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On Double Coverings of Some Rotary Hypermaps*

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Abstract. We present a classification of double coverings of Platonic and dihedral rotary hypermaps and analyse its reflexibility.

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1. Introduction

This paper is based upon results given in [2], [3], [4], [5] and, as such most of the definitions and notations are borrowed from there. Without going into details, we briefly review the theory of hypermaps taking into account [2, 3, 4] and [6]. We leave the reader to [8], [9], [10], [11] for a more general account on hypermaps.

A (orientable) *hypermap* $\mathcal{H}$ is a cellular imbedding of a finite connected hypergraph into a compact orientable surface (real 2-dimensional manifold without boundary). The simply connected regions that result by removing the embedded hypergraph are the *hyperfaces*. The *hypervertices* and the *hyperedges* of $\mathcal{H}$ are the hypervertices and the hyperedges of the hypergraph. We may use the orientability of the surface to describe $\mathcal{H}$ algebraically, by fixing an orientation and considering two permutations R and L that cyclically permute the darts (the edges of the bipartite graph representing the hypergraph) around hypervertices and hyperedges,

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respectively, according to the fixed orientation. This algebraic setup gives rise to a triple $\mathcal{Q} = (D; R, L)$, called an *oriented hypermap*, composed of a finite non-empty set D (the darts) and two permutations R and L of D generating the *monodromy* group $Mon(\mathcal{Q})$ of $\mathcal{Q}$ that acts transitively on D. The orbits of R, L and RL on D describe, respectively, the hypervertices, hyperedges and the hyperfaces of $\mathcal{Q}$, while the *type* of $\mathcal{Q}$ is the triple (l, m, n) where l, m and n are the least common multiple of the length of the orbits of the action of $\langle R\rangle$, $\langle L\rangle$ and $\langle RL\rangle$ on D, respectively.

A *covering* of oriented hypermaps $\mathcal{Q}_1 = (D_1; R_1, L_1)$ to $\mathcal{Q}_2 = (D_2; R_2, L_2)$ is a function $\phi : D_1 \to D_2$ satisfying $R_1\phi = \phi R_2$ and $L_1\phi = \phi L_2$. Every covering is necessarily onto and said to be an *isomorphism* if it is one-to-one. An *automorphism* of the oriented hypermap $\mathcal{Q}$ is an isomorphism from $\mathcal{Q}$ to $\mathcal{Q}$. Topologically, automorphisms of oriented hypermaps are accomplished by orientation preserving self-homeomorphisms of the surface S.

A *rotary* hypermap is a regular oriented hypermap $\mathcal{Q} = (D; R, L)$, that is, an oriented hypermap $\mathcal{Q}$ such that $Aut(\mathcal{Q})$ acts transitively on D. In this case the set D can be replaced by the monodromy group $\langle R, L\rangle$ and the action by the right multiplication (automorphisms correspond to left multiplication). So rotary hypermaps correspond essentially to groups with distinguished ordered pairs of generators. If G_1 and G_2 are two groups generated by R_1, L_1 and R_2, L_2, respectively, then the rotary hypermap $\mathcal{Q}_1 = (G_1; R_1, L_1)$ covers the rotary hypermap $\mathcal{Q}_2 = (G_2; R_2, L_2)$ if and only if the assignment $R_1 \mapsto R_2$, $L_1 \mapsto L_2$, extends to an epimorphism $\pi : G_1 \to G_2$; this epimorphism π induces a covering from $\mathcal{Q}_1$ to $\mathcal{Q}_2$. A rotary hypermap $\mathcal{Q} = (G; R, L)$ is *reflexible* if the two rotary hypermaps $\mathcal{Q}$ and $\mathcal{Q}^{-1} = (G; R^{-1}, L^{-1})$ are isomorphic, that is, if the assignment $R \mapsto R^{-1}$, $L \mapsto L^{-1}$ extends to an automorphism of the monodromy group of $\mathcal{Q}$. Both $\mathcal{Q}$ and $\mathcal{Q}^{-1}$ are cellular imbeddings of the same hypergraph on the same surface but associated to different orientations.

By a 2-*base* we mean a pair of generators. Each 2-base of G gives rise to a rotary hypermap with automorphism group G. Two such 2-bases (a_1, a_2) and (b_1, b_2) of G give rise to isomorphic rotary hypermaps if and only if there is a group automorphism $\phi \in Aut(G)$ with $\phi(a_i) = b_i$, $i = 1, 2$. Denoting by $d_2(G)$ the number of (non-isomorphic) rotary hypermaps with automorphism group G one has (see [12])

$$d_2(G) = \frac{|B_2(G)|}{|Aut(G)|}\,, \tag{1}$$

where $B_2(G)$ is the set of 2-bases of G. We say that $\pi : \hat{G} \to G$ is a *covering* (*p-covering*) if π is an epimorphism (with kernel of size p). When there is a covering $\pi : \hat{G} \to G$, we say that $\hat{G}$ is a *covering of* G. If $\pi : \hat{G} \to G$ is a p-covering with characteristic kernel K, then, as shown in [5], the function ζ^π sending each $\psi \in Aut(\hat{G})$ to the function $\psi\zeta^\pi : G \to G$ determined by $\pi(\psi\zeta^\pi) = \psi\pi$ is an homomorphism from $Aut(\hat{G})$ to $Aut(G)$. As defined in [5], if ζ^π is an epimorphism with kernel of size q, then π is said to be a *q-strong p-covering* (or just a *strong covering*). For a p-covering $\pi : \hat{G} \to G$, when p is prime, the number of 2-bases of $\hat{G}$ is the total number of liftings of 2-bases of G minus those that generate subgroups isomorphic to G, that is, $|B_2(\hat{G})| = p^2|B_2(G)| - |\mathcal{P}_2(\pi)||B_2(G)|$, where $\mathcal{P}_n(\pi)$ is the set of n-generated subgroups of $\hat{G}$ isomorphic (via π) to G. Then, for q-strong p-coverings $\pi : \hat{G} \to G$, we have the following relation between $d_2(\hat{G})$ and $d_2(G)$:

$$d_2(\hat{G}) = \frac{p^2 - |\mathcal{P}_2(\pi)|}{q}\, d_2(G)\,. \tag{2}$$

More generally, for this case (p prime), if $H \in \mathcal{P}_n(\pi)$, then H isomorphic to G, in which case $HK = \hat{G}$ and $H \cap K = \{1\}$, that is, $\hat{G}$ is isomorphic to a semi-direct product $G \ltimes K$. On the other hand, it is clear that if $\hat{G} = H \ltimes K$, for some $H < \hat{G}$, then $H \in \mathcal{P}_n(\pi)$. This shows that $\mathcal{P}_n(\pi) \neq \emptyset$ if and only if $\hat{G} = H \ltimes K$, for some $H < \hat{G}$. In particular, for double coverings ($p = 2$) we have that $\mathcal{P}_n(\hat{G}) \neq \emptyset$ if and only if $\hat{G} \cong G \times C_2$. Thus if $\hat{G} \not\cong G \times C_2$ is a q-strong double covering of G, then

$$d_2(\hat{G}) = \frac{4}{q}\, d_2(G)\,. \tag{3}$$

We review now some results of [5] which will be necessary for the following sections. If $\pi : \hat{G} \to G$ is a double covering with kernel $K = \{1, i\}$ and $P = \langle x_1, \ldots, x_n \mid w_1, \ldots, w_m\rangle$ is a presentation of G, then there is $J = (j_1, \ldots, j_m) \in K^m$ such that

$$P_J = \langle x_1, \ldots, x_n, i \mid w_1 j_1, \ldots, w_m j_m, i^2, [i, x_1], \ldots, [i, x_n]\rangle$$

is a presentation of $\hat{G}$. To simplify notation, we write $W = (w_1, \ldots, w_m)$ and

$$P_J = \langle x_1, \ldots, x_n, i \mid WJ\rangle\,,$$

where we omit the obvious relators i^2 and $[i, x_1], \ldots, [i, x_n]$. If $J = (1, \ldots, 1)$, then P_J is a presentation of $G \times C_2$. If $J \neq (1, \ldots, 1)$, then $i = w$, for some $w \in \{w_1, \ldots, w_m\}$, and we write

$$P_J = \langle x_1, \ldots, x_n \mid WJ\rangle\,.$$

If $J \neq (1, \ldots, 1)$, then P_J is a presentation in $x_1, \ldots, x_n$. If i is the unique central involution of $\hat{G}$, we write -1 instead of i. We see $\{x_1, \ldots, x_n\}$ as an ordered set $X = (x_1, \ldots, x_n)$ so to bring the cartesian product notation to presentations. The set $\mathcal{F} = \{P_J \mid J \in K^m\}$ gives presentations of double coverings of G including eventually presentations of G. For every $I \in K^n$, by changing generators $Y = IX$, both P_J and $P_{JW(I)}$ are presentations of the same double double covering of G or eventually of G. We have then an equivalence relation defined on $\mathcal{F}$ given by $P_J \approx P_{J'}$ if and only if $J' = JW(I)$ for some $I \in K^n$. This equivalence relation partitions $\mathcal{F}$ in equivalence classes $[P_J] = \{P_{JW(I)} : I \in K^n\}$ called *presentation classes*. Let $\pi : \hat{G} \to G$ be a double covering. Then $\hat{G}$ has a presentation P_J for some $J \in K^m$. If $J \neq (1, \ldots, 1)$, then π is a strong double covering if and only if $[P_J]$ is the only presentation class of $\hat{G}$. In this case π is q-strong, where q is the size of the set $\{I \in K^n : W(I) = (1, \ldots, 1)\}$ [5].

2. Associates and derivatives hypermaps

Consider the free-product $\Delta = C_2 * C_2 * C_2$ generated by R_0, R_1 and R_2. The even subgroup $\Delta^+ = \langle R_1R_2, R_2R_0\rangle$ consisting of the words in Δ of even length, is a free group of rank 2. Denote by $X = R_1R_2$ and $Y = R_2R_0$ the "canonical" generators of Δ^+. Each oriented hypermap $\mathcal{Q} = (D; R, L)$ corresponds to a finite transitive permutation representation

$$\varrho : \Delta^+ \to Mon(\mathcal{Q}),\ X \mapsto R,\ Y \mapsto L\,.$$

Under this representation $\mathcal{Q} \cong (\Delta^+/_r H; H_{\Delta^+} X, H_{\Delta^+} Y)$, where H is the stabilizer in Δ^+ of a fixed dart $d \in D$ under the ϱ-induced action of Δ^+ on D.

2.1. Associates

Each automorphism ϕ of Δ preserving Δ^+ gives rise to an operation D_ϕ transforming an oriented hypermap $\mathcal{Q} = (\Delta^+/H; HR_1R_2, HR_2R_0)$ into its *ϕ-dual* $D_\phi(\mathcal{Q}) = (\Delta^+/(H\phi); (H\phi)(R_1R_2\phi), (H\phi)(R_2R_0\phi))$. In particular, the six permutations σ of the symmetric group $S_3 = S_{\{0,1,2\}}$ induce six Δ^+-preserving automorphisms of Δ by transforming the triple (R_0, R_1, R_2) into the triple $(R_{0\sigma^{-1}}, R_{1\sigma^{-1}}, R_{2\sigma^{-1}})$. This gives rise to six *σ-duals* $D_\sigma(\mathcal{Q}) = (G; R_{\sigma^{-1}}, L_{\sigma^{-1}})$. Each σ-dual merely permutes the hypercells (hypervertices, hyperedges and hyperfaces) of the hypermap. The σ-duals, where $\sigma \in S_3$, of a given oriented hypermap $\mathcal{Q}$ form the set $Ass(\mathcal{Q})$ of the *associates* of $\mathcal{Q}$. Some of them may be isomorphic. For instance, $\mathcal{Q} \cong D_\sigma(\mathcal{Q})$ if and only if the assignment $R \mapsto R_{\sigma^{-1}},\ L \mapsto L_{\sigma^{-1}}$ extends to an automorphism of G. Moreover, for any $\sigma, \alpha \in S_3$, we have $D_{\sigma\alpha}(\mathcal{Q}) = D_\sigma D_\alpha(\mathcal{Q})$.

Remark 2.1. The symmetric group S_3 acts on the left via D_σ on the set of oriented hypermaps. The set $Ass(\mathcal{Q})$ is just the orbit of $\mathcal{Q}$ under this action. The isomorphism relation between oriented hypermaps is a G-set congruence, that is, $\mathcal{Q}_1 \cong \mathcal{Q}_2$ implies $D_\sigma(\mathcal{Q}_1) \cong D_\sigma(\mathcal{Q}_2)$, $\sigma \in S_3$. Hence S_3 also acts on the set of congruence classes and in particular on $Ass(\mathcal{Q})/\cong$, the set of G-set congruence classes of $Ass(\mathcal{Q})$. The set $\Sigma(\mathcal{Q}) = \{\sigma \in S_3 : \mathcal{Q} \cong D_\sigma(\mathcal{Q})\}$ is just the stabilizer of the congruence class of $\mathcal{Q}$ under the action of S_3 on $Ass(\mathcal{Q})/\cong$. The number of non-isomorphic hypermaps in $Ass(\mathcal{Q})$ is the size of an orbit in $Ass(\mathcal{Q})/\cong$, which is $|S_3 : \Sigma(\mathcal{Q})| = 6/|\Sigma(\mathcal{Q})|$ by the orbit-stabilizer theorem.

Remark 2.2. A rotary hypermap $\mathcal{Q} = (\Delta^+/H; HR_1R_2, HR_2R_0)$ is reflexible if and only if H is a normal subgroup of Δ. Since the hypermap subgroup of the dual $D_\phi(\mathcal{Q})$ is $H\phi$, $D_\phi(\mathcal{Q})$ is reflexible if and only if $\mathcal{Q}$ is reflexible. In particular, $\mathcal{Q}$ is reflexible if and only if it has a reflexible associate, in which case $\mathcal{Q}$ and all its associates and duals are reflexibles.

2.2. Derivatives

Let $\mathcal{Q} = (G; R, L)$ be a rotary hypermap with monodromy group $G = \langle R, L\rangle$ and let $\xi \in G$ be a central element of order q. Each pair $I = (\xi^i, \xi^j) \in \langle\xi\rangle^2$ gives rise to a rotary hypermap $\mathcal{Q}^I = (G^I, \xi^i R, \xi^j L)$, where $G^I = \langle \xi^i R, \xi^j L \rangle$. Such $\mathcal{Q}^I$ will be called a *derivative* of $\mathcal{Q}$. We note that for every $I, J \in C_q^2$, we have $(\mathcal{Q}^I)^J = \mathcal{Q}^{IJ}$ and that some of the derivatives of $\mathcal{Q}$ may give rise to duals or associates of $\mathcal{Q}$, some may even "degenerate" making G^I smaller than G. For example, consider the rotary hypermaps $\mathcal{GD}^0$ and $\mathcal{A}_4^{+0}$ (see Table 4 for a combinatorial description) with monodromy group $A_5 \times C_2$ and cyclic centre $C_2 = \langle -1 \rangle$. The derivative $\mathcal{GD}^{0(-1,1)}$ is the dual $D_{(02)}(\mathcal{GD}^0)$, while the derivative $\mathcal{A}_4^{+0\,(1,-1)}$ degenerates to $\mathcal{A}_4^+$.

We say that $g \in G = \langle R, L \rangle$ is *auto-reflexible* if g can be written as a word $g = w(R, L)$ such that $g^{-1} = w(R^{-1}, L^{-1})$. It is not guaranteed that any g is auto-reflexible, even if g is a central element of G or G is a *reflexible* group, that is, the assignment $R \mapsto R^{-1}$, $L \mapsto L^{-1}$ extends to an automorphism of G. For example, the reflexible group

$$G = \langle R, L \mid R^4, L^2, (R^{-1}L)^{16}, R^{-1}(LR)^4 R(RL)^4, R^{-1}LR^2L(RL)^2R^2LR^{-1}L(R^2L)^2 \rangle$$

of order 512 has center $C_2 \times C_2$. The central involution $(LR)^2(LR^2)^2(LR^3)^2$ is not auto-reflexible.

Theorem 2.3. *If $\xi \in Z(G)$ is auto-reflexible, then for any $I \in \langle \xi \rangle^2$, $\mathcal{Q}$ is reflexible if and only if $\mathcal{Q}^I$ is reflexible.*

Proof. It is enough to prove that "$\mathcal{Q}$ is reflexible" implies "$\mathcal{Q}^I$ is reflexible" since $I^{-1} \in \langle \xi \rangle^2$. If $\mathcal{Q}$ is reflexible, then there is an automorphism ψ of G sending the pair (R, L) to (R^{-1}, L^{-1}). Then $\xi\psi = \xi^{-1}$, and therefore ψ sends the pair $I(R, L)$ to the pair $I^{-1}(R^{-1}, L^{-1}) = (I(R, L))^{-1}$. Hence $\mathcal{Q}^I$ is also reflexible. $\square$

3. Regular Platonic double coverings

By a *rotary Platonic group* we mean a group of all orientation preserving automorphisms of a Platonic solid. The rotary Platonic groups are A_4, S_4 and A_5. According to [5] their double coverings are q-strong, where $q = 1$ for double coverings of A_4 and A_5, and $q = 2$ for double coverings of S_4.

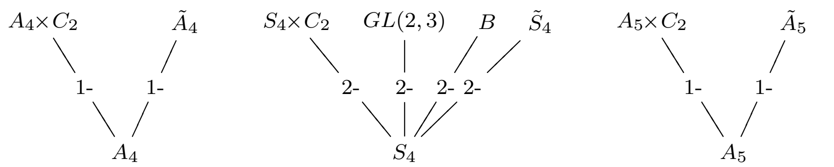

Diagram 1. The double coverings of the rotary Platonic groups

Here $\tilde{A}_4$, $\tilde{S}_4$ and $\tilde{A}_5$ denote, respectively, the binary tetrahedral group, the binary octahedral group and the binary icosahedral group. The group B is described in [5].

As $Aut(A_4) \cong Aut(S_4) \cong S_4$, and $Aut(A_5) \cong S_5$, in view of (1), a straightforward computation of the number of 2-bases of A_4, S_4 and A_5 gives $d_2(A_4) = 4$, $d_2(S_4) = 9$, and $d_2(A_5) = 19$. Let $\hat{G} = G \times C_2$, where G is a rotary Platonic group, and let $\pi : \hat{G} \to G$ be the canonical epimorphism. If $G = A_4$ or A_5, then G is simple and therefore $G \times \{1\}$ is the unique subgroup of $\hat{G}$ that projects onto G via π. Hence $|\mathcal{P}_2(\pi)| = 1$ and by (2) $d_2(\hat{G}) = 3\,d_2(G)$. If $G = S_4$, then $\hat{G}$ has three normal subgroups of index 2, namely $H_1 = S_4 \times \{1\}$, $H_2 = A_4 \times C_2$ and $H_3 = S_4^+ \times \{1\} \cup S_4^- \times \{-1\}$, where $S_4^+ = A_4$ is the subgroup of even permutations of S_4 and $S_4^- = S_4 \setminus A_4$ is the subset of the odd permutations of S_4. Then $|\mathcal{P}_2(\pi)| = 2$, since only H_1 and H_3 are isomorphic to S_4. According to (2), $d_2(S_4 \times C_2) = d_2(S_4)$. For double coverings $\hat{G} \not\cong G \times C_2$ of a rotary Platonic group G, (3) gives $d_2(\hat{G}) = 4\,d_2(G)$, for $G = A_4$ and A_5, and $d_2(\hat{G}) = 2d_2(G)$, for $G = S_4$. The following table summarizes these results.

G	$d_2(G)$	q	$G \times C_2$	$\lvert\mathcal{P}_2(\pi)\rvert$	$d_2(G \times C_2)$	$\hat{G} \not\cong G \times C_2$	$\lvert\mathcal{P}_2(\pi)\rvert$	$d_2(\hat{G})$
A_4	4	1	$A_4 \times C_2$	1	12	$\tilde{A_4}$	0	16
S_4	9	2	$S_4 \times C_2$	2	9	$\tilde{S_4}, GL(2,3), B$	0	18
A_5	19	1	$A_5 \times C_2$	1	57	$\tilde{A_5}$	0	76

Table 1. The number of double coverings of rotary Platonic hypermaps

3.1. The rotary hypermaps with automorphism group $G \times C_2$

There are 4 rotary hypermaps with automorphism group A_4 and 12 rotary hypermaps with automorphism group $A_4 \times C_2$. According to the classification given in [3] there are 13 reflexible hypermaps with automorphism group S_4, four of which are orientable with rotation group A_4. These four, or more precisely their oriented versions, are our rotary hypermaps with automorphism group A_4. Analogously, according to [3] there are 39 reflexible hypermaps with automorphism group $S_4 \times C_2$, 21 of which are orientable. Among these twenty one, 12 have rotation group $A_4 \times C_2$. These correspond to our 12 rotary hypermaps with automorphism group $A_4 \times C_2$. The next table gives a combinatorial description of them. As with all subsequent tables, these are divided in two wings: the right wing describing the Platonic hypermaps $\mathcal{Q}$ with automorphism group G, and carrying the same notation as in [3], and the left wing describing their rotary double coverings $\hat{\mathcal{Q}}$ with automorphism group $\hat{G}$. Usually hypermaps come with more than one associate; when this is the case, we only display one of them, which is always chosen to match a Platonic solid or a more familiar hypermap. The number of non-isomorphic associates is given in the column labelled by σ. In the last two columns of each wing we display the type (l, m, n) and the genus g.

$\hat{\mathcal{Q}}$	σ	l	m	n	g
$\mathcal{H}_1^{\hat{0}}$	6	3	2	6	1
$\mathcal{H}_1^{\hat{1}}$	3	6	2	6	3
$\mathcal{H}_2^{\hat{0}}$	3	3	6	6	5

$\mathcal{Q}$	l	m	n	g
$\mathcal{H}_1 = \mathcal{S}_1 = \mathcal{T}$	3	2	3	0
$\mathcal{H}_2 = \mathcal{S}_2 = W^{-1}(\mathcal{T}^{\hat{0}})$	3	3	3	1

Table 2. The 12 rotary hypermaps with automorphism group $A_4 \times C_2$

The 9 rotary hypermaps with automorphism group $S_4 \times C_2$ are the rotary double coverings obtained by a direct product $\mathcal{H} \times \mathcal{B}^{+\hat{1}}$ of $\mathcal{H}$, one of the 9 reflexible hypermaps consisting by $\mathcal{O}$ (the octahedron), $W^{-1}(\mathcal{O}^{\hat{0}})$ and their associates, all with automorphism group $S_4 \times C_2$ and orientation preserving automorphism group S_4, with $\mathcal{B}^{+\hat{1}}$, a reflexible hypermap with 4 "flags" (i.e. two darts), as described in [1] (p. 67). Alternatively, they are also the orientable double coverings of the 18 non-orientable reflexible hypermaps with automorphism group $S_4 \times C_2$ described in [3]. These 18 non-orientable hypermaps come in pairs, each pair having the same orientable double covering. As they are a direct product of a reflexible hypermap with $\mathcal{B}^{+\hat{1}}$, which is also reflexible, all these 9 rotary hypermaps are reflexible.

$\hat{\mathcal{Q}}$	σ	l	m	n	g
$\mathcal{O}^{\hat{0}} \cong \mathcal{O}^{\hat{1}}$	6	4	2	6	3
$\mathcal{S}_4^{+\hat{0}} \cong \mathcal{S}_4^{+\hat{1}}$	3	4	4	6	9

$\mathcal{Q}$	l	m	n	g
$\mathcal{S}_3^+ = \mathcal{O}$	4	2	3	0
$\mathcal{S}_4^+ = W^{-1}(\mathcal{O}^{\hat{0}})$	4	4	3	3

Table 3. The 9 rotary hypermaps with automorphism group $S_4 \times C_2$

Similarly, the 57 rotary hypermaps with automorphism group $A_5 \times C_2$ are the orientable double coverings ${\mathcal{A}_k}^{\hat{i}+}$ of the 114 non-orientable reflexible hypermaps with automorphism group $A_5 \times C_2$ described in §6.2 of [3]. As above, these 114 non-orientable hypermaps come in pairs, $\mathcal{A}_k^i$ and $\mathcal{A}_k^{\hat{i}}$, each pair having the same orientable double covering $\mathcal{A}_k^{i+} \cong \mathcal{A}_k^{\hat{i}+}$. As $\mathcal{A}_k^{+i} \cong \mathcal{A}_k^{i+} \cong \mathcal{A}_k^{\hat{i}+} \cong \mathcal{A}_k^{+\hat{i}}$, these 57 rotary hypermaps are the double coverings ${\mathcal{A}_k}^{+i}$ of the 19 orientable reflexible hypermaps ${\mathcal{A}_k}^{+}$, $k = 1, \dots, 6$, with automorphism group $A_5 \times C_2$ and rotation group A_5 described in [3] and resumed in the right wing of Table 4. As both ${\mathcal{A}_k}^{+}$ and $\mathcal{B}^i$ are reflexible these 57 rotary hypermaps are also reflexible.

$\hat{\mathcal{Q}}$	σ	l	m	n	g
$\mathcal{D}^0$	6	3	2	10	5
$\mathcal{D}^1$	6	6	2	10	15
$\mathcal{D}^2$	6	6	2	5	9
$\mathcal{GD}^0$	6	5	2	10	13
$\mathcal{GD}^1$	3	10	2	10	19
$\mathcal{A}_3^{+0}$	6	3	6	10	25
$\mathcal{A}_3^{+2}$	3	6	6	5	29
$\mathcal{A}_4^{+0}$	6	5	10	6	33
$\mathcal{A}_4^{+2}$	3	10	10	3	29
$\mathcal{A}_5^{+0}$	6	5	10	6	33
$\mathcal{A}_5^{+2}$	3	10	10	3	29
$\mathcal{A}_6^{+0}$	3	5	10	10	37

$\mathcal{Q}$	l	m	n	g
$\mathcal{A}_1^+ = \mathcal{D}$	3	2	5	0
$\mathcal{A}_2^+ = \mathcal{GD}$	5	2	5	4
$\mathcal{A}_3^+ = W^{-1}(\mathcal{D}^{\hat{0}})$	3	3	5	5
$\mathcal{A}_4^+ = W^{-1}(\mathcal{I}^{\hat{0}})$	5	5	3	9
$\mathcal{A}_5^+ = W^{-1}(\mathcal{M}_5)$	5	5	3	9
$\mathcal{A}_6^+ = W^{-1}(\mathcal{GD}^{\hat{0}})$	5	5	5	13

Table 4. The 57 rotary hypermaps with automorphism group $A_5 \times C_2$

Remark 3.1. The results of this paragraph can also be achieved by proceeding in the same way as in the following Subsections 3.2, 3.3 and 3.4, using the presentations $\langle X, Y \mid X^3 = -1, Y^n = (XY)^2 = 1\rangle$ of $A_4 \times C_2$ $(n = 3)$, $S_4 \times C_2$ $(n = 4)$

and $A_5 \times C_2$ ($n = 5$). The reflexivity of these rotary hypermaps comes as a bonus from the classification given in [3], since the number of them matches the number of orientable and reflexible hypermaps with corresponding orientation preserving automorphism group.

3.2. The rotary hypermaps with automorphism group $\tilde{G}$

• $\tilde{G} = \tilde{A}_4$: The four rotary hypermaps with automorphism group A_4 correspond (according to [3]) to the following pairs of generators: (x, y), $(y, (xy)^{-1})$, $((xy)^{-1}, x)$ and (x^{-1}, y^x), where $x = (1\ 2\ 3)$ and $y = (2\ 3\ 4)$. These have types $(3, 3, 2)$, $(3, 2, 3)$, $(2, 3, 3)$ and $(3, 3, 3)$, respectively, being the first three associates. The presentation pair (x, y) of the presentation $\langle x, y \mid x^3, y^3, (xy)^2\rangle$ of A_4 lifts in $\tilde{A}_4$ to four generating pairs $(\pm X, \pm Y)$, where (X, Y) is a presentation pair of the presentation $\langle X, Y \mid X^3 = Y^3 = (XY)^2 = -1\rangle$ of $\tilde{A}_4$. They give rise to rotary hypermaps of type $(6, 6, 4)$, $(3, 6, 4)$, $(6, 3, 4)$ and $(3, 3, 4)$. Including associates we count 12 non-isomorphic rotary hypermaps.

The 2-base (x^{-1}, y^x) of A_4 lifts to four pairs $(\pm X^{-1}, \pm Y^X)$ of generators of $\tilde{A}_4$ which give rise to hypermaps of type $(6, 6, 6)$, $(3, 6, 3)$, $(6, 3, 3)$ and $(3, 3, 6)$ (note that the order of $X^{-1}Y^X$ is the order of Y since $X^{-1}Y^X = -XYX = (XY)^{-1}X = Y^{-1}$). Including associates we count 4 non-isomorphic rotary hypermaps.

Summing up, they totalise 16 rotary hypermaps with automorphism group $\tilde{A}_4$.

$\hat{\mathcal{Q}}$	σ	l	m	n	g
$Bin(\mathcal{T})_1$	3	6	4	6	6
$Bin(\mathcal{T})_2$	6	3	4	6	4
$Bin(\mathcal{T})_3$	3	3	4	3	2
$Bin(W^{-1}(\mathcal{T}^{\hat{0}}))_1$	1	6	6	6	7
$Bin(W^{-1}(\mathcal{T}^{\hat{0}}))_2$	3	3	3	6	3

$\mathcal{Q}$	l	m	n	g
$\mathcal{T}$	3	2	3	0
$W^{-1}(\mathcal{T}^{\hat{0}})$	3	3	3	1

Table 5. The 16 rotary hypermaps with automorphism group $\tilde{A}_4$

• $\tilde{G} = \tilde{S}_4$: Twelve of the eighteen rotary hypermaps with automorphism group $\tilde{S}_4$ arise by lifting the presentation pair $(x, y) = ((1\ 2\ 3), (1\ 4\ 3\ 2))$ of $\langle x, y \mid x^3, y^4, (xy)^2\rangle$ to the four generating pairs $(\pm X, \pm Y)$ of $\tilde{S}_4$, where (X, Y) is a presentation pair of $\langle X, Y \mid X^3 = Y^4 = (XY)^2 = -1\rangle = \tilde{S}_4$. The two pairs $(X, \pm Y)$ give rise to isomorphic hypermaps of type $(6, 8, 4)$, while the two pairs $(-X, \pm Y)$ give rise to isomorphic hypermaps of type $(3, 8, 4)$. Among their associates are $Bin(\mathcal{O})_1$ and $Bin(\mathcal{O})_2$, double coverings of the octahedron $\mathcal{O}$, and $Bin(\mathcal{C})_1 = D_{(0\ 2)}(Bin(\mathcal{O})_1)$, $Bin(\mathcal{C})_2 = D_{(0\ 2)}(Bin(\mathcal{O})_2)$, double coverings of the cube $\mathcal{C} = D_{(0\ 2)}(\mathcal{O})$. The remaining six hypermaps arise by lifting (xy^{-1}, y) (this permutation pair corresponds to $\bar{\mathcal{G}}_4$ in page 8 of [3]) to the 2-bases $(\pm XY^{-1}, \pm Y)$ of $\tilde{S}_4$. The two pairs $(XY^{-1}, \pm Y)$ give rise to isomorphic hypermaps of type $(8, 8, 6)$, while the two pairs $(-XY^{-1}, \pm Y)$ give rise to isomorphic hypermaps of type $(8, 8, 3)$. Note that $XY^{-1} = X(-XYX) = X^{-1}YX$, and so the order of XY^{-1} is the order of Y.

$\hat{\mathcal{Q}}$	σ	l	m	n	g
$Bin(\mathcal{O})_1$	6	8	4	6	12
$Bin(\mathcal{O})_2$	6	8	4	3	8
$Bin(\mathcal{S}_4^+)_1$	3	8	8	6	15
$Bin(\mathcal{S}_4^+)_2$	3	8	8	3	11

$\mathcal{Q}$	l	m	n	g
$\mathcal{S}_3^+ = P\mathcal{O}^+ = \mathcal{O}$	4	2	3	0
$\mathcal{S}_4^+ = W^{-1}(\mathcal{O}^{\hat{0}})$	4	4	3	3

Table 6. The 18 rotary hypermaps with automorphism group $\tilde{S}_4$

• $\tilde{G} = \tilde{A}_5$: The 19 orientable reflexible hypermaps $\mathcal{A}_k^+$, $k = 1, \dots, 6$, and their associates, with automorphism group $A_5 \times C_2$, have rotation group $A_5 = Mon(\mathcal{A}_k^+)^+ = Mon(\mathcal{A}_k)^+ = Mon(\mathcal{A}_k)$, and so their oriented versions are our rotary hypermaps with automorphism group A_5.

The hypermap $D_{(0\,2)}(\mathcal{A}_1)$, an associate of the projective dodecahedron $\mathcal{A}_1 = P\mathcal{D}$ obtained by transposing 0-faces and 2-faces of $P\mathcal{D}$, is one of the 19 non-orientable hypermaps with automorphism group A_5 (see Table 4 in [3] for a complete list). Its orientable double covering is given by the pair of permutations $(x, y) = ((1\ 2\ 3), (1\ 5\ 3\ 4\ 2))$ which is a presentation pair of the presentation $\langle x, y \mid x^3, y^5, (xy)^2 \rangle$ of A_5. This pair (x, y) lifts to the four pairs $(\pm X, \pm Y)$ of generators of $\tilde{A}_5$, where (X, Y) is a presentation pair of the presentation $\langle X, Y \mid X^3 = Y^5 = (XY)^2 = -1 \rangle$ of $\tilde{A}_5$. These give rise to hypermaps of type $(6, 10, 4)$, $(3, 10, 4)$, $(6, 5, 4)$ and $(3, 5, 4)$. Including associates, there are 24 non-isomorphic rotary hypermaps.

Denoting by $\Delta^+(a, b, c)$ the group with presentation $\langle s, t \mid s^a, t^b, (st)^c \rangle$, where $a, b, c \in \mathbb{N}$, the hypermap $\mathcal{A}_2^+ = \mathcal{GD}$ arises from the epimorphism $\Delta^+(5, 2, 5) \to A_5$, $s \mapsto y^2$, $t \mapsto xy$. The 2-base (y^2, xy) of A_5 lifts to 2-bases $(\pm Y^2, \pm XY)$ of $\tilde{A}_5$ giving rise to hypermaps of type $(5, 4, 5)$, $(10, 4, 10)$, $(5, 4, 10)$ and $(10, 4, 5)$, since $|Y^2XY| = |-YX^{-1}| = |-(YX)^{-1}X| = |-Y|$. Twelve non-isomorphic rotary hypermaps are counted including associates.

The hypermap $\mathcal{A}_3^+$ arises from the epimorphism $\Delta^+(3, 3, 5) \to A_5$, $s \mapsto x$, $t \mapsto xy^2$. The 2-base (x, xy^2) lifts to 2-bases $(\pm X, \pm XY^2)$ of $\tilde{A}_5$ giving rise to hypermaps of type $(6, 3, 5)$, $(3, 3, 10)$, $(6, 6, 10)$ and $(3, 6, 5)$, since $|X^2Y^2| = |(-Y^{-1}XY^{-1})^{X^{-1}}| = |(-Y^{-1})^{XY^{-2}X^{-1}}|$. This also gives 12 non-isomorphic rotary hypermaps (including associates).

The hypermap $\mathcal{A}_4^+$ arises from the epimorphism $\Delta^+(5, 5, 3) \to A_5$, $s \mapsto xy^{-1}$, $t \mapsto y$. The 2-base (xy^{-1}, y) lifts to 2-bases $(\pm XY^{-1}, \pm Y) = (\pm Y^X, \pm Y)$ of $\tilde{A}_5$ giving rise to hypermaps of type $(10, 10, 6)$, $(5, 10, 3)$, $(10, 5, 3)$ and $(5, 5, 6)$. Including their associates we get 12 non-isomorphic rotary hypermaps.

The hypermap $\mathcal{A}_5^+$ arises from the epimorphism $\Delta^+(5, 5, 3) \to A_5$, $s \mapsto xy^{-2}$, $t \mapsto y^2$. The 2-base (xy^{-2}, y^2) lifts to 2-bases $(\pm XY^{-2}, \pm Y^2)$ of $\tilde{A}_5$, and as the order of XY^{-2} is the order of Y (since $XY^{-2} = Y^{XY^{-2}X}$) they give rise to hypermaps of type $(10, 5, 6)$, $(5, 5, 3)$, $(10, 10, 3)$ and $(5, 10, 6)$. Another 12 rotary hypermaps are counted (including associates).

Finally the hypermap $\mathcal{A}_6^+$ arises from the epimorphism $\Delta^+(5,5,5) \to A_5$ sending s to xy^{-1} and t to y^{-1}. The 2-base (xy^{-1}, y^{-1}) lifts to 2-bases $(\pm XY^{-1}, \pm Y^{-1}) = (\pm Y^X, \pm Y^{-1})$ of $\tilde{A}_5$ giving rise to hypermaps of type $(10,10,10)$, $(5,10,5)$, $(10,5,5)$ and $(5,5,10)$. This contributes with 4 rotary hypermaps.

$\hat{\mathcal{Q}}$	σ	l	m	n	g
$Bin(\mathcal{D})_1$	6	6	4	10	30
$Bin(\mathcal{D})_2$	6	3	4	10	20
$Bin(\mathcal{D})_3$	6	6	4	5	24
$Bin(\mathcal{D})_4$	6	3	4	5	14
$Bin(\mathcal{GD})_1$	3	5	4	5	22
$Bin(\mathcal{GD})_2$	3	10	4	10	34
$Bin(\mathcal{GD})_3$	6	5	4	10	28
$Bin(\mathcal{A}_3^+)_1$	6	6	3	5	19
$Bin(\mathcal{A}_3^+)_2$	3	3	3	10	15
$Bin(\mathcal{A}_3^+)_3$	3	6	6	10	35
$Bin(\mathcal{A}_4^+)_1$	3	10	10	6	39
$Bin(\mathcal{A}_4^+)_2$	6	5	10	3	23
$Bin(\mathcal{A}_4^+)_3$	3	5	5	6	27
$Bin(\mathcal{A}_5^+)_1$	6	5	10	6	33
$Bin(\mathcal{A}_5^+)_2$	3	5	5	3	17
$Bin(\mathcal{A}_5^+)_3$	3	10	10	3	29
$Bin(\mathcal{A}_6^+)_1$	3	5	5	10	31
$Bin(\mathcal{A}_6^+)_2$	1	10	10	10	43

$\mathcal{Q}$	l	m	n	g
$\mathcal{A}_1^+ = \mathcal{D}$	3	2	5	0
$\mathcal{A}_2^+ = \mathcal{GD}$	5	2	5	4
$\mathcal{A}_3^+ = W^{-1}(\mathcal{D}^{\hat{0}})$	3	3	5	5
$\mathcal{A}_4^+ = W^{-1}(\mathcal{I}^{\hat{0}})$	5	5	3	9
$\mathcal{A}_5^+ = W^{-1}(\mathcal{M}_5)$	5	5	3	9
$\mathcal{A}_6^+ = W^{-1}(\mathcal{GD}^{\hat{0}})$	5	5	5	13

Table 7. The 76 rotary hypermaps with automorphism group $\tilde{A}_5$

3.3. The rotary hypermaps with automorphism group $GL(2,3)$

Similarly to $\tilde{S}_4$, twelve of the eighteen rotary hypermaps with automorphism group $GL(2,3)$ arise by lifting the presentation pair $(x,y) = ((1\ 2\ 3), (1\ 4\ 3\ 2))$ of the presentation of S_4 given in 3.2 to the four pairs $(\pm X, \pm Y)$ of generators of $GL(2,3)$, where (X,Y) satisfy the relations of $\langle X, Y \mid X^3 = Y^4 = -1,\ (XY)^2 = 1\rangle$. The two pairs $(X, \pm Y)$ give rise to isomorphic hypermaps of type $(6,8,2)$, while the two pairs $(-X, \pm Y)$ give rise to isomorphic hypermaps of type $(3,8,2)$. The remaining six arise by lifting (xy^{-1}, y) to 2-bases $(\pm XY^{-1}, \pm Y) = (\mp Y^X, \pm Y)$ of $GL(2,3)$. The two pairs (XY^{-1}, Y) and $(-XY^{-1}, -Y)$ give rise to isomorphic hypermaps of type $(8,8,6)$, while the two pairs $(-XY^{-1}, Y)$ and $(XY^{-1}, -Y)$ give rise to isomorphic hypermaps of type $(8,8,3)$.

$\hat{\mathcal{Q}}$	σ	l	m	n	g
$\mathcal{L}(\mathcal{O})_1$	6	8	2	6	6
$\mathcal{L}(\mathcal{O})_2$	6	8	2	3	2
$\mathcal{L}(\mathcal{S}_4^+)_1$	3	8	8	6	15
$\mathcal{L}(\mathcal{S}_4^+)_2$	3	8	8	3	11

$\mathcal{Q}$	l	m	n	g
$\mathcal{S}_3^+ = P\mathcal{O}^+ = \mathcal{O}$	4	2	3	0
$\mathcal{S}_4^+ = W^{-1}(\mathcal{O}^{\hat{0}})$	4	4	3	3

Table 8. The 18 rotary hypermaps with automorphism group $GL(2,3)$

3.4. The rotary hypermaps with automorphism group B

As in previous sections, let (X,Y) be a presentation pair for $\langle X,Y \mid X^3 = (XY)^2 = -1,\, Y^4 = 1\rangle$, lifted from the presentation pair $(x,y) = ((1\,2\,3),(1\,4\,3\,2))$ of the presentation of S_4 given in Subsection3.2. The two pairs $(X,\pm Y)$ give rise to isomorphic hypermaps of type $(6,4,4)$, while the two pairs $(-X,\pm Y)$ give rise to isomorphic hypermaps of type $(3,4,4)$. Now, contrary to the previous cases where associates of the same type were isomorphic, the 6 associates either of $(B;X,Y)$ or of $(B;-X,Y)$ are not isomorphic since the assignment $X \mapsto X^{-1}$, $Y \mapsto XY$ does not extend to an automorphism of B. The remaining 6 arise by lifting (xy^{-1},y) to the four generating pairs $(\pm XY^{-1},\pm Y) = (\pm Y^X,\pm Y)$. The two pairs (XY^{-1},Y) and $(-XY^{-1},-Y)$ give rise to isomorphic rotary hypermaps of type $(4,4,6)$, while the two pairs $(-XY^{-1},Y)$ and $(XY^{-1},-Y)$ give rise to isomorphic rotary hypermaps of type $(4,4,3)$.

$\hat{\mathcal{Q}}$	σ	l	m	n	g
$\mathcal{B}(\mathcal{O})_1$	6	4	4	6	9
$\mathcal{B}(\mathcal{O})_2$	6	4	4	3	5
$\mathcal{B}(\mathcal{S}_4^+)_1$	3	4	4	6	9
$\mathcal{B}(\mathcal{S}_4^+)_2$	3	4	4	3	5

$\mathcal{Q}$	l	m	n	g
$\mathcal{S}_3^+ = P\mathcal{O}^+ = \mathcal{O}$	4	2	3	0
$\mathcal{S}_4^+ = W^{-1}(\mathcal{O}^{\hat{0}})$	4	4	3	3

Table 9. The 18 rotary hypermaps with automorphism group B

4. Reflexible double coverings of the Platonic rotary hypermaps

It was shown in §9 of [3] that the Platonic hypermaps are reflexible. Now we show that the rotary Platonic double coverings are also reflexible (Theorem 4.1). As we have seen in 3.1, the rotary hypermaps with automorphism group $A_4 \times C_2$, $S_4 \times C_2$ and $A_5 \times C_2$ are reflexible. For the remaining cases we list (Table 10) the rotary hypermaps up to an associate and a derivative. On the middle right of the table are the relations that define their automorphism groups. The choice of the rotary hypermap representing the class of associates and derivatives was carefully made in order to get a direct application of Theorem 2.3.

$\hat{\mathcal{Q}}$	*Type*	Relations	$Aut(\mathcal{Q})$
$D_{(12)}(Bin(\mathcal{T})_1)$	$(6,6,4)$	$X^3 = Y^3 = (XY)^2 = -1$	$\tilde{A}_4$
$Bin(W^{-1}(\mathcal{T}^{\hat{0}}))_1$	$(6,6,6)$	$X^3 = Y^3 = (X^{-1}Y)^2 = -1$	$\tilde{A}_4$
$D_{(012)}(Bin(\mathcal{O})_1)$	$(6,8,4)$	$X^3 = Y^4 = (XY)^2 = -1$	$\tilde{S}_4$
$Bin(\mathcal{S}_4^+)_1$	$(8,8,6)$	$Y^4 = (XY)^3 = (XY^2)^2 = -1$	$\tilde{S}_4$
$D_{(12)}(Bin(\mathcal{D})_1)$	$(6,10,4)$	$X^3 = Y^5 = (XY)^2 = -1$	$\tilde{A}_5$
$Bin(\mathcal{GD})_1$	$(5,4,5)$	$X^5 = (YX^2)^3 = 1$, $Y^2 = -1$	$\tilde{A}_5$
$Bin(\mathcal{A}_3^+)_1$	$(6,3,5)$	$X^3 = ((YX^{-1})^2Y)^2 = -1$, $(X^{-1}Y)^5 = 1$	$\tilde{A}_5$
$Bin(\mathcal{A}_4^+)_1$	$(5,5,6)$	$Y^5 = 1$, $(XY)^3 = (XY^2)^2 = -1$	$\tilde{A}_5$
$Bin(\mathcal{A}_5^+)_1$	$(10,5,6)$	$Y^5 = 1$, $(XY)^3 = (XY^{-1})^2 = -1$	$\tilde{A}_5$
$Bin(\mathcal{A}_6^+)_1$	$(10,10,10)$	$Y^5 = (XY^{-1})^3 = (XY^{-2})^2 = -1$	$\tilde{A}_5$
$D_{(012)}(\mathcal{L}(\mathcal{O})_1)$	$(6,8,2)$	$X^3 = Y^4 = -1$, $(XY)^2 = 1$	$GL(2,3)$
$\mathcal{L}(\mathcal{S}_4^+)_1$	$(8,8,6)$	$Y^4 = (XY)^3 = -1$, $(XY^2)^2 = 1$	$GL(2,3)$
$D_{(012)}(\mathcal{B}(\mathcal{O})_1)$	$(6,4,4)$	$X^3 = (XY)^2 = -1$, $Y^4 = 1$	B
$\mathcal{B}(\mathcal{S}_4^+)_1$	$(4,4,6)$	$Y^4 = 1$, $(XY)^3 = (XY^2)^2 = -1$	B

Table 10. The rotary Platonic double coverings up to an associate and a derivative

Theorem 4.1. *Every rotary Platonic double covering is reflexible.*

Proof. It is a matter of routine to check that these 14 rotary hypermaps are reflexible. On the other hand, the central involution -1 being of the form $(X^nY^m)^k$, for some $n, m, k \in \mathbb{N}$, is clearly auto-reflexible. According to Theorem 2.3 and Remark 2.2 we conclude that all their associates and derivatives are also reflexible. □

5. Double coverings of the dihedral rotary hypermaps

Consider the following presentations for D_k: $P_1 = \langle x, y \mid x^k, y^2, (xy)^2\rangle$, $P_2 = \langle x, y \mid x^2, y^k, (xy)^2\rangle$ and $P_3 = \langle x, y \mid x^2, y^2, (xy)^k\rangle$. Any pair (a, b) generating D_k is a presentation pair of one of P_1, P_2 or P_3. Hence

$$|B_2(D_k)| = \begin{cases} |S_{P_1}| = |Aut(D_k)| & \text{if } k = 2 \\ |S_{P_1}| + |S_{P_2}| + |S_{P_3}| = 3|Aut(D_k)| & \text{if } k > 2\,, \end{cases}$$

and therefore $d_2(D_k) = \begin{cases} 1 & \text{if } k = 2 \\ 3 & \text{if } k > 2 \end{cases}$, where in the case of $k > 2$ the 3 hypermaps are the 3 associates $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ (with monodromy groups P_1, P_2, P_3 resp.) of the map $\mathcal{Q}_1$ with 2 vertices, k edges and k faces on the sphere. The double coverings $\hat{G}$ of D_k are (see [5]) D_{2k}, Q_{2k}, if k is odd, and D_{2k}, Q_{2k}, $(C_{\frac{k}{2}} \rtimes C_4) \rtimes C_2$, $Q_k \rtimes C_2$, $C_k \rtimes C_4$, and $D_k \times C_2$, if k is even, where Q_{2k} is the dicyclic group, often called the generalised quaternion group, of order $4k$ with presentation $\langle x, y \mid x^k = y^2, x^y = x^{-1}\rangle$. Of these only D_{2k} and Q_{2k} are q-strong coverings, 2-strong if k is odd and 4-strong otherwise (see [5]). Thus by (3) we have for these two cases:

$$d_2(D_{2k}) = \begin{cases} 1 & \text{if } k = 2 \\ 3 & \text{if } k > 2 \end{cases}$$

and

$$d_2(Q_{2k}) = \frac{4d_2(D_k)}{q} = \begin{cases} 2d_2(D_k) = 6 & \text{if } k \text{ is odd} \\ d_2(D_k) = 3 & \text{if } k > 2 \text{ is even} \\ d_2(D_2) = 1 & \text{if } k = 2 \end{cases}.$$

Fix a presentation pair (x, y) of P_3. For D_{2k}, (x, y) lift to a presentation pair (X, Y) of the presentation $P_{(1,1,i)} = \langle X, Y \mid X^2 = Y^2 = 1, (XY)^k = i \rangle = D_{2k}$ which gives rise to a rotary hypermap $\hat{\mathcal{Q}}$ of type $(2, 2, 2k)$. In this case the other lifts (iX, Y), (X, iY) and (iX, iY), when they generate D_{2k}, are also presentation pairs of $P_{(1,1,i)}$ and so they give rise to rotary hypermaps isomorphic to $\hat{\mathcal{Q}}$. For Q_{2k}, (x, y) lift to a presentation pair (X, Y) of the presentation $P_{(i,i,i)} = \langle X, Y \mid X^2 = Y^2 = (XY)^k = i \rangle = Q_{2k}$. If k is even, the lifts (X, Y), (iX, Y), (X, iY) and (iX, iY) are presentation pairs of $P_{(i,i,i)}$ hence giving rise to isomorphic rotary hypermaps of type $(4, 4, 2k)$. If k is odd, the lifts (X, Y), (iX, iY) are presentation pairs of $P_{(i,i,i)}$, while the lifts (iX, Y), (X, iY) are presentation pairs of $P_{(i,i,1)} = \langle X, Y \mid X^2 = Y^2 = i, (XY)^k = 1 \rangle = Q_{2k}$. These give rise to 2 rotary hypermaps of type $(4, 4, 2k)$ and $(4, 4, k)$.

$\hat{\mathcal{Q}}$	$\hat{G}$	σ	l	m	n	g
$\mathcal{Q}_4^0$	Q_4	1	4	4	4	2
$\downarrow$						
$\mathcal{D}_2$	D_2	1	2	2	2	0

$\hat{\mathcal{Q}}_k$, $k > 2$	$\hat{G}$	σ	l	m	n	g
$\mathcal{D}_{2k}$	D_{2k}	3	2	2	$2k$	0
$\mathcal{Q}_{2k}^0$, k even	Q_{2k}	3	4	4	$2k$	k
$\mathcal{Q}_{2k}^1$, k odd	Q_{2k}	3	4	4	$2k$	k
$\mathcal{Q}_{2k}^2$, k odd	Q_{2k}	3	4	4	k	$k-1$
$\downarrow$						
$\mathcal{D}_k$	D_k	3	2	2	k	0

Table 11. The rotary double coverings $\hat{\mathcal{Q}}$ of $\mathcal{D}_k$ with $Aut(\hat{\mathcal{Q}})$ strong covering D_k

The other double coverings $\hat{G}$ are not strong which means that there is more than one presentation class containing presentations of $\hat{G}$. The direct product $D_k \times C_2$ (k even), being not 2-generated, does not give rise to rotary hypermaps. Thereby this case will be excluded in the following discussion. In general different presentation classes may not give rise to dual hypermaps, but in our case the non-strong double coverings have only two presentation classes giving rise to dual hypermaps. Looking at [5] we see that $P_{(1,i,1)}$, $P_{(1,i,i)}$ and $P_{(i,i,1)}$ are presentations of $(C_{\frac{k}{2}} \rtimes C_4) \rtimes C_2$, $Q_k \rtimes C_2$ and $C_k \rtimes C_4$ (k even), respectively. In the particular case of $k = 2$, $(C_1 \rtimes C_4) \rtimes C_2 = D_4$ and $Q_2 \rtimes C_2 = C_2 \rtimes C_4 = C_4 \times C_2$. So the number of rotary hypermaps doubly covering $\mathcal{D}_k$ with $Aut(\hat{\mathcal{Q}})$ not strongly covering D_k is $3 + 3 = 6$ if $k = 2$, $3 + 6 + 3 = 12$ if $k = 4$ and $6 + 6 + 3 = 15$ if $k > 4$.

Note that $\mathcal{N}_4^3$ has 3 (non-isomorphic) associates of type $(4, 4, 4)$. This is a consequence of the fact that there is no automorphism of $P_{(i,i,1)}$ sending (X, Y) to (X^{-1}, XY).

Theorem 5.1. *Any double covering of a rotary dihedral hypermap is reflexible.*

$\hat{\mathcal{Q}}$	$\hat{G}$	σ	l m n	g
$\mathcal{D}_4$	D_4	3	2 2 4	0
$\mathcal{N}_2$	$C_4 \times C_2$	3	4 4 2	1
$\downarrow$				
$\mathcal{D}_2$	D_2	1	2 2 2	0

$\hat{\mathcal{Q}}_k$, $k > 2$ even	$\hat{G}$	σ	l m n	g
$\mathcal{N}_k^1$	$(C_{\frac{k}{2}} \rtimes C_4) \rtimes C_2$	3 or 6	2 4 k	$\frac{k-2}{2}$
$\mathcal{N}_k^2$	$Q_k \rtimes C_2$	6	2 4 $2k$	$\frac{k}{2}$
$\mathcal{N}_k^3$	$C_k \rtimes C_4$	3	4 4 k	$k-1$
$\downarrow$				
$\mathcal{D}_k$	D_k	3	2 2 k	0

Table 12. The rotary double coverings $\hat{\mathcal{Q}}$ of $\mathcal{D}_k$ with $Aut(\hat{\mathcal{Q}})$ not strongly covering D_k, k even

Proof. One easily sees that if (X, Y) is a presentation pair of P_J, then (X^{-1}, Y^{-1}) is also a presentation pair of P_J. This proves that the hypermaps in Tables 11 and 12 are reflexible. $\square$

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