two certificates represent the transformation necessary to get $w''$ from $w'$ which is isomorphic to a sublist of $w'$ and a reference to AssFTDCCompleteN$(n, ((w'', P')), I)$ with certificate. Also the verified assertions AssFTDCCompleteN$(n, ((w', P_1)), I)$ and AssFTDCCompleteN$(n, ((w', P_2)), I)$ yield a certificate for AssFTDCCompleteN$(n, ((w', P')), I)$ if $P' \subset P_1 \cup P_2$.

But the main idea behind the generation of the list $C^n$ is represented by a certificate which considers for fixed $(i, j) \in \mathbb{P}_n$ all possibilities for $(i, j, \varrho)$ to be added into $w$ or a similar position-metric-list, with $\varrho \in G$. If $w' = (w'_1, \ldots, w'_k)$ is not equilateral, then there are $4k$ possibilities for both variants $\varrho = S$ and $\varrho = B$, namely $(w'_1 \cup \{(i, j, \varrho)\}, w'_2, \ldots, w'_k)$, $(w'_1, \{(i, j, \varrho)\}, w'_2, \ldots, w'_k)$, $(w'_1, w'_2 \cup \{(i, j, \varrho)\}, w'_3, \ldots, w'_k)$, $(w'_1, w'_2, \{(i, j, \varrho)\}, w'_3, \ldots, w'_k)$, ..., $(w'_1, \ldots, w'_{k-1}, w'_k \cup \{(i, j, \varrho)\})$ and $(w'_1, \ldots, w'_k, \{(i, j, \varrho)\})$, and the same again with $(j, i, \varrho)$ instead of $(i, j, \varrho)$. For each of these $8k$ new position-metric-lists $w^x$ we need a statement AssFTDCCompleteN$(n, ((w^x, P')), I)$ with certificates. Again, the situation is a little more complicated for equilateral $w'$. Then there are another $4k$ possibilities by first switching all distances in $w'$ and afterwards inserting $(i, j, \varrho)$ and $(j, i, \varrho)$, where $\varrho$ is the distance occurring in $w'$.

We achieved the following numbers of realizable and unrealizable parameter-metric-position-lists representing candidates of full 2-distance configurations:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $\lvert C^n\rvert$ | | 26 | 121 | 132 | 22 | 9 | 3 | 1 |
| $\lvert U^n\rvert$ | | 24 | 106 | 108 | 0 | 0 | 0 | 0 |
| $\lvert Q^n\rvert$ | 1 | 4 | 19 | 35 | 22 | 9 | 3 | 1 |

### 6.4. Embedding systems for candidates of full 2-distance configurations

Sections 2 and 3 described the main tools and techniques for embedding problems in Minkowski planes.

Using Theorem 17 together with Lemma 34 we can build for each candidate $(w^*, P^*)$ in our list $I^n$ the parameterized system $\mathrm{SysEmP}(\rho(w^*, X), l(w^*))$ where

$$\rho(w, X) : (i, j) \mapsto \begin{cases} 0 & \text{if } i = j, \\ X & \text{if } \{i, j\} \in L(w) := \{\, \{x, y\} \mid \exists k : (x, y, B) \in w_k \,\}, \\ 1 & \text{otherwise}. \end{cases}$$

Then $(w^*, r)$ is realizable if and only if $\mathrm{SysEmP}(\rho(w^*, r), l(w^*))$ is admissible. We need certificates for all $r \in P^*$ for which this system is not admissible.

To obtain the classification of affine 2-distance configurations we have to solve these systems completely. After identifying solutions in $L(\mathrm{SysEmP}(\rho(w^*, r), l(w^*)))$ which represent affinely equivalent 2-distance sets due to symmetry, this corresponds directly to the set of all affine 2-distance configurations contained in the full 2-distance configuration represented by $(w^*, r)$.

$\mathrm{SysEmP}(\rho(w^*, X), l(w^*))$ has homogeneous polynomial restrictions in the unknown variables $\underline{b}$ with coefficients in $\mathbb{Z}[X]$, i.e., the coefficients depend polynomially on the real parameter $r$.

Instead of solving these systems directly, we solved first the linear subsystems $\mathrm{SysEmPLin}(\rho(w^*, X), l(w^*))$. Due to the very special metrics the solution sets of $\mathrm{SysEmPLin}(\rho(w^*, X), l(w^*))$ turned out to be really nice. First, with only one exception the dimension of the solution set is at most 3. Second, with the same exception this solution set was also the solution set of the original system: $L(\mathrm{SysEmPLin}(\rho(w^*, X), l(w^*))) = L(\mathrm{SysEmP}(\rho(w^*, X), l(w^*)))$. This was a big surprise for $n \geq 5$, while for $n \leq 4$ both systems have the same restrictions! This result was obtained by checking each quadratic restriction $\mathrm{GPR}_{I,J}$ in $\mathrm{SysEmP}(\rho(w^*, X), l(w^*))$ for common validity in the polyhedral cone $L(\mathrm{SysEmPLin}(\rho(w^*, X), l(w^*)))$, see Section 6.6.2.

### 6.5. Realizability of candidates of full 2-distance configurations

Now we take a closer look on how to decide whether or not a candidate $(w, r)$ of a full 2-distance configuration is realizable. Since realizable candidates can easily be verified with an example 2-distance set with polynomial coordinates, our main focus is on certificates for the assertion $\mathrm{AssFTDCUnrealizable}(w, P)$ for $P \subset \mathbb{R}$.

Luckily, $\mathrm{SysEmPLin}(\rho(w, r), l(w))$ is not admissible for all $r \in P$ for all $(w, P)$ which occurred in $U^n$.

But for linear systems of equations and inequalities it is easy to find certificates using Farkas' Lemma and dual linear systems.

Note that the real parameter $X = r$ in SysEmPLin$(\rho(w, X), l(w))$ needs some extra thoughts to adapt the well developed techniques to solve linear systems to parameterized linear systems. Using Cramer's rule we can express the solutions of linear equations with polynomial coefficients by rational functions in the parameter. For homogeneous linear systems we even get polynomial solutions. The second basic operation to solve linear systems of equations and inequalities is to determine the sign of coefficients of the restriction functions and of the evaluated restrictions at solutions like above. In our setting, this means to determine the sign of $f(X)$ or of $f(r)$ for $f \in \mathbb{Z}[X]$. This yields at most three semi-algebraic subsets of $\mathbb{R}$ such that if $r$ is within these sets sign $f(r)$ is constants and known. Finally, we get a partitioning of $P$ into intervals (including singletons) such that within each interval a classical algorithm to solve the systems can use polynomial arithmetic and determine uniquely the sign of all expressions where this is necessary.

**Remark 47.** One possibility to obtain this partitioning and the corresponding polynomial solutions is to repeatedly

1. evaluate the parameterized system at (rational) values $r$ for which the solution is still unknown,
2. then solve the evaluated system using rational or algebraic arithmetic and comparison,
3. then generalize the solution using polynomial arithmetic, and finally
4. determine the largest parameter set within which the polynomial solution is still valid.

**Definition 48.** *A certificate $Z^{\mathrm{impl},u,(E,W,S)}$ for the assertion that $u \in W \cup S$ is an implicit equation of the system $(E, W, S)$ in $X$ is a vector $Z^{\mathrm{impl},u,(E,W,S)} \in \mathbb{R}^{E \cup W \cup S}$.*

$Z^{\mathrm{impl},u,(E,W,S)}$ ensures that $u : X \to \mathbb{R}$ is an implicit equation of $(E, W, S)$, i.e., that $u(x) = 0$ for all $x \in L(E, W, S)$, if $\sum_{i \in E \cup W \cup S} Z_i^{\mathrm{impl},u,(E,W,S)} \cdot i(x) = \mathbf{0}$ for all $x \in X$, $Z_i^{\mathrm{impl},u,(E,W,S)} \geq 0$ for all $i \in W \cup S$, and finally $Z_u^{\mathrm{impl},u,(E,W,S)} > 0$. For parameterized system such as SysEmPLin$(\rho(w, X), l(w))$ this becomes $Z^{\mathrm{impl},u(X),(E(X),W(X),S(X))} \in \mathbb{Z}[X]^{E(X) \cup W(X) \cup S(X)}$ with the obvious meaning that its evaluation at $X = r$ is a certificate that $u(r)$ is an implicit equation of the evaluated system $(E(r), W(r), S(r))$ for all $r$ belonging to an explicitly given set $P$. In our implementation we represent this vector by its nonzero components and we use integer indices, thus $Z^{\mathrm{impl},u(X),(E(X),W(X),S(X))} \in \mathbb{Z}[X]^k$ with attached combinatorial descriptions of the $k$ involved restrictions of the system.

**Definition 49.** *A certificate for the assertion that $(E, W, S)$ is not admissible is any $s \in S$ together with a certificate $Z^{\mathrm{impl},s,(E,W,S)}$ that $s$ is an implicit equation of $(E, W, S)$.*

If the certificate $Z^{\mathrm{impl},s,(E,W,S)}$ is valid then in fact $(E,W,S)$ is not admissible, since $s(x) > 0$ and $s(x) = 0$ for all $x \in L(E,W,S)$.

Note that for all homogeneous linear systems which are not admissible such a certificate exists and can be calculated.

Now a certificate for AssFTDCUnrealizable$(w,P)$ consists of one or more certificates for the assertion that SysEmPLin$(\rho(w,r), l(w))$ is not admissible for all $r \in P_i$, with $P \subset \bigcup_i P_i$, possibly together with basic certificates for the correct sign of polynomials in some intervals, see Section 6.2.

### 6.6. Affine 2-distance configurations

There are four steps to obtain the classification of affine 2-distance configurations.

First, we determine and verify a complete description of $H := L(\mathrm{SysEmPLin}$ $(\rho(w,r), l(w))) \neq \emptyset$, see 6.6.1.

Second, we will investigate the solution set $X := L(\mathrm{SysEmP}(\rho(w,r), l(w)))$ from $H$, see 6.6.2 and 6.6.3.

Then we have to reduce symmetrical copies of the same affine 2-distance configuration from $X$. For the symmetries in connection with permutations of the points see 6.6.4.

And last, we will transform the reduced solution set $Y \subset \mathbb{R}^{m(d(l(w)),n)}$ into a set $Z \subset (\mathbb{R}^2)^n$ consisting of all coordinates of exactly one (labeled) 2-distance set for every affine 2-distance configuration, see Section 6.7. These sets were illustrated in Section 5.3.

#### 6.6.1. Complete solution set of linear systems

Let us denote SysEmPLin$(\rho(w,r), l(w)) =: (E,W,S)$. Then $H = L(E,W,S)$ corresponds to a subset of the face lattice of the polyhedral cone $L^* := L(E, W \cup S, \emptyset) = \mathrm{cl}\, H$.

We can described $L(E,W,S)$ as the disjoint union of relative interiors $C$ of polyhedral cones $C'$. For $C = \mathrm{rel\,int}\, C'$ we know two representations, as $C = L(E', \emptyset, S')$, $C' = L(E', S', \emptyset)$ with $E' \subset E \cup W \cup S$ and $S' \subset W \cup S$, as well as via a finite list of generators spanning $C'$. The restrictions $u_j \in E \cup W \cup S$ and generators $g_i$ are combinatorially connected via the incidence relation, whether $u_j(g_i) = 0$ or otherwise $u_j(g_i) > 0$.

The description of $L(E,W,S)$ can easily be obtained from the description of $L^*$ via generators and the incidence relation by traversing the face-lattice of $L^*$.

**Definition 50.** *As certificate for the exactness of the description of* $L^* = \mathrm{cone}(g_1, \ldots, g_k) := \{\sum_{i=1}^k \lambda_i g_i : \lambda^i \geq 0\}$ *we use the dimension* $d := \dim L^*$ *and a certificate* $Z_{\dim \leq d}$ *for the assertion that* $\dim L^* \leq d$.

The description of $L^* = \mathrm{cone}(g_1, \ldots, g_k)$ is correct provided that

- $\dim L^* \leq d$ holds true (see below),
- $e(g_i) = 0$ for all $e \in E$,
- $u(g_i) \geq 0$ for all $u \in W \cup S$,

- the boolean incidence matrix $I = (u_j(g_i) = 0)_{i\in[l],j\in[k]}$, where $W \cup S = \{u_1, \dots, u_l\}$, has a property which is special to incidence matrices of $d$-dimensional polyhedral cones, see [6, Definition 5.17: recursive $d$-incidence property].

**Definition 51.** *A certificate for the assertion that the solution set of the linear system $(E, W, S)$ in $\mathbb{R}^m$ is at most $d$-dimensional is a set $Z_{\dim\le d} \subset E \cup W \cup S$ of cardinality $m - d$, together with certificates $Z^{\text{impl},u}$ for all $u \in Z_{\dim\le d} \setminus E$ that $u$ is an implicit equation of $(E, W, S)$.*

$Z_{\dim\le d}$ ensures that $\dim L(E, W, S) \le d$ if all restrictions in $Z_{\dim\le d}$ are linearly independent, and the certificates $Z^{\text{impl},u}$, $u \in Z_{\dim\le d} \setminus E$, are all valid.

Please note that these certificates play an important role already in our algorithm to find the solution of a parametrized linear system. They were used to determine the parameter set where a generalization of a solution of an evaluated system is still valid.

### 6.6.2. Nonlinear restrictions

**Lemma 52.** *For a symmetric bilinear form $q$ we have that $q(\mathbf{x}, \mathbf{x}) = 0$ for all $\mathbf{x} \in \operatorname{cone}\{\mathbf{g}_1, \dots, \mathbf{g}_k\}$ if and only if $q(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{x}, \mathbf{y} \in \{\mathbf{g}_1, \dots, \mathbf{g}_k\}$.*

*Proof.* The sufficiency of the last condition is clear by $q(\sum \lambda_i \mathbf{g}_i, \sum \mu_j \mathbf{g}_j) = \sum_{i,j} \lambda_i \mu_j q(\mathbf{g}_i, \mathbf{g}_j)$, the necessity follows from $q(\mathbf{g}_i, \mathbf{g}_i) = 0 = q(\mathbf{g}_i + \mathbf{g}_j, \mathbf{g}_i + \mathbf{g}_j)$ using the formula

$$q(\mathbf{g}_i, \mathbf{g}_j) = \frac{1}{2} \left( q(\mathbf{g}_i + \mathbf{g}_j, \mathbf{g}_i + \mathbf{g}_j) - q(\mathbf{g}_i, \mathbf{g}_i) - q(\mathbf{g}_j, \mathbf{g}_j) \right) . \tag{36}$$

□

Using (36) we can calculate symmetric bilinear forms $q_1, \dots, q_t$, each representing one quadratic form in $E_{\text{det,red}}(n, 2)$ as $q_i(\mathbf{x}, \mathbf{x}) = \text{GPR}_{I,J}(\mathbf{x})$. Remember that $\text{SysEmP}(\rho(w, r), l(w)) = \text{SysEmPLin}(\rho(w, r), l(w)) \cup (E_{\text{det,red}}(n, d(l(w))), \emptyset, \emptyset)$. If $d(l(w)) = 1$, then obviously $X = H$. So we assume that $d(l(w)) = 2$ for now. With Lemma 52 we can verify whether or not all quadratic restrictions $q_i(x, x) = 0$ are satisfied by all vectors $x \in L^* = \operatorname{cone}(g_1, \dots, g_k) = \operatorname{cl} H$. Since $\operatorname{rel\,int} L^* \subset H \subset L^*$ and quadratic forms are continuous, this test also yields the answer whether or not all solution vectors in $H$ also belong to $X$, i.e., if $H = X$ holds true.

**Investigation 53.** For all full 2-distance configurations $D(T, r)$, $r \in R(T)$, with $T \neq T^5_{10}$ with representation $(w, r)$ we have that

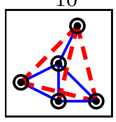
$$L(\text{SysEmPLin}(\rho(w, r), l(w))) = L(\text{SysEmP}(\rho(w, r), l(w))) .$$

Thus, up to one exception we have that $X = H$. In this case we define $d(T, r) := \dim \operatorname{lin}(L(\text{SysEmPLin}(\rho(w, r), l(w)))) - 1$.

### 6.6.3. The exception which is not polyhedral

**Investigation 54.** The only exception with $X \neq H$ happens for the full 2-distance configuration $D(T^5_{10}, 2) = T^5_{10}$. The solution set $H$ of the linear system is 4-dimensional, the corresponding cone $C = \operatorname{cone}\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4\}$ has 4 linearly independent generators. Their convex hull is a 3-dimensional tetrahedron $\Delta := \operatorname{conv}\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4\}$. Since $n = 5$, exactly one quadratic restriction has to be considered. The solution set $X = L(\mathrm{SysEmP}(\rho(w, 2), l(w))) \subset H$ is not convex. But the closure $\operatorname{cl} X$ contains all generators $\mathbf{g}_1, \dots, \mathbf{g}_4$ of $C$ and 4 of the 6 faces generated by the following edges of $\Delta$: $\overline{\mathbf{g}_1\mathbf{g}_2}$, $\overline{\mathbf{g}_2\mathbf{g}_3}$, $\overline{\mathbf{g}_3\mathbf{g}_4}$ and $\overline{\mathbf{g}_4\mathbf{g}_1}$.

$T^5_{10}$



$D(C) = \{2\}$

In the following section about symmetry we can assume $T(w) \neq T^5_{10}$, the next one we will consider $T^5_{10}$ separately, see Section 6.7.1.

### 6.6.4. Reducing the symmetry

**Proposition 55.** *Assume that $(w, r)$ represents a full 2-distance configuration $D = D(T, r)$ with $d(l(w)) = 2$ and that we have two solutions $\mathbf{x}_1, \mathbf{x}_2 \in L(\mathrm{SysEmP}(\rho(w, r), l(w))) = X$. Then there are two embeddings $e^1, e^2 : [n] \to \mathbb{R}^2$ of $([n], \rho(w, r))$ into suitable Minkowski planes with $\mathbf{x}_1 = \underline{b}^{e^1}$ and $\mathbf{x}_2 = \underline{b}^{e^2}$, see (12). The corresponding affine 2-distance configurations $A_1$ and $A_2$, containing the 2-distance sets $\{e^1(1), \dots, e^1(n)\}$ and $\{e^2(1), \dots, e^2(n)\}$ (in suitable planes each), respectively, are well defined by $\mathbf{x}_1$ and $\mathbf{x}_2$.*

*Then $A_1 = A_2$ holds if and only if there is a permutation $\sigma : [n] \to [n]$ and a scalar $\lambda \neq 0$ with $b((i, j), (k, l))(\mathbf{x}_1) = \lambda b((\sigma(i), \sigma(j)), (\sigma(k), \sigma(l)))(\mathbf{x}_2)$ for all $i, j, k, l \in [n]$. In this case we have necessarily for some $s \in \mathbb{Z}$ that $\sigma(w) = \mathrm{Shift}(w, s)$ if $\lambda > 0$, or $\sigma(w) = \mathrm{Shift}(\mathrm{Mirror}(w), s)$ if $\lambda < 0$.*

*Also the converse is true: assume that the permutation $\sigma : [n] \to [n]$ and a scalar $\lambda \neq 0$ satisfy $\sigma(w) = \mathrm{Shift}(w, s)$ if $\lambda > 0$, or $\sigma(w) = \mathrm{Shift}(\mathrm{Mirror}(w), s)$ if $\lambda < 0$ for some $s \in \mathbb{Z}$. Then for each $\underline{b} \in X$ the corresponding $\underline{b}' = \lambda \cdot \sigma(\underline{b}) \in \mathbb{R}^{\mathrm{Seq}_{2,n}}$ with $\sigma(\underline{b})(i, j) := b((\sigma(i), \sigma(n)), (\sigma(j), \sigma(n)))(\underline{b})$ belongs to $X$ as well and determines the same affine 2-distance configuration as $\underline{b}$ does.*

Note that for $d(l(w)) = 1$ there are analogous statements, but actually these two 2-distance configurations, $T^2_1$ and $T^3_3$, are itself affine 2-distance configurations.

**Corollary 56.** *For each realizable candidate $w$, $d(l(w)) = 2$, there is a finite group $G = \{a_1, \dots, a_g\}$ of linear transformations $a_i$ in $\mathbb{R}^{m(2,n)}$ such that (using the notation of Proposition 55) $A_1 = A_2$ holds if and only if there is some $i \in [g]$ and $\lambda > 0$ with $\mathbf{x}_1 = \lambda a_i(\mathbf{x}_2)$. All $a_i$ are orthogonal transformations for the following symmetric and positive definite scalar product $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{I \in ([n]^2)^2} b(I)(\mathbf{x}) \cdot b(I)(\mathbf{y})$, i.e., it is $\langle \mathbf{x}, \mathbf{x} \rangle = \langle a_i(\mathbf{x}), a_i(\mathbf{x}) \rangle$ for all $\mathbf{x} \in \mathbb{R}^{m(2,n)}$.*

*Proof.* We can use $G := \{\, \underline{b} \mapsto \sigma(\underline{b}) : \sigma$ is permutation of $[n], \sigma(w) = \mathrm{Shift}(w,s)$, $s \in \mathbb{Z} \,\} \cup \{\, \underline{b} \mapsto -\sigma(\underline{b}) : \sigma$ is permutation of $[n], \sigma(w) = \mathrm{Shift}(\mathrm{Mirror}(w), s), s \in \mathbb{Z} \,\}$. The orthogonality follows since the sums $\langle \mathbf{x}, \mathbf{x} \rangle$ and $\langle a_i(\mathbf{x}), a_i(\mathbf{x}) \rangle$ have exactly the same summands but in permuted summation order, possibly multiplied by $(-1)^2$. □

If the group $G$ corresponding to $T(w)$ is trivial, i.e., if $G$ only contains the identity, then the set of corresponding affine 2-distance configurations coincides with all positive equivalence classes of $Y := X = L(\mathrm{SysEmP}(\rho(w,r), l(w)))$. Especially $G$ is trivial for $T^5_{10}$, the exceptional 2-distance configuration discussed in 6.6.3.

Otherwise, if $|G| > 1$, we have to find a subset $Y$ of $X$ such that each positive equivalence class of $Y$ corresponds to exactly one equivalence class in $X$ with respect to the relation $\mathbf{x} \equiv \mathbf{x}' \Leftrightarrow \mathbf{x}' = \lambda a_i(\mathbf{x})$ for some $\lambda > 0$ and $i \in [g]$. We will sketch a general method for this task for polyhedral sets

$$X = \dot{\bigcup}_{i\in[k]} \operatorname{rel\,int\,cone} Q_i \,,$$

where all $Q_i$ are finite sets in $\mathbb{R}^{m(2,n)}$ and $\mathbf{0} \notin X$. We assume that the representation of $X$ is closed under $G$: for all $g \in G$ and $i \in [k]$ there is some $j \in [k]$ with $g(\operatorname{cone} Q_i) = \operatorname{cone} Q_j$.

This induces an equivalence relation in $[k]$: $i \equiv j$ if there is some $g \in G$ with $g(\operatorname{cone} Q_i) = \operatorname{cone} Q_j$. For given $X$ and $G$ we can compute these equivalence classes. Special care must be taken for $g \in G$ and $i \in [k]$ with $g(\operatorname{cone} Q_i) = \operatorname{cone} Q_i$. Assume first that $g(\operatorname{cone} Q_i) = \operatorname{cone} Q_i$ only happens if $g(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \operatorname{cone} Q_i$. Choosing exactly one cone $\operatorname{rel\,int\,cone} Q_i$ from every equivalence class yields the desired subset $Y$ of $X$. Note that for aesthetic reasons we did not choose the representatives of the equivalence classes independent from each other at random. Instead, we sorted the classes by decreasing dimension. For each class we have chosen randomly one representative with a maximal number of generators which coincide with generators of formerly chosen representatives.

Now assume contrarily to the above assumption that for some $g \in G$ and $i \in [k]$ we have $g(\operatorname{cone} Q_i) = \operatorname{cone} Q_i$ but $g$ restricted to $\operatorname{cone} Q_i$ is not the identity. Then there is some $q \in Q_i$ with $g(q) \neq q$. Note that $q$ and $g(q)$ are linearly independent since $g$ is orthogonal. Otherwise $g(q) = \lambda q$ for some $\lambda \in \mathbb{R}$, thus $\langle g(q), g(q) \rangle = \langle q, q \rangle = \lambda^2 \langle q, q \rangle$, and $\lambda = \pm 1$, contradicting $g(q) \neq q$ or $\mathbf{0} \notin \operatorname{rel\,int\,cone} Q_i$. Thus we have to split $\operatorname{rel\,int\,cone} Q_i$ into more polyhedral pieces, see for example the affine 2-distance configurations for $T^4_5$ and $T^4_{13}$. We will divide $\operatorname{rel\,int\,cone} Q_i$ by a hyperplane $h = \{\, \mathbf{x} : \langle \mathbf{n}, \mathbf{x} \rangle = 0 \,\}$ having $q$ and $g(q)$ on opposite sides. As normal vector with respect to $\langle \cdot, \cdot \rangle$ we choose the difference $\mathbf{n} := q - g(q)$. Since $g$ is an orthogonal transformation, we get that $\langle \mathbf{n}, (q + g(q)) \rangle = \langle q, q \rangle - \langle g(q), g(q) \rangle = 0$, thus we really have $\langle \mathbf{n}, q \rangle = -\langle \mathbf{n}, g(q) \rangle = \frac{1}{2} \langle \mathbf{n}, \mathbf{n} \rangle \neq 0$. Now we replace every cone $C = \operatorname{rel\,int\,cone} Q_j$ which is intersected by $h$ but not contained in $h$ by three relatively open cones $C \cap \{\, \mathbf{x} : \langle \mathbf{n}, \mathbf{x} \rangle < 0 \,\}$, $C \cap \{\, \mathbf{x} : \langle \mathbf{n}, \mathbf{x} \rangle = 0 \,\}$, and $C \cap \{\, \mathbf{x} : \langle \mathbf{n}, \mathbf{x} \rangle > 0 \,\}$. The same procedure has to be repeated for all hyperplanes which are symmetric to $h$, i.e., we refine our representation of $X$ by splitting all cones by hyperplanes $\{\, \mathbf{x} : \langle \mathbf{n}, g'(\mathbf{x}) \rangle = 0 \,\}$, with $g' \in G$. We obtain $X = \dot{\bigcup}_{i\in[k']} \operatorname{rel\,int\,cone} Q'_i$ for which we start the above procedure again. It is possible that further refinement steps are necessary to reduce the symmetry of basic cones, but only finitely many times.

**Proposition 57.** *Any set $Y$ which was constructed by the above procedure contains exactly one representative of the equivalence relation $x \equiv x' \iff \exists g \in G : x' = g(x)$ in the set $X$. For $X = L(\mathrm{SysEmP}(\rho(w,r), l(w)))$ and the corresponding $G$ from Corollary 56 the positive equivalence classes $[y]$ of $Y$ correspond exactly with all affine 2-distance configurations $A(w,r,y)$ belonging to $D(T(w),r)$.*

To verify the correctness of $Y$ we have to verify that after our refinement the representation $L(\mathrm{SysEmP}(\rho(w,r), l(w))) = \dot{\bigcup}_{i\in[k]} \operatorname{rel\,int\,cone} Q_i$ is still correct. This can be achieved by verifying each single replacement of $C = (C \cap \{\, \mathbf{x} : f(\mathbf{x}) < 0 \,\}) \dot{\cup} (C \cap \{\, \mathbf{x} : f(\mathbf{x}) = 0 \,\}) \dot{\cup} (C \cap \{\, \mathbf{x} : f(\mathbf{x}) > 0 \,\})$ in the disjoint union. Using a representation of the cones by linear restrictions – the $\mathcal{H}$-representation – this can be done combinatorially. The correctness of a parallel representation as $C = \operatorname{rel\,int\,cone} Q_i$, the $\mathcal{V}$-representation, can be verified as described in Section 6.6.1. Additionally we have to identify combinatorially the equivalence classes in the cones due to symmetry.

## 6.7. Construction of polynomial representatives of affine 2-distance configurations

Assume again that $d(l(w)) = 2$, the case $d(l(w)) = 1$ being easy.

Starting from Proposition 57 and the following representation of $Y$ we are looking for a set $Z$ with description and certificates representing all corresponding affine 2-distance configurations exactly once using explicit coordinates.

$$Y = \dot{\bigcup}_{i\in[k]} \operatorname{rel\,int\,cone} Q_i$$

with $k \in \mathbb{N}$, where for all $i \in [k]$

$$Q_i = \{\mathbf{g}_{i,1}(r), \mathbf{g}_{i,2}(r), \ldots, \mathbf{g}_{i,l(i)}(r)\}$$

for $l(i) \in \mathbb{N}$ generators $\mathbf{g}_{i,j} \in \mathbb{Z}[X]^{m(2,n)}$ ($j \in [l(i)]$) and $r \in R$.

In Remark 14 we constructed for each $C = ((a_1, o), (a_2, o)) \in ([n]^2)^2$ with $\mathrm{fsign}^{l(w)}(C) \neq 0$ an embedding $\tilde{e}^{C,\underline{b}} : [n] \to \mathbb{R}^2$ for $\underline{b} \in Y$ which belongs to $A(w, r, \underline{b})$ due to $\underline{b}^{\tilde{e}^{C,\underline{b}}} = b(C)(\underline{b}) \cdot \underline{b}$. In contrast to the construction $e$ in this remark with $\underline{b}^e = \underline{b}$, the coordinates of $\tilde{e}^{C,\underline{b}}$ are affinely linear functions in $\underline{b}$.

Still we have some choices to define $Z$ using $\underline{b} \mapsto \tilde{e}^{C,\underline{b}}$. First, we have to choose a suitable $C = ((a_1, o), (a_2, o)) \in ([n]^2)^2$, which geometrically describes the origin $o$ and the two coordinate axes of a coordinate system. Second, we have to choose for each positive equivalence class of $Y$ only one representative $y$ with $\tilde{e}^{C,y} \in Z$.

Let us first consider the second problem, and assume that $C$ has been chosen with $\mathrm{fsign}^{l(w)}(C) \neq 0$. After scaling all generators $\mathbf{g}_{i,j}$ by suitable polynomials which are not zero in $R$, we can assume that $b(C)(\mathbf{g}_{i,j}) =: u \in \mathbb{Z}[X]$ is constant for all $i \in [k]$ and $j \in [l(i)]$. We define

$$Z := \{\, \tilde{e}^{C,\underline{b}} : \underline{b} \in Y, b(C)(\underline{b}) = u(r)\,\}\,. \tag{37}$$

Geometrically this means to choose $y \in Y$ belonging to the convex hull of the generators. Since $\underline{b} \mapsto \tilde{e}^{C,\underline{b}}$ is affinely linear, cone $Q_i$ will be mapped to the convex hull of $\{\tilde{e}^{C,\mathbf{g}_{i,1}}, \dots, \tilde{e}^{C,\mathbf{g}_{i,l(i)}}\}$.

If we choose carefully $C$, then in most cases only $d := d(T(w), r) \in \{0, 1, 2\}$ of the $2n$ coordinates were not constant for all generators. Only for $T = T_9^6$ with $d(T, r) = 1$ there are two points which can "move" together along a horizontal line segment, always admitting a fixed difference of $x$-coordinates.

So we have chosen some $C$ with a minimal number of non-constant coordinates of $\tilde{e}^{C,\mathbf{g}_{\cdot,\cdot}}$ (with individual scalings for each $C$). Then we can take any $d$ different coordinates as parameters $p_a = b(C^a)(\underline{b})$, and express the coordinates of $\tilde{e}^{C,\mathbf{g}_{i,j}}$ identically as affinely linear functions in $p := (p_1, \dots, p_d)$ whose coefficients are polynomials in $r$ over $\mathbb{Z}$, which yields $S_{T(w),i^*} \in (\mathbb{Z}[r, p_1, \dots, p_d]^2)^n$. We define the parameter set $P(T, i^*, r)$ as the projection of $Z = \bigcup_{i\in[k]} \operatorname{rel\,int\,conv}\{\, \tilde{e}^{C,\mathbf{g}_{i,j}} : j \in [l(i)]\,\}$ to the coordinates which were used to define the parameters.

Obviously, $Z$ defined by (37) represents each affine 2-distance configuration exactly once if $\operatorname{fsign}^{l(w)}(C) \neq 0$. To verify the correctness of its representation

$$Z = \{\, S_{T(w),i^*}(r, p) : p \in P(T, i^*, r)\,\} =: Z'(w, r)$$

where $P(T, i^*, r) = \dot{\bigcup}_{i\in[k]} \operatorname{rel\,int\,conv}\{p_{i,1}(r), p_{i,2}(r), \dots, p_{i,l(i)}(r)\}$ we will use the correspondence $\tilde{e}^{C,\mathbf{g}_{i,j}} = S_{T(w),i^*}(r, p_{i,j})$, $b(C)(\mathbf{g}_{i,j}) = u$ and that the map $p \mapsto S_{T(w),i^*}(r, p)$ is affinely linear and injective. If we have verified that $X = H$, then this representation is correct.

### 6.7.1. The exception $T_{10}^5$ with non-polyhedral $H$

It remains to consider $T_{10}^5$, with $n = 5$ and $r = 2$. Luckily, the same procedure as above worked fine for $T_{10}^5$ as well. Instead of $X$, for which we did not have an explicit representation so far, we used the polyhedral cone $H$ of dimension 4. Choosing a "best" $C = ((a_1, o), (a_2, o)) \in ([n]^2)^2$, we defined $Z$ by using (37) with $Y := H$. Note that there is no further symmetry in $T_{10}^5$. Thus we used the non-injective linear map $\underline{b} \mapsto \tilde{e}^{C,\underline{b}}$ to project the 3-dimensional set $\{\, \underline{b} \in Y : b(C)(\underline{b}) = u(r)\,\}$ to the set $Z$ of dimension $d(T_{10}^5, r) := \dim \operatorname{aff} Z = 2$. Luckily, $\tilde{e}^{C,\cdot}$ maps the 2-dimensional surface $X$ (considered in projective space) bijectively to the polyhedral set $Z$, which is simply the relative interior of a square. So the combinatorics of $H$ changes by this transformation. The verification of this exception can be done manually using topological arguments. Note that the boundary of $Z$ consists of four segments whose preimages belong to the closure of $X$, which was verified by the authors implementation.

Summarizing we set $k(T_{10}^5) := 1$, $R(T_{10}^5, 1) := \{2\}$, $d(T_{10}^5, r) := 2$ for $r = 2$, $S_{T_{10}^5,1} := ((p_1, -1), (1, 0), (0, 0), (0, 1), (2, p_2)) \in (\mathbb{Z}[X, p_1, p_2]^2)^5$ and $P(T_{10}^5, 1, r) := (0, 1)^2 = \operatorname{int\,conv}\{(0, 0),\ (0, 1),\ (1, 0),\ (1, 1)\}$ to get the classification of affine 2-distance configurations which belong to $T_{10}^5$.

### 6.8. Strong 2-distance configurations

Remember that the idea behind the system $\mathrm{SysEmP}(\rho(w,r), l(w))$ was the weak convex position of all the points $\frac{e(i)-e(j)}{\rho(i,j)} \in U \subset \mathbb{R}^2$ with $\rho = \rho(w,r)$. This weak convex position is necessary and also sufficient for the existence of a unit ball $B$ (i.e., a convex centered body in $\mathbb{R}^2$) which contains $U$ in its boundary. Exactly if the unit ball $B$ of the Minkowski plane $\mathbb{M}^2$ satisfies $U \subset \partial B$, then $\rho$ is the metric induced by $S$ in $\mathbb{M}^2$.

In Section 5.4 we have seen how the structure of $U$ determines the conditions which $B$ must satisfy: in which cases $\partial B$ must contain some line segment, in which cases $\partial B$ must contain some vertices or even in which cases $B$ is uniquely determined by $U$. These questions can be answered in terms of whether or not $\mathbf{b} \in \operatorname{rel\,int} \overline{\mathbf{ac}}$ for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in U$. The required information for an affine 2-distance configuration $A(w, r, \underline{b})$ can be extracted easily from the oriented position list $l(w)$ together with the incidence relation of $\underline{b} \in Y$ with the corresponding triangle inequality.

**Proposition 58.** *Let $\rho = \rho(w,r)$, $e : [n] \to \mathbb{R}^2$ and $\underline{b} := \underline{b}^e \in L(\mathrm{SysEmP}(\rho(w,r), l(w)))$.*

*Furthermore, assume that for $I = ((i_1,j_1),(i_3,j_3),(i_2,j_2)) \in (\mathbb{P}_n)^3$ the vectors*

$$\mathbf{a} = \frac{1}{\rho(i_1,j_1)}(e(i_1) - e(j_1)), \mathbf{b} = \frac{1}{\rho(i_2,j_2)}(e(i_2) - e(j_2)), \mathbf{c} = \frac{1}{\rho(i_3,j_3)}(e(i_3) - e(j_3))$$

*and $-\mathbf{a}$ are in this cyclical order and pairwise distinct. Then $\mathbf{b} \in \operatorname{rel\,int} \overline{\mathbf{ac}}$ holds if and only if $\underline{b}$ is incident to $w^I_{\rho,2,\mathrm{fsign}^{l(w)}}$, i.e., if $w^I_{\rho,2,\mathrm{fsign}^{l(w)}}(\underline{b}) = 0$.*

Note that the condition that $\mathbf{a}, \mathbf{b}, \mathbf{c}, -\mathbf{a}$ are in cyclic order and distinct is equivalent to: $(\{(i_1,j_1)\}, \{(i_2,j_2)\}, \{(i_3,j_3)\})$ is a sublist of $l(\mathrm{Shift}(w,s))$ for a suitable $s \in \mathbb{Z}$.

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