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Davis' Convexity Theorem and Extremal Ellipsoids

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Abstract. We give a variety of uniqueness results for minimal ellipsoids circumscribing and maximal ellipsoids inscribed into a convex body. Uniqueness follows from a convexity or concavity criterion on the function used to measure the size of the ellipsoid. Simple examples with non-unique minimal or maximal ellipsoids conclude this article.

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1. Introduction

By a classic result in convex geometry the minimal volume ellipsoid enclosing a convex body $F \subset \mathbb{R}^d$ and the maximal volume ellipsoid inscribed into F are unique [4], [10]. Both ellipsoids are important objects in convex geometry and have numerous applications in diverse fields of applied and pure mathematics (see for example [2], [7] or the introductory sections of [12], [16]). More information on the role of ellipsoids in convex geometry can be found in [14] and [9, Section 3].

In this article we are concerned with uniqueness results for minimal and maximal ellipsoids with respect to size functions different from the volume. The earliest contribution to this topic is [6] who proved uniqueness of the minimal quermass integral ellipsoid among all enclosing ellipsoids with prescribed center. This result can also be deduced from more general findings of [8] and [15]. As to maximal inscribed ellipsoids we are only aware of [11] who shows uniqueness with respect to a vast class of size functions that are defined with the help of an arbitrary convex body.

In this article we provide uniqueness results for minimal enclosing and maximal inscribed ellipsoids for further families of size functions. The basic ideas are similar to that of [4] and [15]. The new results are found by applying them to diverse representations of ellipsoids with the help of symmetric matrices.

After recalling some basic concepts in Section 2 we define the notion of a "size function" and, in Section 3, present several different uniqueness results. In any case it is necessary to study a particular representation of ellipsoids and properties of an "in-between ellipsoid" in this representation. Finally, in Section 4 we describe a few examples of convex bodies and size functions with non-unique extremal ellipsoids.

2. Preliminaries

With the help of a positive semi-definite symmetric matrix $A \in \mathbb{R}^{d\times d}$ and a vector $m \in \mathbb{R}^d$ an ellipsoid can be described as

$$E = \{x \in \mathbb{R}^d \colon (x-m)^T \cdot A \cdot (x-m) - 1 \leq 0\}. \tag{1}$$

The interior of E is the set of all points x that strictly fulfill the defining inequality. In this article we generally admit degenerate ellipsoids with empty interior since they may appear as maximal inscribed ellipsoids. The interior is empty if A is only positive semi-definite and not positive definite. We call the ellipsoid *singular* if this is the case and *regular* otherwise.

The vector m is the coordinate vector of the ellipsoid center. A straight line incident with m and in direction of an eigenvector of A is called an ellipsoid axis, its semi-axis length a_i is related to the corresponding eigenvalue ν_i via $a_i = \nu_i^{-1/2}$. Since A is symmetric and positive definite, there exist d pairwise orthogonal axes with real semi-axis lengths.

Note that equation (1) is not the only possibility for describing ellipsoids. In Section 3 we will encounter several alternatives but all of them use a symmetric matrix and a vector as describing parameters.

There exist different notions for the "size" of an ellipsoid. A natural measure for the size is the ellipsoid's volume, but we may also take the surface area, a quermass integral, a norm on the vector of semi-axis lengths etc. More generally, we consider a non-negative function f on the ordered vector of semi-axis lengths that satisfies a few basic requirements. By $\mathbb{R}_>$ we denote the set of positive, by $\mathbb{R}_\geq$ the set of non-negative reals; $\mathbb{R}^d_\geq$ is the set of vectors $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$ with entries $x_i \in \mathbb{R}_\geq$.

Definition 1. *A function $f \colon \mathbb{R}^d_\geq \to \mathbb{R}_\geq$ is called size function for an ellipsoid if it is continuous, strictly monotone increasing in any of its arguments and symmetric, that is, $f(y) = f(x)$ whenever y is a permutation of x.*

Denote by $e(A)$ the vector of eigenvalues of a symmetric matrix A, arranged in ascending order. Clearly, f can be extended to the space of symmetric, positive semi-definite matrices by letting $f(A) = f \circ e(A)$. Sometimes we will even write $f(E)$ when an ellipsoid E is described by a symmetric matrix A.

Note that f depends only on the eigenvalues of the symmetric matrix A. Hence, it is independent of the position and orientation of E.

3. Uniqueness results

The uniqueness proofs in this article all follow a certain scheme. We want to prove that there exists only one minimal enclosing ellipsoid (with respect to a certain size function f) of a convex body $F \subset \mathbb{R}^d$. Assuming existence of two minimizers E_0 and E_1 we construct an "in-between ellipsoid" E_λ that contains the common interior of E_0 and E_1 (and hence also the set F) and is strictly smaller (measured by the size function f) than E_0 and E_1. Uniqueness results for maximal inscribed ellipsoids can be obtained in similar fashion.

This type of proof requires the construction of an in-between ellipsoid E_λ that contains F (or is contained in F) and is strictly smaller (or larger) than E_0 and E_1. Different constructions of E_λ yield different uniqueness results.

3.1. Image of the unit sphere

An ellipsoid may be viewed as affine image of the unit ball:

$$E = \{y \in \mathbb{R}^d \colon y = P \cdot x + t,\ x \in \mathbb{R}^d,\ \|x\| \le 1\}, \tag{2}$$

where $P \in \mathbb{R}^{d \times d}$ is a (not necessarily regular) matrix and $t \in \mathbb{R}^d$.

The matrix P is not uniquely determined by the ellipsoid. It is still possible to apply an automorphic transformation to the unit sphere before the map $x \mapsto P \cdot x + t$ or an automorphic transformation to the resulting ellipsoid afterwards. By the left polar decomposition there exists a symmetric positive semi-definite matrix S and an orthogonal matrix U such that $P = S \cdot U$. Hence, we may choose P to be symmetric and positive semi-definite.

The ordered vector of semi-axis lengths of E is

$$a = (a_1, \ldots, a_d)^T = (\nu_1, \ldots, \nu_d)^T, \tag{3}$$

where ν_i, $i = 1, \ldots, d$ are the eigenvalues of P. In other words, we have $a = e(P)$. For reasons that will become clear in the course of this text we can also write this with the help of the function

$$w^p \colon \mathbb{R}^d \to \mathbb{R}^d, \quad (x_1, \ldots, x_d)^T \mapsto (|x_1|^p, \ldots, |x_d|^p)^T \tag{4}$$

as

$$a = w^1 \circ e(P) = e(P). \tag{5}$$

Definition 2. (in-between ellipsoid) *We define the in-between ellipsoid E_λ to two ellipsoids E_0 and E_1 with respect to the representation* (2) *as*

$$E_\lambda = \{y \in \mathbb{R}^d \colon y = P_\lambda \cdot x + t_\lambda,\ \|x\| \le 1\}, \quad \lambda \in [0,1] \tag{6}$$

where

$$E_0 = \{P_0 \cdot x + t_0 \colon \|x\| \le 1\}, \quad E_1 = \{P_1 \cdot x + t_1 \colon \|x\| \le 1\}, \tag{7}$$

and

$$P_\lambda = (1-\lambda)P_0 + \lambda P_1, \quad t_\lambda = (1-\lambda)t_0 + \lambda t_1. \tag{8}$$

Note that P_λ is a symmetric, positive semi-definite matrix and E_λ is indeed an ellipsoid.

Lemma 3. *The in-between ellipsoid E_λ, $0 \le \lambda \le 1$, of two ellipsoids E_0 and E_1 is a subset of the convex hull of the two ellipsoids E_0 and E_1, that is*

$$E_\lambda \subset \operatorname{conv}(E_0, E_1). \tag{9}$$

Proof. Let x be an element of E_λ. There exists y with $\|y\| \le 1$ such that $x = P_\lambda \cdot y + t_\lambda$. By the definition of P_λ and t_λ we can write

$$x = (1-\lambda)(P_0 \cdot y + t_0) + \lambda(P_1 \cdot y + t_1) = (1-\lambda)x_0 + \lambda x_1, \tag{10}$$

with $x_0 \in E_0$ and $x_1 \in E_1$. Hence, x is in the convex hull of E_0 and E_1 and we conclude $E_\lambda \subset \operatorname{conv}(E_0, E_1)$. □

This lemma together with the following proposition already yields a first uniqueness result for minimal enclosing ellipsoids.

Proposition 4. (Davis' convexity theorem) *A convex, lower semi-continuous and symmetric function f of the eigenvalues of a symmetric matrix is (essentially strictly) convex on the set of symmetric matrices if and only if its restriction to the set of diagonal matrices is (essentially strictly) convex.*

This proposition was stated and proved by [5] and extended to "essentially strict convexity" by [13]. In Proposition 4 "symmetric" means that the function f is independent of the order of its arguments. The precise definition of "essentially strict convexity" is rather technical and will be omitted since we will use only a weaker version of Lewis' generalization.

We will apply Davis' convexity theorem to size functions of ellipsoids. When proving uniqueness results for minimal ellipsoids we demand strict convexity of f on $\mathbb{R}^d_>$. For maximal inscribed ellipsoids we demand strict concavity on $\mathbb{R}^d_\ge$. Results of [13] then guarantee strict convexity/concavity of $f \circ e$ on the spaces of symmetric matrices with eigenvalues in $\mathbb{R}_>$, $\mathbb{R}_\ge$, respectively.

Theorem 5. *Let f be a size function for ellipsoids such that $f \circ w^1$ is strictly concave on $\mathbb{R}^d_{\geq}$. Further let $F \subset \mathbb{R}^d$ be a compact convex body. Among all ellipsoids that are contained in F there exists a unique ellipsoid that is maximal with respect to f.*

Proof. The existence of a maximal (with respect to f) inscribed ellipsoid follows from the compactness of F and the continuity of $f \circ w^1$. This is explained in great detail in [4].

To proof uniqueness, we assume existence of two f-maximal ellipsoids E_0 and E_1, that is $f(E_0) = f(E_1)$, both contained in F. We compute the in-between ellipsoid E_λ for $0 < \lambda < 1$ as in (6). By Lemma 3 it is contained in the convex hull of E_0 and E_1 and therefore also in F. Looking at the size of the in-between ellipsoid we find

$$f(E_\lambda) = f \circ w^1 \circ e(P_\lambda) = f \circ w^1 \circ e\big((1-\lambda)P_0 + \lambda P_1\big). \tag{11}$$

Because P_λ is a symmetric matrix, we can use Davis' convexity theorem and find, by strict concavity of $f \circ w^1 \circ e$,

$$\begin{aligned} f \circ w^1 \circ e\big((1-\lambda)P_0 + \lambda P_1\big) &> (1-\lambda) f \circ w^1 \circ e(P_0) + \lambda f \circ w^1 \circ e(P_1) \\ &= f(E_0) = f(E_1) \end{aligned} \tag{12}$$

which is a contradiction. □

Remark 6. The maximal ellipsoids with respect to size functions can be computed by a convex program, similar to that described in [3, Section 8.4.2].

3.2. Inverse image of the unit sphere

In this section we view an ellipsoid as the set

$$E = \{x \in \mathbb{R}^d \colon \|P \cdot x + t\| \leq 1\}, \tag{13}$$

where $P \in \mathbb{R}^{d \times d}$ and $t \in \mathbb{R}^d$, that is, as affine pre-image of the unit ball. Again, it is no loss of generality to assume that P is symmetric and positive semi-definite. Since we will use the representation (13) only for deriving uniqueness results for minimal ellipsoids we can even assume that P is positive definite. The ordered vector of semi-axis lengths of E is

$$a = w^{-1} \circ e(P). \tag{14}$$

Definition 7. (in-between ellipsoid) *The in-between ellipsoid E_λ to two ellipsoids E_0 and E_1 with respect to the representation* (13) *is defined as*

$$E_\lambda = \{x \in \mathbb{R}^d \colon \|P_\lambda \cdot x + t_\lambda\| \leq 1\}, \quad \lambda \in [0,1] \tag{15}$$

where

$$E_0 = \{x \in \mathbb{R}^d \colon \|P_0 \cdot x + t_0\| \leq 1\}, \quad E_1 = \{x \in \mathbb{R}^d \colon \|P_1 \cdot x + t_1\| \leq 1\}, \tag{16}$$

and

$$P_\lambda = (1-\lambda)P_0 + \lambda P_1, \quad t_\lambda = (1-\lambda)t_0 + \lambda t_1. \tag{17}$$

Again, P_λ is symmetric and positive definite and E_λ is a non-degenerate ellipsoid.

Lemma 8. *Let E_λ, $0 \le \lambda \le 1$, be the in-between ellipsoid of two ellipsoids E_0 and E_1 defined as in equations* (15)–(17). *Then the in-between ellipsoid E_λ encloses the intersection of E_0 and E_1.*

Proof. If the intersection of E_0 and E_1 is empty, nothing has to be shown. (Note that this case is irrelevant for the proof of the main Theorem 9 below.) Assume therefore that there exists $x \in E_0 \cap E_1$, that is,

$$\|P_i \cdot x + t_i\| \le 1, \quad i \in \{0, 1\}. \tag{18}$$

We then have

$$1 = (1-\lambda) \cdot 1 + \lambda \cdot 1 \ge (1-\lambda)\|P_0 \cdot x + t_0\| + \lambda \|P_1 \cdot x + t_1\|. \tag{19}$$

The triangle inequality implies

$$\begin{aligned} &(1-\lambda)\|P_0 \cdot x + t_0\| + \lambda\|P_1 \cdot x + t_1\| \ge \\ &\|(1-\lambda)\big(P_0 \cdot x + t_0\big) + \lambda\big(P_1 \cdot x + t_1\big)\| = \\ &\|\big((1-\lambda)P_0 + \lambda P_1\big) \cdot x + \big((1-\lambda)t_0 + \lambda t_1\big)\| = \\ &\|P_\lambda \cdot x + t_\lambda\|. \end{aligned} \tag{20}$$

Combining (19) and (20) we see that $\|P_\lambda \cdot x + t_\lambda\| \le 1$. This shows that $x \in E_\lambda$. Hence $E_0 \cap E_1 \subset E_\lambda$ and the proof is complete. □

Theorem 9. *Let f be a size function for ellipsoids such that $f \circ w^{-1}$ is strictly convex on $\mathbb{R}^d_>$. Further let $F \subset \mathbb{R}^d$ be a compact convex body. Among all ellipsoids that contain F there exists a unique ellipsoid that is minimal with respect to f.*

Proof. The existence of a minimal (with respect to f) ellipsoid that encloses F, follows from the compactness of F and the continuity of $f \circ w^{-1}$ (see again [4]).

To proof uniqueness, we assume existence of two f-minimal ellipsoids E_0 and E_1, that is $f(E_0) = f(E_1)$, both containing F. We compute the in-between ellipsoids E_λ for $0 < \lambda < 1$, as in (15). By Lemma 8 it contains the common interior of $E_0 \cap E_1$ and hence also F. Looking at the size of E_λ we find

$$f(E_\lambda) = f \circ w^{-1} \circ e(P_\lambda) = f \circ w^{-1} \circ e\big((1-\lambda)P_0 + \lambda P_1\big). \tag{21}$$

Because P_λ is a symmetric matrix, we can use Davis' convexity theorem (see Proposition 4 on page 266). It implies that $f \circ w^{-1} \circ e$ is strictly convex. Therefore we can write

$$f \circ w^{-1} \circ e\big((1-\lambda)P_0 + \lambda P_1\big) < (1-\lambda) f \circ w^{-1} \circ e(P_0) + \lambda f \circ w^{-1} \circ e(P_1). \tag{22}$$

Because E_0 and E_1 have the same size it follows that

$$f(E_\lambda) = f \circ w^{-1} \circ e(P_\lambda) < f \circ w^{-1} \circ e(P_0) = f(E_0) = f(E_1). \tag{23}$$

We have now that the size of E_λ is smaller than the size of E_0 and E_1. Together with Lemma 8 this constitutes a contradiction to the assumed minimality of E_0 and E_1 and finishes the proof. □

3.3. Extremal affine images of convex unit balls

It is easy to see that the proves of Theorems 5 and 9 remain true if we replace the Euclidean unit ball by an arbitrary centrally symmetric convex body, centered at the origin, and measure its size by the volume. Hence, we can state a much more general result:

Theorem 10. *The volume-minimal circumscribing affine image of an arbitrary convex unit ball to a compact convex body F is unique. The same is true for volume-maximal inscribed affine image of an arbitrary convex unit ball.*

3.4. Algebraic equation

The maybe most straightforward way to represent a non-degenerate ellipsoid $E \subset \mathbb{R}^d$ uses the algebraic equation of E:

$$E = \{x \in \mathbb{R}^d \colon (x-m)^T \cdot A \cdot (x-m) \le 1\}, \tag{24}$$

with a symmetric, positive definite matrix $A \in \mathbb{R}^{d\times d}$ and $m \in \mathbb{R}^d$. The vector a of ordered semi-axis lengths of E is found as

$$a = w^{-1/2} \circ e(A). \tag{25}$$

The equation of E can also be written with the help of a single matrix of dimension $(d+1)\times(d+1)$:

$$E = \{X \in \mathbb{R}^{d+1} \colon X^T \cdot M \cdot X \le 0\}, \tag{26}$$

where

$$X = \begin{pmatrix} 1 \\ x \end{pmatrix}, \quad M = \begin{pmatrix} -1 & -m^T \cdot A' \\ -A' \cdot m & A' \end{pmatrix} \quad \text{and} \quad A' = \frac{A}{1 - m^T \cdot A \cdot m}. \tag{27}$$

If we define the in-between ellipsoid E_λ to two ellipsoids E_0 and E_1 with respect to the representation (26) by building a convex sum of the two homogeneous matrices that define E_0 and E_1,

$$E_\lambda = \{X \in \mathbb{R}^{d+1} \colon X^T \cdot M_\lambda \cdot X \le 0\}, \quad \lambda \in [0,1] \tag{28}$$

where

$$M_\lambda = (1-\lambda)M_0 + \lambda M_1, \tag{29}$$

we arrive at the situation discussed in [15]. The main uniqueness result is

Proposition 11. *Let f be a size function and $f \circ w^{-1/2}$ be a strictly convex function on $\mathbb{R}^d_>$. Further let $F \subset \mathbb{R}^d$ be a compact convex body. Among all ellipsoids that contain F there exists a unique ellipsoid that is minimal with respect to f.*

3.5. Dual equation

An ellipsoid can also be viewed as the set of hyperplanes that intersect the (point-set) ellipsoid in real points. Using hyperplane coordinates, this description is formally the same as in Section 3.4:

$$E = \{u \in \mathbb{R}^d \colon (u-c)^T \cdot B \cdot (u-c) \le 1\}, \tag{30}$$

where $B \in \mathbb{R}^{d\times d}$ is a symmetric, positive semi-definite matrix and $c \in \mathbb{R}^d$. In homogeneous form this is

$$E = \{U \in \mathbb{R}^{d+1} \colon U^T \cdot N \cdot U \le 0\}, \tag{31}$$

where

$$U = \begin{pmatrix} 1 \\ u \end{pmatrix}, \quad N = \begin{pmatrix} -1 & -c^T \cdot B' \\ -B' \cdot c & B' \end{pmatrix}, \quad \text{and} \quad B' = \frac{B}{1 - c^T \cdot B \cdot c}. \tag{32}$$

Translating the center of E to the origin, this description becomes

$$E_o = \{U \in \mathbb{R}^{d+1} \colon U^T \cdot N_o \cdot U \le 0\}, \tag{33}$$

where

$$N_o = \begin{pmatrix} -1 & 0^T \\ 0 & B' \cdot c \cdot c^T \cdot B' + B' \end{pmatrix}. \tag{34}$$

In this representation, the vector of semi-axis lengths is

$$a = w^{1/2} \circ e(B' \cdot c \cdot c^T \cdot B' + B'). \tag{35}$$

Definition 12. (in-between ellipsoid) *We define the in-between ellipsoid E_λ to two ellipsoids E_0 and E_1 with respect to the representation* (31) *by building the convex sum of the two defining homogeneous matrices:*

$$E_\lambda = \{U \in \mathbb{R}^{d+1} \colon U^T \cdot N_\lambda \cdot U \le 0\}, \quad \lambda \in [0,1] \tag{36}$$

where

$$N_\lambda = (1-\lambda)N_0 + \lambda N_1. \tag{37}$$

Note that we have no guarantee that E_λ is really an ellipsoid for all values $\lambda \in [0,1]$. It is, however, an ellipsoid at least in the vicinity of $\lambda = 0$ and $\lambda = 1$ and this is all we need. For reasons of simplicity we will not always mention this explicitly and still refer to E_λ as "in-between ellipsoid".

Lemma 13. *The in-between ellipsoid E_λ of two ellipsoids E_0 and E_1 lies inside the convex hull of E_0 and E_1, that is*

$$E_\lambda \subset \operatorname{conv}(E_0, E_1), \tag{38}$$

at least for values of λ in the vicinity of 0 *and* 1.

In order to prove Lemma 13 it is sufficient to consider the case $d = 2$. This can be seen as follows: Let x be a point in E_λ and take a plane π through x and the centers of E_0 and E_1, respectively. The in-between ellipsoid E_λ intersects π in an ellipse E'_λ that is obtained as in-between ellipse to $\pi \cap E_0$ and $\pi \cap E_1$. Hence, x lies in E_λ if and only if it lies in E'_λ.

The proof for $d = 2$ can be carried out by straightforward computation. It requires, however, a case distinction, is rather technical and does not provide useful insight. Therefore, we omit it at this place. It will be published in the first author's doctoral thesis.

Theorem 14. *Let f be a size function for ellipsoids such that $f \circ w^{1/2}$ is strictly concave on $\mathbb{R}^d_\geq$. Further let $F \subset \mathbb{R}^d$ be a compact convex body. Among all ellipsoids with a fixed center that are inscribed into F there exists a unique ellipsoid that is maximal with respect to f.*

Once we have realized that we can describe E_0 and E_1 by homogeneous matrices

$$N_i = \begin{pmatrix} -1 & 0^T \\ 0 & B_i \end{pmatrix}, \quad i = 0, 1 \tag{39}$$

the proof is quite similar to the proof of Theorem 5.

Remark 15. The uniqueness results of Theorems 5, 9, and 14 also hold if we look for extremal ellipsoids only among ellipsoids with prescribed axes. Theorems 5 and 9 remain true if the center is prescribed.

4. Non-uniqueness results

In this section we give two simple examples of size functions and convex sets such that the corresponding extremal ellipsoids are not unique. In view of our results, the size functions lack a convexity or concavity property. While non-uniqueness in both examples is rather obvious we feel the need to publish them since we are not aware of a single similar counter-example. Only [1] mentions the non-uniqueness of maximal inscribed circles. A trivial example is two congruent circles inscribed into their convex hull.

Minimal ellipsoids with non-convex size function

Denote by $F \subset \mathbb{R}^2$ the set of four points with coordinates $(\pm 1, \pm 1)$ and let f be the non-convex size function

$$f\colon \mathbb{R}^2_\geq \to \mathbb{R}_\geq,\ (a, b) \mapsto \max\{a, b\} + 16 \min\{a, b\}.$$

If the f-minimal ellipse to F was unique it must have four axis of symmetry and therefore it must be the circle C through the points of F. But the size of the two ellipses E_1 and E_2

$$E_1\colon \Big(\frac{32}{257}2^{\frac{1}{3}} - \frac{4}{257}2^{\frac{2}{3}} + \frac{1}{257}\Big)x^2 + \Big(-\frac{32}{257}2^{\frac{1}{3}} + \frac{4}{257}2^{\frac{2}{3}} + \frac{256}{257}\Big)y^2 - 1 \le 0$$
$$E_2\colon \Big(-\frac{32}{257}2^{\frac{1}{3}} + \frac{4}{257}2^{\frac{2}{3}} + \frac{256}{257}\Big)x^2 + \Big(\frac{32}{257}2^{\frac{1}{3}} - \frac{4}{257}2^{\frac{2}{3}} + \frac{1}{257}\Big)y^2 - 1 \le 0$$

is smaller than the size of the circle (compare Figure 1):

$$f(E_1) = f(E_2) \approx 19.9248 < f(C) \approx 24.0416$$

The ellipses E_1 and E_2 are the minimizers of f among all ellipses E_λ through the four points of F. Figure 1, right, displays the plot of the size function for all ellipses in the pencil of conics spanned by these points.

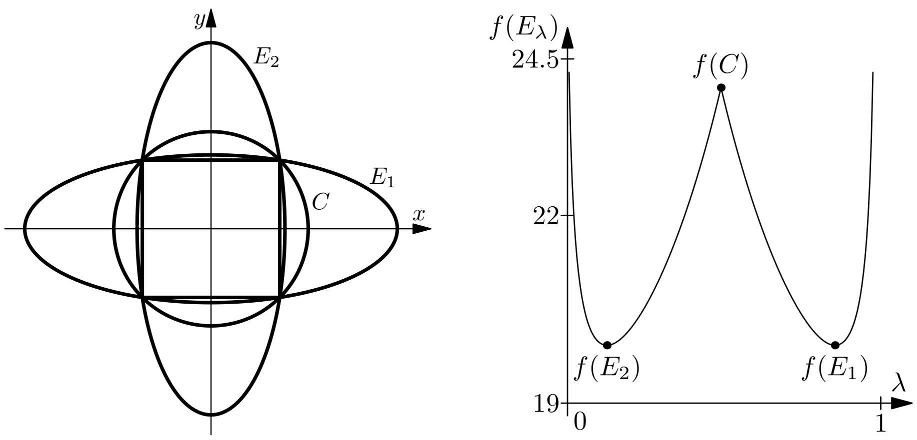

Figure 1. Non-unique minimal ellipsoids through the vertices of a square

Maximal ellipsoids with non-concave size function

Let $F \subset \mathbb{R}^2$ be the equilateral triangle with side length 1 (see Figure 2). The size function under consideration is the arc-length of an ellipse. We will demonstrate that the inscribed ellipse of maximal arc length is not unique. This is particularly interesting since the minimal arc-length enclosing ellipse is known to be unique, see [6],[8],[15].

The arc-length of an ellipse with semi-axis length a and b can be expressed in terms of the complete elliptic integral of first kind

$$f(a,b) = 4\max\{a,b\}E(1 - \frac{\min\{a,b\}}{\max\{a,b\}}) \quad \text{where} \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

If the maximal arc-length ellipse contained in F was unique it must share the triangle's symmetries. Therefore, it must be the in-circle C. But the arc-length of the ellipsoid E_s that degenerates to the triangle side on the x-axis is greater than that of the circle: $f(E_s) = 2 > \pi/\sqrt{3} = f(C)$, see Figure 2. This shows that the maximal arc-length ellipse inscribed into an equilateral triangle is not unique. The plot in Figure 2, right, depicts the size function of the drawn inscribed ellipses. The circle corresponds to the kink in the graph.

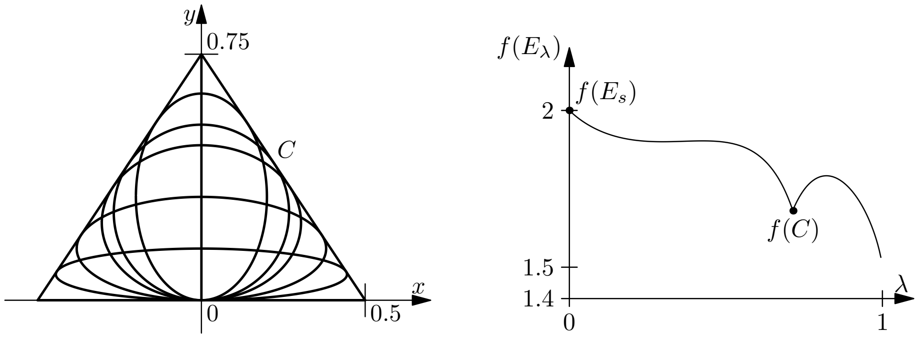

Figure 2. The arc-length of some ellipses inscribed into an equilateral triangle

5. Conclusion

We studied uniqueness results of minimal circumscribed and maximal inscribed ellipsoids. Uniqueness can be guaranteed if the function used for measuring the ellipsoid size satisfies a certain convexity or concavity condition. Summarizing our findings we can state that the minimal enclosing ellipsoid with respect to a size function f is unique if $f \circ w^p$ is convex for $p \in \{-1, -1/2\}$. The maximal inscribed ellipsoid is unique if $f \circ w^p$ is concave for $p = 1$ or for $p = 1/2$ if the center is prescribed. Uniqueness for $p = 1/2$ under general assumptions is still an open question.

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