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Notes on the Algebra and Geometry of Polynomial Representations

Gennadiy Averkov[1]

Institute of Algebra and Geometry, Faculty of Mathematics
Otto-von-Guericke University of Magdeburg
Universitätsplatz 2, 39106 Magdeburg, Germany
e-mail: gennadiy.averkov@googlemail.com[2]

Abstract. The paper deals with a semi-algebraic set A in $\mathbb{R}^d$ constructed by the inequalities $p_i(x) > 0$, $p_i(x) \geq 0$, and $p_i(x) = 0$ for a given list of polynomials $p_1, \dots, p_m$, and presents several statements that fit into the following template. Assume that in a neighborhood of a boundary point the semi-algebraic set A can be described by an irreducible polynomial f. Then f is a factor of a certain multiplicity of some of the polynomials $p_1, \dots, p_m$. Special attention is paid to the case when A is a polytope.

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1. Introduction

In what follows $x := (x_1, \dots, x_d)$ is a variable vector in $\mathbb{R}^d$ ($d \in \mathbb{N}$). As usual, $\mathbb{R}[x] := \mathbb{R}[x_1, \dots, x_d]$ denotes the ring of polynomials in variables $x_1, \dots, x_d$ and coefficients in $\mathbb{R}$. A subset A of $\mathbb{R}^d$ is said to be *semi-algebraic* if

$$A = \left\{ x \in \mathbb{R}^d : \Phi\big((\operatorname{sign} p_1(x) \in E_1), \dots, (\operatorname{sign} p_m(x) \in E_m)\big) \right\}, \qquad (1.1)$$

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[2] http://fma2.math.uni-magdeburg.de/∼averkov

where Φ is a boolean formula, $p_1, \dots, p_m \in \mathbb{R}[x]$, and $E_1, \dots, E_m$ are non-empty subsets of $\{0, 1\}$; see also [1], [8], and [6]. We call (1.1) a *representation* of A by polynomials $p_1, \dots, p_m$. We distinguish the following particular types of semi-algebraic sets:

$$(p_1, \dots, p_m)_{\geq 0} := \left\{x \in \mathbb{R}^d : p_1(x) \geq 0, \dots, p_m(x) \geq 0\right\}, \tag{1.2}$$
$$(p_1, \dots, p_s)_{>0} := \left\{x \in \mathbb{R}^d : p_1(x) > 0, \dots, p_m(x) > 0\right\}, \tag{1.3}$$
$$Z(p_1, \dots, p_m) := \left\{x \in \mathbb{R}^d : p_1(x) = 0, \dots, p_m(x) = 0\right\}. \tag{1.4}$$

Sets representable by (1.2), (1.3), and (1.4), respectively, are called *elementary closed semi-algebraic*, *elementary open semi-algebraic*, and *algebraic*, respectively.

A subset P of $\mathbb{R}^d$ is said to be a *polytope* if P is the convex hull of a non-empty and finite set of points; see [14]. It is known that a set P in $\mathbb{R}^d$ is a polytope if and only if P is non-empty, bounded, and can be represented by $P = (p_1, \dots, p_m)_{\geq 0}$, where $p_1, \dots, p_m \in \mathbb{R}[x]$ $(m \in \mathbb{N})$ are of degree one (the so-called *H-representation*). Thus, polytopes are just special elementary closed semi-algebraic sets. The study of polynomial representations of polygons and polytopes was initiated in [7] and [12]; see also the survey [13]. In [12] it was noticed that, if a d-dimensional polytope P is represented by

$$P = (q_1, \dots, q_m)_{\geq 0} \tag{1.5}$$

with $q_1, \dots, q_m \in \mathbb{R}[x]$, then $m \geq d$. In [9] it was conjectured that every d-dimensional polytope in $\mathbb{R}^d$ can be represented by (1.5) with $m = d$. This conjecture has recently been confirmed by L. Bröcker [10]; see also [4], [3], and [5] for further related results. We refer to [1, Chapter 5] and [8, §6.5 and §10.4] for results on minimal representations of general elementary semi-algebraic sets. In this paper we derive necessary conditions on representations of polytopes consisting of d polynomials.

Theorem 1.1. *Let P be a d-dimensional polytope in $\mathbb{R}^d$ with m facets such that*

$$P = (p_1, \dots, p_m)_{\geq 0} = (q_1, \dots, q_d)_{\geq 0},$$

where $p_1, \dots, p_m, q_1, \dots, q_d \in \mathbb{R}[x]$ and $p_1, \dots, p_m$ are of degree one. Then every p_i, $i \in \{1, \dots, m\}$, is a factor of precisely one polynomial q_j with $j \in \{1, \dots, d\}$. Furthermore, for i and j as above, the factor p_i of q_j is of odd multiplicity. □

Theorem 1.1 improves Proposition 2.1(i) from [12]. In [7] it was shown that every convex polygon P in $\mathbb{R}^2$ can be represented by two polynomials. We are able to determine the precise structure of such minimal representations.

Theorem 1.2. *Let P be a convex polygon in $\mathbb{R}^2$ with $m \geq 7$ edges and let*

$$P = (p_1, \dots, p_m)_{\geq 0} = (q_1, q_2)_{\geq 0},$$

where $p_1, \dots, p_m, q_1, q_2 \in \mathbb{R}[x]$ and $p_1, \dots, p_m$ are of degree one. Then there exist $k_1, \dots, k_m \in \mathbb{N}$ and $g_1, g_2 \in \mathbb{R}[x]$ such that $\{q_1, q_2\} = \{p_1^{k_1} \cdot \dots \cdot p_m^{k_m} g_1, g_2\}$ and the following conditions are fulfilled:

1. $k_1, \dots, k_m$ *are odd ;*
2. g_1, g_2 *are not divisible by* p_i *for every* $i \in \{1, \dots, m\}$*;*
3. $g_2(y) = 0$ *for every vertex* y *of* P*.* □

It is not hard to see that the set $(q_1, q_2)_{\geq 0}$ in Theorem 1.2 does not depend on the concrete choice of odd numbers $k_1, \dots, k_m$. More precisely, for g_1, g_2 as in Theorem 1.2 we have $P = (p_1 \cdot \dots \cdot p_m\, g_1, g_2)_{\geq 0}$. In [7] the polynomials q_1, q_2 representing P were defined in such a way that $g_1 = 1$ and $k_1 = \dots = k_m = 1$ see Figure 1 for an illustration of this result and Theorem 1.2. We also remark that the assumption $m \geq 7$ cannot be relaxed in general, since Theorem 1.2 would not hold if P is a centrally symmetric hexagon. In fact, assume that P is a centrally symmetric hexagon and $p_1, \dots, p_6$ are polynomials of degree one such that $Z(p_1) \cap P, \dots, Z(p_6) \cap P$ are consecutive edges of P. Then $P = (q_1, q_2)_{\geq 0}$ for $q_1 := p_1\, p_3\, p_5$ and $q_2 := p_2\, p_4\, p_6$; see Figure 2. It will be seen from the proof of Theorem 1.2 that the assumption $m \geq 7$ can be relaxed to $m \geq 5$ for the case when P does not have parallel edges.

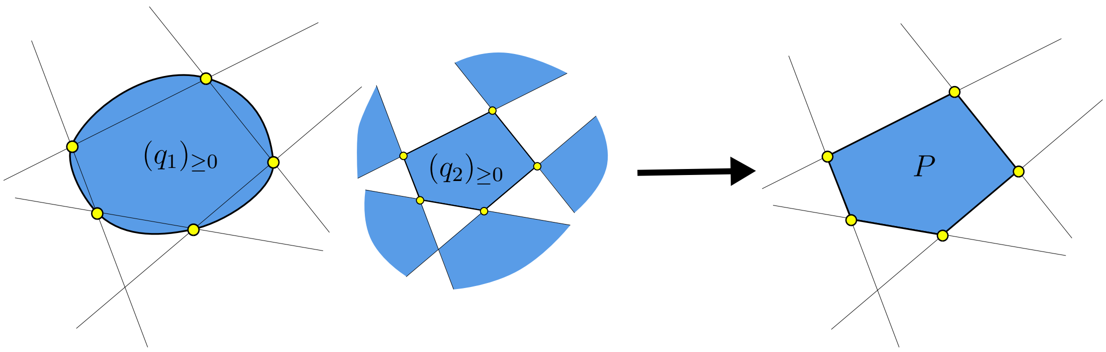

Figure 1. Illustration to Theorem 1.2 and the result on representation of convex polygons by two polynomials

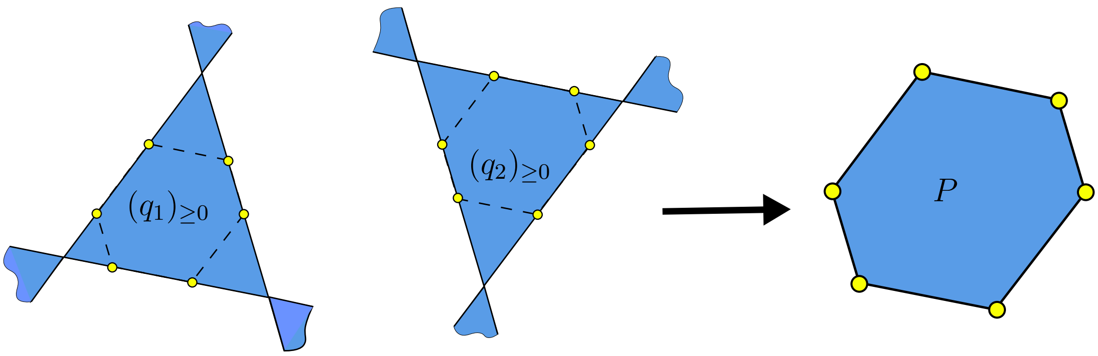

Figure 2. Centrally symmetric hexagon P represented by $P = (q_1, q_2)_{\geq 0}$ for $q_1 = p_1\, p_3\, p_4$ and $q_2 = p_2\, p_4\, p_6$

Theorems 1.1 and 1.2 are obtained as corollaries of the more general Theorem 2.2 given in Section 2. Theorem 2.2 and Corollaries 2.3–2.5 from Section 2 are results analogous to Theorems 1.1 and 1.2 for more general classes of semi-algebraic sets. Even though the mentioned general results are somewhat technical, they can be of independent interest (see also Section 2.2 providing related examples).

2. The proofs

The origin in $\mathbb{R}^d$ is denoted by o. Given $c \in \mathbb{R}^d$ and $\rho > 0$ by $B^d(c, \rho)$ we denote the open Euclidean ball with center c and radius ρ. The abbreviations int and bd stands for the interior and boundary, respectively. By dim we denote the dimension.

We refer to [2] and [11] for standard notions and results from commutative algebra and algebraic geometry. The notion of dimension of a semi-algebraic set can be defined in several equivalent ways; for details see [8, §2.8]. Given a polynomial $p \in \mathbb{R}[x]$, by ∇p we denote the gradient of p. The statement of the following lemma is known (see [8, Theorem 4.5.1]).

Lemma 2.1. *Let f be a polynomial irreducible over $\mathbb{R}[x]$. Then $\dim Z(f) = d-1$ if and only if for some $y \in \mathbb{R}^d$ one has $f(y) = 0$ and $\nabla f(y) \neq o$. Furthermore, if $\dim Z(f) = d-1$ and $p \in \mathbb{R}[x]$, then the following conditions are equivalent:*

(i) $\dim(Z(f) \cap Z(p)) = d - 1$.

(ii) $Z(f) \subseteq Z(p)$.

(iii) *f is a factor of p.* □

In the proofs below we shall deal with polynomials $p_1, \ldots, p_m$. Throughout the proofs $f_1, \ldots, f_n$ will denote the polynomials irreducible over $\mathbb{R}[x]$ which are involved in the *prime factorization* of the product $p_1 \cdot \cdots \cdot p_m$ (see [11, p. 149]), i.e.

$$p_1 \cdot \cdots \cdot p_m = f_1^{s_1} \cdot \cdots \cdot f_n^{s_n}$$

for some $s_1, \ldots, s_n \in \mathbb{N}$ and for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$ the polynomials f_i and f_j do not coincide up to a constant multiple.

Theorem 2.2. *Let A be a semi-algebraic set in $\mathbb{R}^d$ given by (1.1) and let f be a polynomial irreducible over $\mathbb{R}[x]$. Then the following statements hold true.*

I. *One has $\operatorname{bd} A \subseteq \bigcup_{i=1}^m Z(p_i)$.*

II. *If*

$$\dim(\operatorname{bd} A \cap Z(f)) = d - 1, \tag{2.1}$$

then f is a factor of p_i for some $i \in \{1, \ldots, m\}$.

III. *If there exist $a \in Z(f)$ and $\varepsilon > 0$ such that*

$$\dim(Z(f) \cap B^d(a, \varepsilon)) = d - 1, \tag{2.2}$$

$$(f)_{\geq 0} \cap B^d(a, \varepsilon) = A \cap B^d(a, \varepsilon), \tag{2.3}$$

then (2.1) is fulfilled and, moreover, f is an odd-multiplicity factor of p_i for some $i \in \{1, \ldots, m\}$.

IV. *If there exist $a \in Z(f)$ and $\varepsilon > 0$ such that*

$$\begin{aligned} \dim(Z(f) \cap B^d(a,\varepsilon)) &= d-1, \\ (f)_{>0} \cap B^d(a,\varepsilon) &= A \cap B^d(a,\varepsilon), \end{aligned}$$

then (2.1) *is fulfilled and, moreover, f is an odd-multiplicity factor of p_i for some $i \in \{1,\dots,m\}$.* □

Proof. For $x \in \mathbb{R}^d$ we define

$$\Psi(x) := \Phi\big((\operatorname{sign} p_1(x) \in E_1), \dots, (\operatorname{sign} p_m(x) \in E_m)\big).$$

Part I: Let $x_0 \notin \bigcup_{i=1}^m Z(p_i)$, that is $p_i(x_0) \neq 0$ for every $i = 1,\dots,m$. Then there exists an $\varepsilon > 0$ such that the sign of every $p_i(x)$, $i \in \{1,\dots,m\}$, remains constant on $B^d(x_0,\varepsilon)$. It follows that $\Psi(x)$ is constant for $x \in B^d(x_0,\varepsilon)$. Consequently, either $B^d(x_0,\varepsilon) \subseteq A$ or $B^d(x_0,\varepsilon) \cap A = \emptyset$. Hence x_0 is either an interior or an exterior point of A, and we get the conclusion of Part I.

Part II: By Part I we have $\operatorname{bd} A \subseteq \bigcup_{i=1}^n Z(f_i)$. Consequently

$$\begin{aligned} d-1 &\overset{(2.1)}{=} \dim(\operatorname{bd} A \cap Z(f)) \le \dim\left(\left(\bigcup_{i=1}^m Z(p_i)\right) \cap Z(f)\right) \\ &= \max_{1 \le i \le m} \dim(Z(p_i) \cap Z(f)) \le d-1. \end{aligned}$$

Hence $\dim Z(f) = d-1$ and for some $i \in \{1,\dots,m\}$ one has $\dim(Z(p_i) \cap Z(f)) = d-1$. Then Lemma 2.1 yields the assertion of Part II.

Part III: Let $a \in Z(f)$ and $\varepsilon > 0$ satisfy (2.2) and (2.3). From (2.2) it follows that $\dim Z(f) = d-1$. By Lemma 2.1, there exists $a' \in Z(f) \cap B^d(a,\varepsilon)$ such that $\nabla f(a') \neq o$. We choose $\varepsilon' > 0$ such that $B^d(a',\varepsilon') \subseteq B^d(a,\varepsilon)$ and $\nabla f(x) \neq o$ for every $x \in B^d(a',\varepsilon')$. Let us show that

$$Z(f) \cap B^d(a',\varepsilon') \subseteq \operatorname{bd} A. \tag{2.4}$$

Consider an arbitrary point $x \in Z(f) \cap B^d(a',\varepsilon')$. In view of (2.3) we have $x \in A$. On the other hand, since $f(x) = 0$ and $\nabla f(x) \neq o$, there exists a sequence $\left(x^k\right)_{k=1}^{+\infty}$ of points from $B^d(a',\varepsilon')$ such that $f(x^k) < 0$ for every $k \in \mathbb{N}$ and $x^k \to x$, as $k \to +\infty$. Since $x^k \notin (f)_{\ge 0}$ and $x^k \in B^d(a,\varepsilon)$, in view of (2.3) it follows that $x^k \notin A$ for every $k \in \mathbb{N}$. Hence, x is a point of A and is a limit of a sequence of points lying outside A. The latter implies (2.4). Since $f(a') = 0$ and $\nabla f(x) \neq o$ for every $x \in Z(f) \cap B^d(a',\varepsilon')$ it follows that $Z(f) \cap B^d(a',\varepsilon')$ has dimension $d-1$. Consequently, we have

$$\begin{aligned} d-1 = \dim(Z(f) \cap B^d(a',\varepsilon')) &\overset{(2.4)}{=} \dim(Z(f) \cap \operatorname{bd} A \cap B^d(a',\varepsilon')) \\ &\le \dim(Z(f) \cap \operatorname{bd} A) \le \dim(Z(f)) = d-1. \end{aligned}$$

Hence $\dim(Z(f) \cap \operatorname{bd} A) = d-1$. By Part II, it follows that f coincides, up to a constant multiple, with f_i for some $i \in \{1,\dots,n\}$. Without loss of generality we assume that $f = f_1$. By Lemma 2.1, we can choose $a'' \in Z(f) \cap B^d(a', \varepsilon')$ such that $f_i(a'') \neq 0$ for $i \in \{2, \dots, n\}$. This means the sign of the polynomials f_i, $i = \{2, \dots, n\}$, remains constant on $B^d(a'', \varepsilon'')$. We prove the statement of Part III by contradiction. Assume that whenever f is factor of p_i, $i \in \{1, \dots, m\}$, this factor is of even multiplicity. Since $\nabla f(a'') \neq o$, we can choose x_0, $y_0 \in B^d(a'', \varepsilon'')$ such that $f(x_0) > 0$ and $f(y_0) < 0$. Since the signs of $f_2, \dots, f_n$ do not change on $B^d(a'', \varepsilon'')$ and since $f_1 = f$ appears with an even multiplicity only, we obtain $\operatorname{sign} p_j(x_0) = \operatorname{sign} p_j(y_0)$ for every $j = 1, \dots, m$. Hence $\Psi(x_0) = \Psi(y_0)$. But by (2.3), $x_0 \in A$ and $y_0 \notin A$, which implies that $\Psi(x_0) \neq \Psi(y_0)$, a contradiction.

The proof of Part IV is omitted, since it is analogous to the proof of Part III. □

Let us give an informal interpretation of Theorem 2.2. Let f be an irreducible polynomial such that $Z(f)$ is a $(d-1)$-dimensional algebraic surface. Consider a semi-algebraic set A given by (1.1). If the boundary of A coincides locally with a part of $Z(f)$, then f is a factor of some p_i. If A coincides locally with a part of $(f)_{\geq 0}$, then f is an odd-multiplicity factor of some p_i. Furthermore, if in a neighborhood of a boundary point the set A coincides locally with a part of $Z(f)$, then f is a factor of at least two different polynomials p_i or an even-multiplicity factor of at least one polynomial p_i.

We remark that (2.2) cannot be replaced by the weaker condition $\dim Z(f) = d-1$ and $Z(f) \cap B^d(a, \varepsilon) \neq \emptyset$, since the algebraic set $Z(f)$ corresponding to an irreducible polynomial f can have "parts" of dimensions strictly smaller than $\dim Z(f)$. In fact, for $d = 2$ the irreducible polynomial $f(x) := x_1^2 + x_2^2 - x_1^3$ generates the cubic curve $Z(f)$ with isolated point at the origin. For $d = 3$, for the irreducible polynomial $f(x) = x_3^2 x_1 - x_2^2$ the set $Z(f)$ is the well-known Whitney umbrella, which is a two-dimensional algebraic surface with the one-dimensional "handle" $Z(x_2, x_3)$.

Corollary 2.3. *Let A be a semi-algebraic set given by*

$$A = \left\{ x \in \mathbb{R}^d : \Phi\big((p_1(x) \geq 0), \dots, (p_m(x) \geq 0)\big) \right\},$$

where Φ is a boolean formula and $p_1, \dots, p_m \in \mathbb{R}[x] \setminus \{0\}$, and let f be a polynomial irreducible over $\mathbb{R}[x]$. Then the following statements hold true.

I. *If there exist $b \in Z(f)$ and $\varepsilon > 0$ such that*

$$\dim(Z(f) \cap B^d(b, \varepsilon)) = d - 1, \tag{2.5}$$
$$Z(f) \cap B^d(b, \varepsilon) = A \cap B^d(b, \varepsilon), \tag{2.6}$$

then (2.1) is fulfilled, and furthermore f is a factor of p_i and p_j for some i, $j \in \{1, \dots, m\}$ with $i \neq j$ or f is an even-multiplicity factor of p_i for some $i \in \{1, \dots, m\}$.

II. *If there exist a, $b \in \mathbb{R}^d$ and $\varepsilon > 0$ such that equalities (2.2), (2.3), (2.5), and (2.6) are fulfilled, then f is a factor of p_i and an odd-multiplicity factor of p_j for some i, $j \in \{1, \dots, m\}$ with $i \neq j$.*

Proof. Part I: Let $b \in Z(f)$ and $\varepsilon > 0$ satisfy (2.5) and (2.6). From (2.5) it follows that $\dim Z(f) = d - 1$. By Lemma 2.1, there exists $b' \in Z(f) \cap B^d(b, \varepsilon)$ such that $\nabla f(b') \neq o$. Choose $\varepsilon' > 0$ such that $B^d(b', \varepsilon') \subseteq B^d(b, \varepsilon)$ and $\nabla f(x) \neq o$ for every $x \in B^d(b', \varepsilon')$. Using arguments analogous to those from the proof of Theorem 2.2(III) we show the inclusion $Z(f) \cap B^d(b', \varepsilon') \subseteq \operatorname{bd} A$ and (2.1). Hence, by Theorem 2.2(II), f coincides, up to a constant multiple, with f_i for some $i \in \{1, \ldots, n\}$. Without loss of generality we assume that $f = f_1$. If f is a factor of p_i and p_j for some $i, j \in \{1, \ldots, m\}$ with $i \neq j$, we are done. We consider the opposite case, that is, for some $i \in \{1, \ldots, m\}$ the polynomial f is a factor of precisely one polynomial p_i with $i \in \{1, \ldots, m\}$, say p_1. We show by contradiction that in this case the factor f of p_1 has even multiplicity. Assume the contrary, i.e., the factor f of p_1 has odd multiplicity. Analogously to the arguments from the proof of Theorem 2.2, we choose $b'' \in Z(f)$ and $\varepsilon'' > 0$ such that $B^d(b'', \varepsilon'') \subseteq B^d(b', \varepsilon')$ and $f_i(x) \neq 0$ for every $i \in \{2, \ldots, n\}$ and every $x \in B^d(b'', \varepsilon'')$. By the choice of b'' and ε'' we have $\operatorname{sign} p_i(x) = \operatorname{sign} p_i(b'')$ for all $i \in \{2, \ldots, m\}$ and $x \in B^d(b'', \varepsilon'')$. Since $\nabla f(b'') \neq o$, there exist points $x_0, y_0 \in B^d(b'', \varepsilon'')$ such that $f(x_0)\, f(y_0) < 0$. Then $p_1(x_0)\, p_1(y_0) < 0$. Consequently, either $p_1(x_0) > 0$ or $p_1(y_0) > 0$. Without loss of generality we assume that $p_1(x_0) > 0$. It follows that $(p_i(x_0) \geq 0) \equiv (p_i(b'') \geq 0)$ for $i = 1, \ldots, m$. Hence $x_0 \in A$. But since $f(x_0) \neq 0$, in view of (2.6), we get $x_0 \notin A$, a contradiction.

Part II: By Theorem 2.2 (III) f is a factor of odd multiplicity of some p_i with $i \in \{1, \ldots, m\}$. Furthermore, for some $j \in \{1, \ldots, m\}$ with $i \neq j$ the polynomial f is a factor of p_j, since otherwise we would get a contradiction to Part I. $\square$

Corollary 2.4. *Let* $p_1, \ldots, p_m \in \mathbb{R}[x] \setminus \{0\}$ *and* $A := (p_1, \ldots, p_m)_{\geq 0}$. *Let* f *be a polynomial irreducible over* $\mathbb{R}[x]$. *Assume that there exist* $b \in Z(f)$ *and* $\varepsilon > 0$ *such that equalities* (2.5) *and* (2.6) *are fulfilled and additionally*

$$\dim(\operatorname{int} A \cap Z(f)) = d - 1. \tag{2.7}$$

Then f *is a factor of* p_i *for some* $i \in \{1, \ldots, m\}$ *and, for every* $i \in \{1, \ldots, m\}$ *such that* p_i *is divisible by* f, *the factor* f *of* p_i *has even multiplicity.*

Proof. By Corollary 2.3 (I), f is a factor of some p_i, say p_1. Without loss of generality we assume that $f_1 = f$. Let us show that the factor f of p_1 is of even multiplicity. Assume the contrary. In view of Lemma 2.1, we can choose $a' \in \operatorname{int} A \cap Z(f)$ such that $\nabla f(a') \neq o$. We fix $\varepsilon' > 0$ such that $\nabla f(x) \neq o$ for every $x \in B^d(a', \varepsilon')$. By Lemma 2.1 we can choose $a'' \in B^d(a', \varepsilon')$ such that $f_i(a'') \neq 0$ for every $i \in \{2, \ldots, n\}$. Fix $\varepsilon'' > 0$ such that for every $i \in \{2, \ldots, n\}$ the sign of f_i remains constant on $B^d(a'', \varepsilon'')$. Since $\nabla f(a'') \neq o$, there exist x_0 and y_0 in $B^d(a'', \varepsilon'')$ with $f(x_0)\, f(y_0) < 0$. Hence $p_1(x_0)\, p_1(y_0) < 0$, and we get that either x_0 or y_0 does not belong to A, a contradiction. $\square$

Corollary 2.5. *Let* $p_1, \ldots, p_m \in \mathbb{R}[x] \setminus \{0\}$ *and* $A := (p_1, \ldots, p_m)_{>0}$. *Let* f *be a polynomial irreducible over* $\mathbb{R}[x]$. *Assume that there exist* $b \in Z(f)$ *and* $\varepsilon > 0$

such that

$$\dim(\operatorname{bd} A \cap Z(f) \cap B^d(b,\varepsilon)) = d-1, \tag{2.8}$$
$$B^d(b,\varepsilon) \setminus Z(f) = A \cap B^d(b,\varepsilon). \tag{2.9}$$

Then f is a factor of p_i for some $i \in \{1,\dots,m\}$ and, for every $i \in \{1,\dots,m\}$ such that p_i is divisible by f, the factor f of p_i has even multiplicity.

Proof. Equality (2.8) implies (2.1), and hence, by Theorem 2.2 (II), f is a factor of p_i for some $i \in \{1,\dots,m\}$. The rest of the proof is analogous to the proof of Corollary 2.4. □

Now we are able to prove Theorems 1.1 and 1.2 from the introduction.

Proof of Theorem 1.1. Let us prove the first part of the assertion. Assume the contrary, say p_1 is a factor of both q_1 and q_2. Then within the $(d-1)$-dimensional affine space $Z(p_1)$ the facet $P \cap Z(p_1)$ of P is represented by $d-2$ polynomials $q_3,\dots,q_d$ in the following way

$$P \cap Z(p_1) = \{x \in Z(p_1) : q_3(x) \geq 0, \dots, q_d(x) \geq 0\}.$$

This yields a contradiction to the fact that a k-dimensional convex polytope cannot be represented (in the above form) by less than k polynomials; see [12, Corollary 2.2]. The second part of the assertion follows directly from Theorem 2.2 (III). □

Proof of Theorem 1.2. For $j \in \{1,2\}$ denote by I_j the set of indices $i \in \{1,\dots,m\}$ for which p_i is a factor of q_j. By Corollary 2.4 it follows that $I_1 \cup I_2 = \{1,\dots,m\}$. Furthermore, $I_1 \cap I_2 = \emptyset$, by Theorem 1.1. Let us show that either I_1 or I_2 is empty. Assume the contrary. We show that then there exist $i \in I_1$ and $j \in I_2$ such that the edges $Z(p_i) \cap P$ and $Z(p_j) \cap P$ of P are not adjacent and not parallel. Since $m \geq 7$, after possibly exchanging the roles of q_1 and q_2, we may assume that the cardinality of I_2 is at least four. Let us take an arbitrary $i \in I_1$. Then there exist at least two sides of the form $Z(p_j) \cap P$, $j \in I_2$, which are not adjacent to $Z(p_i) \cap P$. One of these sides is not parallel to $Z(p_i) \cap P$. The intersection point y of $Z(p_i)$ and $Z(p_j)$ lies outside P and fulfills the equalities $q_1(y) = q_2(y) = 0$, a contradiction to the inclusion $(q_1,q_2)_{\geq 0} \subseteq P$. Hence I_1 or I_2 is empty. Without loss of generality we assume that $I_2 = \emptyset$.

For $i \in \{1,\dots,m\}$ let k_i be the multiplicity of the factor p_i of p_1. Then $q_1 = p_1^{k_1} \cdot \dots \cdot p_m^{k_m} g_1$ for some polynomial g_1, and statements 1 and 2 follow directly from Theorem 2.2 (III).

It remains to verify condition 2 (which involves $g_2 = q_2$). This condition can be deduced from Proposition 2.1 (ii) in [12], but below we also give a short proof. We argue by contradiction. Let y be a vertex of P with $g_2(v) > 0$. Up to reordering the sequence $p_1,\dots,p_m$ we may assume that $p_1(v) = 0$. Clearly, any point y' lying in $Z(p_1) \setminus P$ and sufficiently close to y fulfills the conditions $q_1(y') = 0$ and $q_2(y') > 0$. Hence $y' \in P$, a contradiction to the inclusion $(q_1,q_2)_{\geq 0} \subseteq P$. □

3. Examples to Theorem 2.2 and its corollaries

Each of the examples below is supplied with a figure referring to the case $d = 2$. Let

$$A := \{x \in \mathbb{R}^d \,:\, x_d > 0 \text{ and } \big((x_1-1)^2 + x_2^2 + \cdots + x_d^2 \le 1 \\ \text{ or } x_1^2 + x_2^2 + \cdots + x_d^2 \le 1\big)\},$$

see Figure 3. By Theorem 2.2, if A is given by (1.1), then the polynomials x_d, $(x_1-1)^2 + x_2^2 + \cdots + x_d^2 - 1$, $(x_1+1)^2 + x_2^2 + \cdots + x_d^2 - 1$ are factors of odd multiplicity of some of the polynomials $p_1, \ldots, p_m$.

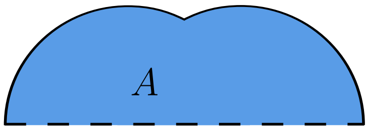

Figure 3. Illustration to Theorem 2.2

The set

$$A := \{x \in \mathbb{R}^d \,:\, (1 - x_1^2 - \cdots - x_d^2)\,(x_d+2)^2 \ge 0\}, \tag{3.1}$$

$$= \{x \in \mathbb{R}^d \,:\, (1 - x_1^2 - \cdots - x_d^2)\,(x_d+2) \ge 0,\ x_d + 2 \ge 0\}, \tag{3.2}$$

depicted in Figure 4 is the disjoint unit of a closed unit ball centered at o and a hyperplane given by the equation $x_d + 2 = 0$. By Corollary 2.3 (I), if A is given by (1.1), then $x_d + 2$ is a factor of at least two polynomials p_i or a factor of even multiplicity of at least one polynomial $p_1, \ldots, p_m$. From (3.1) and (3.2) we see that both of these possibilities are indeed realizable. Figure 5 depicts the semi-algebraic set

$$A := \left\{x \in \mathbb{R}^2 : x_d \ge 0,\ (1 - x_1^2 - \cdots - x_d^2)\, x_d \ge 0\right\},$$

$$= \left\{x \in \mathbb{R}^2 : x_d \ge 0,\ (1 - x_1^2 - \cdots - x_d^2)\, x_d^2 \ge 0\right\}. \tag{3.3}$$

By Corollary 2.3 (II), if A is given by (1.1) with $E_1 = \ldots = E_m = \{0, 1\}$, the polynomial x_d is a factor of at least two polynomials p_i and an odd-multiplicity factor of at least one polynomial p_i. By (3.3) we see that the above conclusion cannot be strengthened. In fact, (3.3) provides a representation $A = (p_1, p_2)_{\ge 0}$ such that x_d is an odd-multiplicity factor of precisely one polynomial p_i.

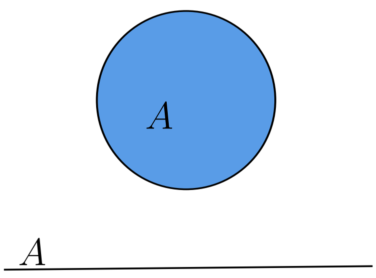

Figure 4. Illustration to Corollary 2.3(I)

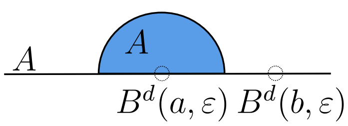

Figure 5. Illustration to Corollary 2.3(II)

Figure 6 presents the semi-algebraic set

$$A := \left\{x \in \mathbb{R}^d : (1 - x_1^2 - \cdots - x_d^2)\, x_d^2 \geq 0\right\},$$

which serves as an illustration of Corollary 2.4. By Corollary 2.4, if $A = (p_1, \ldots, p_m)_{\geq 0}$ for $p_1, \ldots, p_m \in \mathbb{R}[x]$, some of these polynomials are divisible by x_d, and furthermore, if p_i is divisible by x_d, the multiplicity of the factor x_d of p_i is even. Figure 7 depicts the semi-algebraic set

$$A := \left\{x \in \mathbb{R}^2 : (1 - x_1^2 - \cdots - x_d^2)\, x_d^2 > 0\right\}$$

illustrating Corollary 2.5. By Corollary 2.5, if $A = (p_1, \ldots, p_m)_{>0}$ for some polynomials $p_1, \ldots, p_m \in \mathbb{R}[x]$, then x_d is a factor of at least one p_i and f cannot be a factor of p_i of odd multiplicity. We notice that Corollary 2.5 is in a certain sense an analogue of Corollary 2.4 for elementary open semi-algebraic sets (since the conclusions of both corollaries are the same).

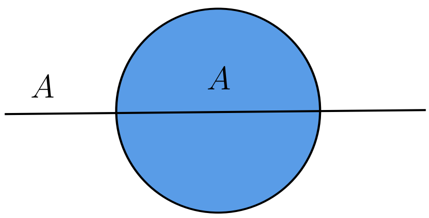

Figure 6. Illustration to Corollary 2.4

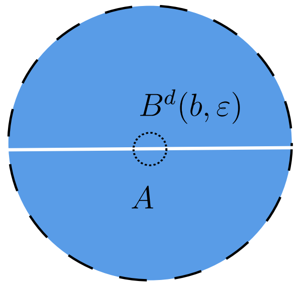

Figure 7. Illustration to Corollary 2.5

Finally, we present examples of semi-algebraic sets for which we can verify that they are not elementary semi-algebraic (see also similar examples given in [1, p. 24]). We define the closed semi-algebraic set

$$\begin{aligned} A := \big\{x \in \mathbb{R}^d \ :\ & x_d = 0 \text{ or } (x_1 - 3)^2 + x_2^2 + \cdots + x_d^2 \leq 1 \\ & \text{or } \left(x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1 \text{ and } x_d \geq 0\right)\big\}, \end{aligned}$$

see Figure 8. We can show that A is not elementary closed. In fact, let us assume the contrary, that is $A = (p_1, \ldots, p_m)_{\geq 0}$ for some polynomials $p_1, \ldots, p_m \in \mathbb{R}[x]$. Then, by Theorem 2.2 (III) applied for $a = o$ and $0 < \varepsilon < 1$, we get that x_d is a factor of odd multiplicity of p_i for some $i \in \{1, \ldots, m\}$. Since (2.7) is fulfilled for $f = x_d$, we can apply Corollary 2.4 obtaining that x_d is a factor of even multiplicity of p_i, a contradiction. Now we introduce the open semi-algebraic set

$$\begin{aligned} A := \big\{x \in \mathbb{R}^d \ :\ & x_1^2 + x_2^2 + \cdots + x_d^2 < 1 \text{ and } x_d > 0 \\ & \text{or } (x_1 - 3)^2 + x_2^2 + \cdots + x_d^2 < 1 \text{ and } x_d \neq 0\big\}, \end{aligned}$$

see Figure 9. By Theorem 2.2 (IV) and Corollary 2.5 (applied for $f(x) = x_d$) A is not elementary open.

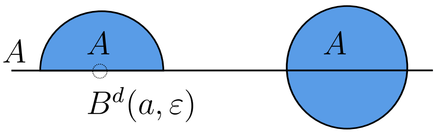

Figure 8. A closed semi-algebraic set which is not elementary closed

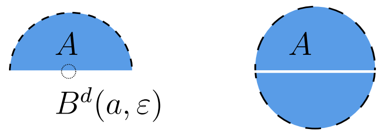

Figure 9. An open semi-algebraic set which is not elementary open

References

[1] Andradas, C.; Bröcker, L; Ruiz, J. M.: *Constructible sets in real geometry.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge **33**, Springer-Verlag, Berlin 1996. Zbl 0873.14044

[2] Ash, R. B.: *Basic abstract algebra.* Dover Publications Inc., Mineola, NY 2007. Zbl 1121.00001

[3] Averkov, G.: *Representing elementary semi-algebraic sets by a few polynomial inequalities: A constructive approach.* Preprint 2008. ArXiv:0804.2134.

[4] Averkov, G.; Henk, M.: *Representing simple d-dimensional polytopes by d polynomial inequalities.* (2007 submitted.)

[5] Averkov, G.; Henk, M.: *Three-dimensional polyhedra can be described by three polynomial inequalities.* Preprint 2008. ArXiv:0807.2137.

[6] Basu, S.; Pollack, R.; Roy, M.-F.: *Algorithms in Real Algebraic Geometry.* (2nd ed.) Algorithms and Computation in Mathematics **10**, Springer-Verlag, Berlin 2006. Zbl 1102.14041

[7] Bernig, A.: *Constructions for the theorem of Bröcker and Scheiderer.* Master's thesis, Universität Dortmund, 1998.

[8] Bochnak, J.; Coste, M.; Roy, M.-F.: *Real Algebraic Geometry.* Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge **36**, Springer-Verlag, Berlin 1998. Zbl 0912.14023

[9] Bosse, H.; Grötschel, M.; Henk, M.: *Polynomial inequalities representing polyhedra.* Math. Program. Ser. A **103**(1) (2005), 35–44. Zbl 1140.90528

[10] Bröcker, L.: *Remarks on the description of polytopes by few polynomials.* Manuscript 2008.

[11] Cox, D.; Little, J.; O'Shea, D.: *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra.* Undergraduate Texts in Mathematics, Springer-Verlag, New York 1992. Zbl 0756.13017

[12] Grötschel, M.; Henk, M.: *The representation of polyhedra by polynomial inequalities.* Discrete Comput. Geom. **29**(4) (2003), 485–504. Zbl 1128.14308

[13] Henk, M.: *Polynomdarstellungen von Polyedern*. Jahresber. Dtsch. Math.-Ver. **109**(2) (2007), 51–69. Zbl 1135.52004

[14] Ziegler, G. M.: *Lectures on Polytopes*. Graduate Texts in Mathematics **152**, Springer-Verlag, New York 1995. Zbl 0823.52002

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