An exposé on discrete Wiener chaos expansions

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Abstract

In this note we review and compare different versions of expansions in discrete Wiener chaos. We relate these expansions to the Rota-Wallstrom combinatorial approach to stochastic integration and to extended Haar systems. At the end we present some simple applications.

1 Preliminaries

Discrete stochastic calculus has seen a revival during the last decade or so, and several lines of work have been developed. We can begin tracing them with the very short note [M] by Meyer, in which he made use of ideas already proposed Kroeken in [K] to prove that for a Markov chain $(\{X_n\}_{n\leq N}, \Omega, \{\mathcal{F}_n\}_{n\leq N}, P)$ every $H \in L_2(\Omega, \mathcal{F}, P)$ has an expansion in an appropriate Wiener chaos associated to the chain. These ideas were very much extended by Privault and Schoutens in [PS]. From a seemingly unrelated direction comes the work by Holden, Lindstrøm, Øksendal and Ubøe see [HLOU-1] and [HLOU-2] in which the role of the Wick product is emphasized, and see as well the note by Leitz-Martini [L-M] who filled in the proofs in these papers and nicely filled in a gap in the relationship between discrete Brownian motion and discrete white noise.

From still another direction, comes the work by Feinsilver and Schott, see [F], [FS-1] and [FS-2] in which the relationship between discrete exponential martingales related to Bernoulli and other random flights and Krawtchouk polynomials is explored. Along these lines falls the note by Akahori [A]. And there is still another line of work, explored by Biane in [B], in which a chaotic representation for finite Markov chains is obtained. A last-but-not-least loose strand is the combinatorial approach to repeated integration by Rota and Wallstrom [RW].

Our aim here is to find an underlying thread through all of these papers. Since there are several different frameworks, we shall present results that capture the essence of those papers in such a way that comparisons and relationships can be made. This paper is organized as follows. In the remainder of this section the basic assumptions and notations are introduced. In sections 2 and 3 the basic discrete exponential martingale is explored from two points of view, and the expansion in Krawtchouk polynomials is used as a stepping stone for the expansions in discrete chaos. Section 4 is devoted to a discrete version of the work by Biane and section 5 is devoted to a very rapid review of the combinatorial approach to iterated integration of Rota and Wallstrom, which should eventually provide the common framework unifying the different expansions in discrete chaos. Section 6 is devoted to a version of the Clark-Ocone formula and in section 6 we show that the Walsh functions are an extended Haar system in the sense of Gundy. To finish we present a list of simple examples.

We shall consider a time homogeneous Markov chain $(\{X_n\}_{n\leq N}, \Omega, \{\mathcal{F}_n\}_{n\leq N}, P)$ with state space either a finite set $S \subset \mathbb{R}$ or a lattice on the real line. In the first case we shall assume that either $P(X_1 = s_j \,|\, X_0 = s_i) > 0$ for any i, j as in Meyer or either $P_{ij} = P(X_1 = s_j \,|\, X_0 = s_i) > 0$ only for $i \neq j$. Now and then we shall use the interval notation $[1, N] = \{1, \ldots, N\}$.

When the state space is a lattice in the real line, we shall assume that the chain is generated in the usual way from a sequence of i.i.d. finite random variables $\{\xi_n \,|\, n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ such that ξ_1 takes values in the set $\{0, \pm h, \pm 2h, \ldots, \pm Kh\}$ or in the set $\{\pm h, \pm 2h, \ldots, \pm Kh\}$. The corresponding chain is described by $X_n = X_0 + \sum_{k=1}^{n} \xi_k$. Also, throughout we shall consider a finite time horizon N. In this fashion our set up sits in between that used in [M], [PS], [FS-2] and [HLOU]. The essential assumption is that the cardinality of the set $\{s \in S \,|\, P(X_1 = s \,|X_0 = s') > 0\} = d$ is constant for all $s' \in S$. This simplifies the notation considerably.

A basic construct both in [M] and [PS] is contained in the simple

Lemma 1.1 *Let X_n be a Markov chain with state space S. For each $s \in S$ there is a family of polynomials $\phi_s^l : S \to \mathbb{R}$ for $l = 0, 1, \ldots, d-1$ such that*
i) The $\phi_s^l(s')$ are of degree l in $s' \in S$, and $\phi_s^0 \stackrel{\text{def}}{=} 1$.
ii) They are orthogonal with respect to $P_{ss'} = P(X_1 = s \,|\, X_0 = s')$, that is

$$\sum_t \phi_s^l(s')\phi_s^k(s')P_{ss'} = E[\phi_{X_0}^l(X_1)\phi_{X_0}^k(X_1)] = \delta_{lk}. \tag{1}$$

Proof. Is simple. For $s \in S$ consider $\{t^k \,|\, t \in S,\ t \neq s\}$. Since the Vandermonde matrix has a non-null determinant, these determine independent vectors in $\mathbb{R}^d$, and an application of the Gram-Schmidt procedure with respect to the scalar product $\langle x, y\rangle_s := \sum_{t\neq s} x_t y_t P_{st}$ yields the desired conclusion.

Comments. Note that the ϕ_s^l are orthogonal to $\phi_s^0 = 1$. This implies that the processes $\phi_{X_{k-1}}^l(X_k)$ are martingales.

The variation on this theme corresponding to the case considered by [FS-1], [HLOU-2] and [L-M] is even simpler, for now the ϕ_s^l are independent of s, which is due to the fact that $P(X_n = t \,|\, X_{n-1} = s) = P(\xi_n = t - s)$. The analogous result is

Lemma 1.2 *Let X_n be a Markov chain generated by an i.i.d. sequence $\{\xi_n\}$. There is a family of polynomials $\phi^l : S \to \mathbb{R}$ for $l = 0, 1, \ldots, d-1$ such that*
i) The $\phi^l(t)$ are of degree l in $t \in S$, and $\phi^0 \stackrel{\text{def}}{=} 1$.
ii) They are orthogonal with respect to $P(s) = P(\xi_1 = s)$, that is

$$\sum_t \phi^l(s)\phi^k(s)P(s) = E[\phi^l(\xi_1)\phi^k(\xi_1)] = \delta_{lk}. \tag{2}$$

Comment. The case considered by [FS-1] concerns only (the non-normalized) version of $\phi^1(s) = s - E\xi_1$. Similarly, [HLOU-2] and [L-M] consider $\{\xi_1 = \pm 1\}$ only.

We should also mention the elegant presentation by Emery in [E-1] and [E-2]. His setup is a bit more general, and his emphasis is on different issues. He considers a process $X = (\Omega, \mathcal{F}_{n\in\mathbb{Z}}, X_{n\in\mathbb{Z}}, P)$ (which he calls a "novation") such that

$$E[X_{n+1}\,|\,\mathcal{F}_n] = 0 \;\text{ and }\; E[X_{n+1}^2\,|\,\mathcal{F}_n] = 1$$

He explores, among other things, the connection between the predictable representation property and the chaotic representation property. An important difference is that for his purpose, he has to consider $\mathbb{Z}$ as time index set.

2 Exponential martingales and Krawtchouk-like polynomials

What comes up next mimics the usual procedure employed to relate the exponential martingales associated with Brownian motion to the Hermite polynomials. Here we shall follow the lead of Feinsilver who relates Krawtchouk polynomials to the discrete version of the exponential martingale. But it will not be until we introduce Wick products that we shall understand why the discrete exponential martingale is the same thing as the continuous exponential martingale.

Definition 2.1 *For $z = (z_1, \ldots, z_{d-1}) \in \mathbb{R}^{d-1}$, not all z_i vanishing simultaneously, and $\tau \in \mathbb{R}$ define*

$$\mathcal{E}_n(z,\tau) = \prod_{k=1}^{n}(1 + \tau Z_k) \tag{3}$$

where τ will play the role of a placeholder or accounting device and where

$$Z_k = \sum_{j=1}^{d-1} z_j \phi^j_{X_{k-1}}(X_k) \;\text{ or }\; Z_k = \sum_{j=1}^{d-1} z_j \phi^j(\xi_k) \tag{4}$$

The following is a simple consequence of property (ii) of the definition of the ϕ's.

Lemma 2.1 *The process $\{\mathcal{E}_n \mid 1 \leq n\}$ is a martingale with respect to each $P^s = P[\cdot \mid X_0 = s]$.*

Now, let us fix the time horizon at N and notice that

$$\mathcal{E}_N(z,\tau) = \sum_{n=0}^{N} \tau^n \sum_{1 \leq k_1 < \ldots < k_n \leq N} \prod_{i=1}^{n} Z_{k_i}$$

and here the account keeping role of τ becomes clear. We may drop it at any time. Clearly,

$$\prod_{i=1}^{n} Z_{k_i} = \prod_{i=1}^{n} (\sum_{j=1}^{d-1} z_j \phi^j_{X_{k_i-1}}(X_{k_i})) = \sum_{(j_1,\ldots,j_n)} \prod_{i=1}^{n} z_{j_i} \phi^{j_i}_{X_{i-1}}(X_{k_i})$$

and similarly for the i.i.d. case we have

$$\prod_{i=1}^{n} Z_{k_i} = \prod_{i=1}^{n} (\sum_{j=1}^{d-1} z_j \phi^j(\xi_{k_i})) = \sum_{(j_1,\ldots,j_n)} \prod_{i=1}^{n} z_{j_i} \phi^{j_i}(\xi_{k_i})$$

where in both cases $(j_1, \ldots, j_n) \in \{1, \ldots, d-1\}^n$, and in analogy with the expansion of the standard exponential martingale in terms on Hermite polynomials, we can write

$$\mathcal{E}_N(z,\tau) = \sum_{n=0}^{N} \tau^n H_n(z)$$

where either

$$\begin{aligned} H_n(z) &= \sum_{1 \leq k_1 < \ldots < k_n \leq N} \sum_{(j_1,\ldots,j_n)} \prod_{i=1}^{n} z_{j_i} \phi^{j_i}_{X_{k_i-1}}(X_{k_i}) \\ &\stackrel{\text{def}}{=} \sum_{1 \leq k_1 < \ldots < k_n \leq N} \sum_{(j_1,\ldots,j_n)} (\prod_{i=1}^{n} z_{j_i}) H(k_1, j_1; \ldots; k_n, j_n) \end{aligned}$$

and for the case of independent increments, the analogous expression is

$$\begin{aligned} H_n(z) &= \sum_{1 \leq k_1 < \ldots < k_n \leq N} \sum_{(j_1,\ldots,j_n)} \prod_{i=1}^{n} z_{j_i} \phi^{j_i}(\xi_{k_i}) \\ &\stackrel{\text{def}}{=} \sum_{1 \leq k_1 < \ldots < k_n \leq N} \sum_{(j_1,\ldots,j_n)} (\prod_{i=1}^{n} z_{j_i}) H(k_1, j_1; \ldots; k_n, j_n) \end{aligned}$$

We also have:

Lemma 2.2 *The family of polynomials*

$$\{\prod_{i=1}^{n} z_{j_i}\phi^{j_i}_{X_{k_i-1}}(X_{k_i})\,|\,1\le k_1<\ldots<k_n\le N\,;(j_1,\ldots,j_n)\in[1,d-1]^n,\ n\ge 1\}$$

and correspondingly, the family

$$\{\prod_{i=1}^{n} z_{j_i}\phi^{j_i}(\xi_{k_i})\,|\,1\le k_1<\ldots<k_n\le N\,;(j_1,\ldots,j_n)\in[1,d-1]^n,\ n\ge 1\}$$

is orthonormal and complete in $L_2(P^s)$ for any $s\in S$.

Proof Proceeds by induction on n based on the orthonormality properties of the ϕ^j_i (or correspondingly the ϕ^j). It all hinges on the fact that when the products are different, there will always be a factor of the type $\phi^j_{X_{k-1}}(X_k)$ for which

$$E^s[\phi^j_{X_{k-1}}(X_k)\,|\,\mathcal{F}_{k-1}] = E^{X_{k-1}}[\phi^j_{X_{k-1}}(X_1)] = 0$$

for any $1\le j\le d-1$. And clearly a similar argument holds for the i.i.d. case. *Comment.* When the time horizon is N, we can identify Ω with $\{\omega:[1,N]\to S\}$ and $L_2(P)$ with $\mathbb{R}^M$ with $M=d^N$ with a conveniently defined scalar product. What matters here is that there are $\sum_{n=1}^{N}\binom{N}{n}(d-1)^n$ polynomials $H(k_1,j_1;\ldots;k_n,j_n)$, and adding $\phi^0_s \stackrel{\text{def}}{=} 1$ to the list, we end up with $((d-1)+1)^N = d^N$ orthogonal polynomials. Thus they span $L_2(P)$. This is Meyer's argument.

Corollary 2.1 *With the notations introduced above, we have*

$$\mathcal{E}(z,\tau) = \sum_{n=0}^{N}\tau^n \sum_{1\le k_1<\ldots<k_n\le N}\ \sum_{(j_1,\ldots,j_n)} (\prod_{i=1}^{n} z_{j_i}) H(k_1,j_1;\ldots;k_n,j_n).$$

Comment. This is the analogue to the expansion of $\exp(\tau B(T)-\tau^2 T/2)$ in terms of Hermite polynomials. More about this correspondence in the next section. Now that we have identified a candidate for a basis, we can extend the previous lemma to

Proposition 2.1 *Let s be any point in S and $Z\in L_2(\Omega,\mathcal{F}_N,P^s)$. Then*

$$Z = E^s[Z] + \sum_{1\le k_1<\ldots<k_n\le N}\ \sum_{(j_1,\ldots,j_n)} E[ZH(k_1,j_1;\ldots;k_n,j_n)]H(k_1,j_1;\ldots;k_n,j_n) \tag{5}$$

is the expansion of Z in discrete X-chaos.

Comment. Clearly a similar result can be formulated for random variables in terms of direct sums of products of the type $L_2(\Omega_1, P_1) \otimes \ldots \otimes L_2(\Omega_m, P_m)$ corresponding to the physicists' multi-particle expansions.

To obtain the predictable representation property from the chaotic representation property, it suffices to have in mind the following

Lemma 2.3 *Let $Z \in L_2(\Omega, P^s)$, $s \in S$. If $Z \in \mathcal{F}_m$ for some fixed $m < N$, then*

$$Z = E^s[Z] + \sum_{1 \leq k_1 < \ldots < k_n \leq m} \sum_{(j_1, \ldots, j_n)} E[ZH(k_1, j_1; \ldots; k_n, j_n)] H(k_1, j_1; \ldots; k_n, j_n) \tag{6}$$

Proof It suffices to note that any $H(k_1, j_1; \ldots; k_n, j_n)$ such that for some k_i on we have $m < k_i < \ldots < k_n \leq N$, satisfies $E^s[H(k_1, j_1; \ldots; k_n, j_n) \mid \mathcal{F}_m] = 0$. This in turn follows from the definition of $H(k_1, j_1; \ldots; k_n, j_n)$ and lemmas (1.1)-(1.2), that is on the fact that, for example, for any $k > m$ we have $E^s[\phi^j_{X_{k-1}}(X_k) \mid \mathcal{F}_m] = 0$.

Corollary 2.2 *With the same notation as above, for any $1 \leq m \leq N$,*

$$E[Z \mid \mathcal{F}_m] = \sum_{1 \leq k_1 < \ldots < k_n \leq m} \sum_{(j_1, \ldots, j_n)} E[ZH(k_1, j_1; \ldots; k_n, j_n)] H(k_1, j_1, \ldots; k_n, j_n)$$

The following is now a simple consequence of the previous lemma. It asserts that the chaos representation property follows from the predictable representation property.

Proposition 2.2 *Let $\{M_n \mid 0 \leq n \leq N\}$ be an $(\Omega, \mathcal{F}_n, P^s)$-martingale, $s \in S$. Then there exist processes $\{Z^j_n \mid 1 \leq n \leq N, 1 \leq j \leq d-1\}$, which are previsible, (i.e. $Z^j_n \in \mathcal{F}_{n-1}$) such that*

$$M_n = M_{n-1} + \sum_j Z^j_n \phi^j_{X_{n-1}}(X_n)$$

or the corresponding version for the process with i.i.d. increments

$$M_n = M_{n-1} + \sum_j Z^j_n \phi^j(\xi_n).$$

Proof It consists in isolating the last term in (6) and appropriately identifying the Z^j_n's.

Let us now examine an interesting property of the martingales associated to the polynomials $\phi^j_y(x)$. Let us put

$$Z^j_0 = 0, \quad \text{and} \quad Z^j_n = \sum_{i=1}^{n} \phi^j_{X_{i-1}}(X_i).$$

An elementary computation confirms

Lemma 2.4 *The* $\{\{Z^j_n \,|\, 0 \leq n \leq N\} \,|\, 1 \leq d-1\}$ *is a collection of mutually orthogonal martingales. That is, for different* i, j *,* $\{Z^i_n Z^j_n \,|\, 1 \leq n \leq N\}$ *is a* P^s*-martingale for each* s*.*

But more interesting is

Proposition 2.3 *The collection* $\{\{Z^j_n \,|\, 0 \leq n \leq N\} \,|\, 1 \leq j \leq d-1\}$ *is a linearly independent collection.*

Proof It suffices to verify that if a_j are constants, then if for any $n \geq 1$ the sum $\sum_{j=1}^{d-1} a_j \Delta Z^j_n = 0$, then the coefficients a_j all vanish. But this is a trivial consequence of the orthogonality of the polynomials with respect to $P^s[\,\cdot\,|\,\mathcal{F}_{n-1}]$.

This means that the orthogonal projections onto the linear subspace of $L_2(\Omega, \mathcal{F}, P^s)$

$$\mathcal{M} := \{\sum_{k=1}^{n}\sum_{j=1}^{d-1} a^j_k \Delta Z^j_k \,|\, \{\{a^j_k \,|\, 1 \leq k \leq N\} \,|\, \text{predictable } 1 \leq j \leq d-1\};\ n \leq N\}$$

are unique. This and proposition 2.2 round up the issues associated with the representation problem. Let us close this section by briefly particularizing the above results.

The simplest case corresponds to the case of simple random walks. This is the case considered in [HLOU-2] and [L-M]. We have $X_n = X_0 + \sum_1^n \xi_k$ with $P(\xi_k = \pm 1) = 1/2$. The orthogonal basis associated to P has two vectors: $\phi^0(s) = 1$ and $\phi^1(s) = s$ and the discrete exponential martingale is $\mathcal{E}(z) = \prod_1^n (1 + z\xi_k)$. This can be rewritten as

$$\mathcal{E}_N(z) = (1+z)^{\sum \xi_k}(1-z)^{N-\sum \xi_k} = \sum_{n=0}^{N} z^n K_n(\sum \xi_k)$$

which is the usual expansion of $\mathcal{E}_N(z)$ in terms of Krawtchouk polynomials.

The other simple chain process with i.i.d. increments corresponds to taking $\xi_k \in \{0, \pm h, \ldots, \pm hK\}$ with probabilities $p_0 = P(\xi_1 = 0), \ldots, p_K = P(\xi_1 = \pm hK)$. (Certainly this is not mandatory, but it makes the state space of $X_n = X_0 + \sum \xi_k$ a lattice in $\mathbb{R}$). Also, if $\phi^j(s)$ are the polynomials described in section 1, clearly $\phi^1(s) = s/\sigma$, where $\sigma^2 = \mathrm{Var}(\xi)$. To relate to the results in [FS1] we consider $\mathcal{E}_n(z) = \prod_1^n (1 + z\phi^1(\xi_k))$. Introducing (the number of times each step is taken) $n(\pm hk)=\sum_{j=1}^{N} I_{\{\pm hk\}}(\xi_j)$, then

$$\mathcal{E}(z)=\prod_{j=1}^{K}(1+\frac{z}{\sigma}hj)^{n(jh)}(1-\frac{z}{\sigma}hj)^{n(-jh)}=\sum_{n=0}^{N}(\frac{z}{\sigma})^n K_n(n(\pm 1),\ldots,n(\pm hk))$$

where the polynomials K_n are now related to Lauricella polynomials. See [FS] for more about this. An interesting related question also considered by Feinsilver, see [F] for instance: When are the polynomials K_n iterated integrals? The answer is: they are iterated integrals only when the variables ξ_n are of Bernoulli type. This fact relates to similar results obtained in [E2] and [PS].

3 Wick products, exponential martingales, and repeated integrals

Let us now examine in what sense the $\mathcal{E}(z,\tau)$ introduced in the previous section is an exponential martingale. What comes up next is an amplification of a side remark in [HLOU-2] that shows part of the relevance of Wick products in probability theory, in particular that the Wick exponential $e^{\diamondsuit Z}$ of a Gaussian variable Z is given by $e^{Z-||Z||^2/2}$.

In order to easily relate the results of different authors, we shall consider the case of processes with i.i.d. increments and begin with

Definition 3.1 *Let ϕ_s^j be the polynomials defined in Lemma 1.2. Let $\{k_i,\ j_i\,|\,1\le i\le n$ and $1\le j_i\le d-1\}$, and $\{l_i,\ j_i\,|\,1\le i\le m$ and $1\le j_i\le d-1\}$ be two sets of indices. We assume that the first indices are all different whereas there may be repetitions among the second indices. The Wick product of $\prod_{i=1}^n \phi^{j_i}(\xi_{k_i})$ and $\prod_{t=1}^m \phi^{j_t}(\xi_{l_t})$ is defined by*

$$\prod_{i=1}^{n}\phi^{j_i}(\xi_{k_i})\diamondsuit\prod_{t=1}^{m}\phi^{j_t}(\xi_{l_t})=\prod_{i=1}^{n}\phi^{j_i}(\xi_{k_i})\prod_{t=1}^{m}\phi^{j_t}(\xi_{l_t}) \tag{7}$$

whenever $\{k_1,\ldots,k_n\}\cap\{l_1,\ldots,l_m\}=\emptyset$ or equal to 0 otherwise, independently of the superscripts.

Comment. Note that with this definition, for example, $\prod_{i=1}^n \phi^{j_i}(\xi_{k_i}) = \phi^{j_1}(\xi_{k_1})\diamondsuit\ldots\diamondsuit\phi^{j_n}(\xi_{k_n})$.

Proposition 3.1 *Consider the sequence $\{c_{n,j}\,|\,n\ge 1,\ 1\le j\le d-1\}$ such that $\sum_{n,j}|c_{n,j}|^2<\infty$. Define the process Z by setting $Z_0=0$ and $Z_n=\sum_{m\le n}\sum_{j=1}^{d-1}c_{m,j}\phi^j(\xi_m)$. Also, write $Z_n=\sum_{k=1}^{n}\Delta Z_k$ where $\Delta Z_k=\sum_{j=1}^{d-1}c_{k,j}\phi^j(\xi_k)$. Then* *i)* $\Delta Z_k \diamondsuit \Delta Z_l = 0$ *whenever* $k = l$.

ii) For every $n \geq 1$ *and* $1 \leq N$, $Z_n^{\diamondsuit N} = 0$ *whenever* $N > n$, *and when* $N \leq n$ *the recurrence* $Z_n^{\diamondsuit N} = N \sum_{k=1}^{n} Z_{k-1}^{\diamondsuit (N-1)} \Delta Z_k$ *holds.*

iii) $\exp^{\diamondsuit}(Z_n) \stackrel{\text{def}}{=} \sum_{N=0}^{\infty} \frac{Z_n^{\diamondsuit N}}{N!} = \prod_{k=1}^{n}(1 + \Delta Z_k)$

Comment. It now appears that the name "exponential martingale" assigned to equation (3) is fully justified from two different points of view.

Proof (i) The identity $\Delta Z_k \diamondsuit \Delta Z_l = 0$ whenever $k = l$ follows from the definition of Wick product. If we rewrite it as $\Delta Z_k \diamondsuit \Delta Z_l = \Delta Z_k \Delta Z_l (1 - \delta_{kl})$ we readily obtain

$$Z_n^{\diamondsuit 2} = 2 \sum_{k=1}^{n} (\sum_{j=1}^{k-1} \Delta Z_j) \Delta Z_k = \sum_{k=1}^{n} Z_{k-1} \Delta Z_k$$

.

This remark is just the first inductive step in the proof of (ii) by induction on N. By definition

$$Z_n^{\diamondsuit N} = \sum_{\{k_1, \ldots, k_N\}}{}^{*} \prod_{i=1}^{N} \Delta Z_{k_i}$$

where Σ^* denotes the sum over the collection of all subsets of $\{1, \ldots, n\}$ with N elements. Notice that when $N > n$ this collection is empty, and therefore we set the sum equal to 0. Re-ordering the factors of each term in the sum, we obtain

$$Z_n^{\diamondsuit N} = N! \sum_{k_1 < \ldots < k_N} \prod_{i=1}^{N} \Delta Z_{k_i}$$

or equivalently

$$Z_n^{\diamondsuit N} = N \sum_{k=1}^{n} Z_{k-1}^{\diamondsuit (N-1)} \Delta Z_k$$

and the reader should not forget that actually the summation has a smaller range due to the fact that $Z_n^{\diamondsuit N} = 0$ whenever $N > n$.

To obtain (iii) is now just a computation. Let us set $M_n = \exp^{\diamondsuit}(Z_n)$. Thus

$$M_n = 1 + \sum_{N=1}^{\infty} \frac{Z_n^{\diamondsuit N}}{N!} = 1 + \sum_{N=1}^{\infty} \frac{1}{(N-1)!} \sum_{j=1}^{n} Z_{j-1}^{\diamondsuit (N-1)} \Delta Z_j$$

which after interchanging the summations becomes $\sum_{j=1}^{n} \exp^{\diamondsuit}(Z_{(j-1)}) \Delta Z_j$. Clearly this implies that $M_n = M_{n-1} + M_{n-1}\Delta Z_n$ which after a simple iteration becomes $M_n = \exp^{\diamond}(Z_n) = \prod_{j=1}^{n}(1+\Delta Z_j)$.

4 Martingales associated with jumps of discrete Markov chains

Let us assume as in section 1 that the state space is a lattice in the real line and that the increments of the chain are in $S = \{0, \pm h, \ldots \pm Kh\}$. What really matters for us now is that the cardinality of $\{s \in S \,|\, P(X_1 = s \,|\, X_0 = s') > 0\} = d$ for every $s' \in S$. If we set $T_1 = \min\{n : \; X_n \neq X_0\}$ then the following are well known results

$$P^s(T_1 = k) = p_{ss}^{k-1}(1 - p_{ss}), \;\; P^s(T_1 > 1) = p_{ss}$$

and

$$P^s(T_1 > n + k \,|\, T_1 > k) = P^s(T_1 > n).$$

For $a \neq 0$ consider the following compensated sum of jumps

$$J_n^a = \sum_{k=1}^{n}\{I_{\{X_n = X_{n-1}+a\}} - p(X_{n-1}, X_{n-1}+a)\}, \;\; J_0^a = 0$$

where for notational convenience we use $p(s, s')$ instead of $p_{s,s'}$.
Since for any $s \in S$, $E^s[(I_{\{X_n = X_{n-1}+a\}} - p(X_{n-1}, X_{n-1}+a)) | \mathcal{F}_{n-1}] = 0$, clearly $\{J_n^a \,|\, 1 \leq n \leq N\}$ is a P^s-martingale for every a. But since we are in discrete time we have for $s \in S$

$$E^s[(\Delta J_n^a)^2 \,|\, \mathcal{F}_{n-1}] = p(X_{n-1}, X_{n-1}+a)(1 - p(X_{n-1}, X_{n-1}+a))\sigma^2(X_{n-1}, a).$$

Therefore, if we set

$$M_0^a = 0 \;\text{ and }\; M_n^a = \sum_{k=1}^{n}\sigma^{-1}(X_{n-1}, a)\Delta J_n^a$$

then $\{M_n^a \,|\, 1 \leq n \leq N\}$ is a P^s-martingale for any $s \in S$ such that
$E^s[(\Delta M_n^a)^2 \,|\, \mathcal{F}_{n-1}] = 1$. Regretfully in discrete time the M^a are not orthogonal, but nevertheless we have

Proposition 4.1 *With the notations introduced above, for every $a \in S$, the collection*
$\{J_n^a \,|\, a \neq 0, \; 1 \leq n \leq N\}$ *is linearly independent.*

Proof It suffices to verify that if the real numbers $\{\alpha_a \,|\, a \neq 0\}$ are such that $\sum_{a\neq 0} \alpha_a \Delta J^a_n = 0$ then $\alpha_a = 0$ for each $a \neq 0$. Think of α_a as the values of a function $A(X_n)$ (just set $A(X_n) = \alpha_a$ on $\{X_n = X_{n-1} + a\}$) and rewrite the last identity as $A(X_n) - E^s[A(X_n) \,|\, \mathcal{F}_{n-1}] = 0$. This means that the random variable $A(X_n)$ is constant $= \alpha_0$. To find the constant note that

$$\alpha_0 = \alpha_0 \sum_{a\neq 0} P^s[X_n = X_{n-1} + a \,|\, \mathcal{F}_{n-1}] = \alpha_0(1 - p(X_{n-1}, X_{n-1}))$$

which, due to our assumption, implies that $\alpha_0 = 0$.
Comment. This shows that when $p(s,s) = 0$ for all s, then $\sum_{a\neq 0} \Delta J^a_n = I_{\{X_n = X_{n-1}\}} = 0$, that is the martingales J^a are not linearly independent.

Let us now examine some discrete time stochastic difference equations satisfied by X. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded Borel-measurable function (but we shall only need the values of g at a finite set of points), then observe that

$$\begin{aligned} g(X_n) - g(X_{n-1}) &= g(X_n) - g(X_{n-1}) - E^s[g(X_n) - g(X_{n-1}) \,|\, \mathcal{F}_{n-1}] \\ &\quad + E^s[g(X_n) - g(X_{n-1}) \,|\, \mathcal{F}_{n-1}] \\ &= \sum_{a\neq 0} L_a g(X_{n-1}) \Delta M^a_n + Ag(X_{n-1}) \end{aligned}$$

where the new symbols are defined by

$$L_a g(x) = \sigma(x,a)(g(x+a) - g(x)) \quad \text{and} \quad Ag(x) = \sum_{a\neq 0} p(x, x+a)(g(x+a) - g(x)).$$

Even though the operator A coincides with that considered by Biane, the operator L_a does not. This is because we work in discrete time.

Consider now, for every $n \leq N$, the process $\{Y_k = P_{n-k}(X_k) \,|\, k \leq n\}$. Use the Markov property to rewrite it as $Y_k = E^{X_k}[g(X_{n-k})] = E^s[g(X_n) \,|\, \mathcal{F}_k]$ which specifies P as an operator and clearly shows that Y is a P^s-martingale for every s, and certainly $Y_n - Y_0 = \sum_{k=1}(Y_k - Y_{k-1})$. Note now that for any $1 \leq k \leq n$,

$$\begin{aligned} Y_k - Y_{k-1} &= P_{n-k}g(X_k) - P(P_{n-k}g)(X_{k-1}) \\ &= P_{n-k}g(X_k) - P_{n-k}g(X_{k-1}) - \\ &\quad \{P(P_{n-k}g)(X_{k-1}) - P_{n-k}g(X_{k-1})\} \\ &= \sum_{a\neq 0} (P_{n-k}g)(X_{k-1}) \Delta J^a_k \\ &= \sum_{a\neq 0} L_a(P_{n-k}g)(X_{k-1}) \Delta M^a_n \end{aligned}$$

from which we finally obtain

$$Y_n = g(X_n) = P_n(X_0) + \sum_{a\neq 0}\sum_{k=1}^{n} L_a(P_{n-k}g)(X_{k-1})\Delta M_n^a.$$

This is the basic relationship from which Biane obtains the chaos representation property for M. To relate this to what we did in section 2, we should express the M^a 's in terms of the Z^a's, but we shall not pursue this here.

5 Rota-Wallstrom combinatorial approach to stochastic integration

Even though Rota and Wallstrom do not bother about discrete chaos in [RW], that is, they do not verify which of their examples of iterated stochastic integrals provide a basis for the corresponding $L_2(\Omega, \mathcal{F}, P)$, they do present a very powerful and general theory of iterated stochastic integrals. Here we shall recall a few of their results worded as closely as possible to our setup.

We have considered iterated integrals in a very simple set: $[1, N] = \{1, \ldots, N\}$ whereas they consider iterated integrals in very general sets (S, Σ). To consider k-iterated integrals begin with the set $[1, N]^{[1,k]}$ on which the obvious σ-algebra is placed. What before we called ΔZ_k we shall now denote by ζ_k, that is, we will be given a process $\{\zeta_k \,|\, 1 \leq k \leq N\}$, which we use to define a measure on $[1, N]$ by setting $\zeta(A) = \sum_{k\in A} \zeta_k$. This certainly is both a stochastic integral on $[1, N]$ and a random measure (on $\mathcal{P}([1, N])$).

We also need to recall that to every $A \subset [1, N]^{[1,k]}$ and to every partition π of $[1, k]$ we have

$$\begin{aligned} A_{\geq\pi} &\stackrel{\text{def}}{=} \{(n_1, \ldots, n_k) \,|\, n_i = n_j \text{ if } i \sim_\pi j\} \\ A_\pi &\stackrel{\text{def}}{=} \{(n_1, \ldots, n_k) \,|\, n_i = n_j \text{ if and only if } i \sim_\pi j\} \end{aligned}$$

Now [RW] define the following two measures on $[1,\ N]^{[1,k]}$:

$$\begin{aligned} \zeta_\pi^k(A) &\stackrel{\text{def}}{=} (\otimes\zeta_{i\leq k})(A_{\geq\pi}) \\ \mathrm{St}_\pi^k(A) &\stackrel{\text{def}}{=} (\otimes\zeta_{i\leq k})(A_\pi) \end{aligned}$$

which are related according to the fundamental (Proposition 1 in [RW])

Proposition 5.1 *The measures ζ_π^k and St_π^k are related as follows*

$$\zeta_\pi^k(A) = \sum_{(\sigma\geq\pi)} \mathrm{St}_\pi^k(A_{\geq\pi})$$

$$\mathrm{St}_\pi^k(A) = \sum_{\sigma\geq\pi} \mu(\pi, \sigma)\zeta_\sigma^k.$$

Here $\sigma \geq \pi$ for two partitions means that every block of π is contained in some block of σ (or π is finer than σ) and μ denotes the Möbius function of the lattice of partitions of $[1, k]$. (The reader can check with Rota's [R] for the basics about Möbius functions.) When $\pi = \hat{0}$ is the finest partition (all of its blocks are the one point subsets of $[1, k]$), we have

Theorem 5.1 *With the notations introduced above*

$$\mathrm{St}_{\hat{0}}^k(A) = \sum_{\sigma \geq \pi} \mu(\hat{0}, \sigma)\zeta_\sigma^k.$$

Comments. The measure $\mathrm{St}_{\hat{0}}^k$ is called the stochastic measure of degree k, or the stochastic measure associated to $\{\zeta_k\}$. If $f : [1, N]^{[1,k]} \to \mathbb{R}$ is any function

$$\int f(n_1, \ldots, n_k) \, \mathrm{St}_{\hat{0}}^k(n_1, \ldots, n_k)$$

is said to be the k-th iterated integral of f with respect to $\mathrm{St}_{\hat{0}}^k$. This measure is easily seen to be symmetric. That is, if $\alpha : [1, k] \to [1, k]$ is any permutation of $[1, k]$, and we set

$$\alpha(A) \stackrel{\text{def}}{=} \{(n_{\alpha(1)}, \ldots, n_{\alpha(k)}) \,|\, (n_1, \ldots, n_k) \in A\}$$

then $\mathrm{St}_{\hat{0}}^k(\alpha(A)) = \mathrm{St}_{\hat{0}}^k(A)$.

Comments. Consider a finite set $S = \{s_1, \ldots, s_d\}$ of cardinality d, and let $\{\psi_s \,|\, s \in S\}$ be any collection of random variables indexed by S. Define a random measure on S by $\zeta(A) = \sum_{s \in A} \psi_s$. Then

$$\mathrm{St}_{\hat{0}}^k(A^k) = \sum_{\{s_1, \ldots, s_k\}} \prod_{j=1}^k \psi_{s_j}$$

the summation being over all subsets of S of size k, is the k-th iterated integral of ζ. Notice that for $k > d$, the integral should be set equal to zero, for there are no subsets of S of cardinality larger than d.

To understand $\mathrm{St}_{\hat{0}}^k$ better, we recall that a set $A_1 \times \ldots \times A_k$ is a triangular set whenever the A_i's are pairwise disjoint sets. For any (and only for) triangular set $A_1 \times \ldots \times A_k$, we put

$$\mathrm{perm}(A_1, \ldots, A_k) = \bigcup_\alpha \alpha(A_1 \times \ldots \times A_k)$$

where the union turns out to be disjoint and obviously $\mathrm{perm}(A_1, \ldots, A_k)$ is symmetric.

Actually, these sets generate the σ-algebra of symmetric subsets of $[1, N]^{[1,k]}$. We have (Theorem 5 in [RW])

Theorem 5.2 *Given a measure associated to $\{\zeta_k\}$ on $[1,N]$ there is one and only one measure* $\mathrm{St}_{\hat{0}}^k$ *on the symmetric sets of* $[1,N]^{[1,k]}$ *such that*

$$\mathrm{St}_{\hat{0}}^k(\mathrm{perm}(A_1,\ldots,A_k)) = k!\,\zeta(A_1)\ldots\zeta(A_k).$$

Two other set functions associated to $\{\zeta_k\}$ and St_π^k introduced in [RW] are

Definition 5.1 *For every positive integer k, the k-th diagonal measure Δ_k associated to ζ, is the measure on $[1,N]$ given by*

$$\Delta_k(A) \stackrel{\text{def}}{=} \zeta_{\hat{1}}^k(A^{[1,k]}) = \mathrm{St}_{\hat{1}}^k(A^{[1,k]})$$

where $\hat{1}$ is the partition of $[1,k]$ having only one block.

Comment. It is clear that the Δ_k are measures.

Definition 5.2 *With the notations introduced above, define the stochastic sequence of binomial type ψ_k to be the sequence of random set functions*

$$\psi_k(A) \stackrel{\text{def}}{=} \mathrm{St}_{\hat{0}}^k(A^{[1,k]}).$$

We complete this short tour through [RW] with two interesting results.

Theorem 5.3 *Let ψ_k be the sequence of binomial type associated to the process $\{\zeta_k\}$. If A and B are disjoint subsets of $[1,N]$, then*

$$\psi_k(A\cup B) = \sum_{j=1}^{k} \binom{k}{j} \psi_j(A)\psi_{k-j}(B).$$

Theorem 5.4 *With the notations introduced above, we have, for $A \subset [1,N]$,*

$$\psi_n(A) = \sum_{k\geq 1} (-1)^{(k-1)} \Delta_k(A)\psi_{n-k}(A).$$

A point of connection wth the results of section 2 appears in Example K of [RW]. Consider now the random measure on $[1,N]$ associated to $\{\zeta_k - E[\zeta_1]\}$. If we put $\zeta(A) = \sum_{k\in A}(\zeta_k - e[\zeta_1])$ then $\psi_n(A) = K_n(A)$, the Krawtchouk polynomials mentioned in section 2. An easy way to see this was provided there as well. Consider

$$\prod_{k\in A}(1+\tau(\zeta_k - e[\zeta_1])) = \sum_{n=0}^{|A|}\tau^n \sum_{\{i_j<\ldots<i_n\}} \prod_{j=1}^{n}(\zeta_{i_j} - e[\zeta_1]) = \sum_{n=0}^{|A|}\tau^n\, K_n(\zeta - E[\zeta])$$

which displays $\psi_n(A)$ explicitly.

Comment. The $\psi_n(A)$ are, except for an $n!$ factor, the Wick products of the $\phi(A) = \psi_1(A)$.

6 Gradient, divergence and the Clark-Ocone formula: a variation on the theme

In this section we provide variations on the theme of discrete stochastic calculus. Something in between the presentations of [L-M] and [PS], but before that, let us settle some notational issues. We introduced in section 2 the basis vectors $H(k_1, j_1; \ldots; k_n, j_n)$ for $L_2(\Omega, \mathcal{F}P)$, where $A = \{k_1 < \ldots < k_n\}$ denotes some non-empty subset of $[1, N]$ and $(j_1, \ldots, j_d) \in [1, d-1]^n$. Throughout this section we shall shorten this to $H(A, \mathbf{j}_A)$. We shall have to take out or add an element k from or to A, that is to form $A-\{k\}$ or $A \cup \{k\}$ which we simply denote $A-k$ and $A \cup k$. In these cases $\mathbf{j}_{A-k}$ or $(\mathbf{j}_{A-k}, i)$ will denote the tuple with an entry deleted or added at the obvious position. Let $e_k : [1, N] \to \{0, 1\}$ denote the basis vectors for the set $V \stackrel{\text{def}}{=} \{f : [1, N] \to \mathbb{R}\} \simeq \mathbb{R}^N$, by c the counting measure on $[1, N]$ and by $e_k \otimes H(A, \mathbf{j}_A)$ the standard basis for $L_2([1, N] \times \Omega, c \otimes P)$. With all these notations we state

Definition 6.1 *For $1 \leq k \leq N$, $1 \leq j \leq d-1$, define $D_{k,j} : L_2(\Omega, P) \to L_2([1, N] \times \Omega, c \otimes P)$ by its action on basis vectors $H(A, \mathbf{j}_A)$*

$$D_{k,j}(H(A, \mathbf{j}_A)) = \chi_{\{k \in A\}} \chi_{\{j_k = j\}}\, e_k \otimes H(A-k, \mathbf{j}_{A-k}), \quad \textit{and} \quad D_{k,j}(1) = 0 \ .$$

With this we define the mapping $D\colon L_2(\Omega, P) \to L_2([1, N] \times \Omega, c \otimes P)$ by linearity, that is, if $X = E[X] + \sum_{A, \mathbf{j}_A} X(A, \mathbf{j}_A) H(A, \mathbf{j}_A)$, then

$$D(X) = \sum_{k,j} \sum_{A, \mathbf{j}_A} X(A, \mathbf{j}_A) D_{k,j}(H(A, \mathbf{j}_A)) \ .$$

In the proof of Proposition 6.1 below we shall need the following computation expressing the action of $D_{k,j}$ on the coefficients of the expansion of X:

$$\begin{aligned} D_{k,j}(X) &= \sum_{A, \mathbf{j}_A} X(A, \mathbf{j}_A) \chi_{\{k \in A\}} \chi_{\{j_k = j\}} H(A-k, \mathbf{j}_{A-k}) \\ &= \sum_{A \cap \{k\} = \emptyset} \sum_{\mathbf{j}_A} X(A \cup k, (\mathbf{j}_A, j)) H(A, \mathbf{j}_A). \end{aligned}$$

Comment. Formally speaking, this (and in the proof of Proposition 6.1) should be considered as the projection of $e_k \otimes L_2(\Omega) \subset L_2([1, N] \times \Omega)$ onto $L_2(\Omega, P)$. But adding more precision would make the statement of Proposition 6.1 needlessly cumbersome.

In our finite, discrete time setting there is no problem in identifying $V \in L_2([1, N] \times \Omega, c \otimes P)$ with a sequence $\{V_k \in L_2(\Omega, P) \mid k \in [1, N]\}$, and if V, W are two such vectors, their scalar product is $(V, W)_{L_2([1,N] \times \Omega)} = \sum_k (V_k, W_k)_{L_2(\Omega)}$. If we set

Definition 6.2 *Define the divergence* $\delta : L_2([1,N]\times\Omega, c\otimes P)\to L_2(\Omega)$ *by its action on the basis elements by*

$$\delta(e_k\otimes H(A,\mathbf{j}_A)) = \chi_{\{k\notin A\}}\sum_{i=1}^{d-1} H(A\cup k,(\mathbf{j}_A,i))\ .$$

With these definitions and notations, we leave for the reader to verify that

Proposition 6.1 *The operators* D *and* δ *are adjoint to each other, that is for any* $X\in L_2(\Omega)$ *and* $V\in L_2([1,N]\times\Omega)$ *we have*

$$(D(X),W)_{L_2([1,N]\times\Omega)} = (X,\delta(W))_{L_2(\Omega)}.$$

But a more interesting result is the following version of the Clark-Ocone identity:

Proposition 6.2 *Let* $X\in L_2(\Omega)$. *Then*

$$X = E[X] + \sum_{k,j} E[D_{k,j}(X)|\mathcal{F}_k]\phi_k^j$$

where by ϕ_k^j *we mean either* $\phi^j_{X_{k-1}}(X_k)$ *or* $\phi_k(\xi_k)$.

Proof Proceeds very much along the lines of the same proof in [L-M]. Notice that according to Corollary 2.2

$$E[D_{k,j}(X)\,|\,\mathcal{F}_k] = \sum_{A\subset[1,k)}\sum_{\mathbf{j}_A} X(A\cup k,(\mathbf{j}_A,j))H(A,\mathbf{j}_A)$$

therefore,

$$\begin{aligned}\sum_{k,j} E[D_{k,j}(X)\,|\,\mathcal{F}_k]\phi_k^j &= \sum_{A\subset[1,k)}\sum_{\mathbf{j}_A} X(A\cup k,(\mathbf{j}_A,j))H(A\cup k,(\mathbf{j}_A,j))\\ &= \sum_{A,\mathbf{j}_A} X(A\cup k,(\mathbf{j}_A,j))H(A\cup k,(\mathbf{j}_A,j)) = X - E[X].\end{aligned}$$

To finish we direct the reader to still another version of the Clark-Ocone formula that appeared in Lindstrøm's [L].

7 Extended H-systems

After Proposition 2.3 we mentioned the particular case in which the underlying i.i.d collection consists of random variables ξ_k taking values in $\{-1,1\}$ with equal probability. This collection generates the orthonormal set $\{H(A) =$

$\prod_{k \in A} \xi_k \,|\, A \subset [1, N]\}$, which according to Proposition 2.2 is a basis for $L_2(\Omega,$ $\mathcal{F}$, $P)$. That is, for any random variable X we have

$$X = \sum_{A \subset [1,N]} X(A)H(A) \quad \text{with} \quad X(A) = E[AH(A)]. \tag{8}$$

In this section we present a variation on the theme of Gundy's ([Gu]) and we shall see that the Walsh functions $H(A)$ determine a Haar system. Let us begin with

Definition 7.1 *An orthonormal system* $\{H(A) \,|\, A \subset [1, N]\}$ *is an extended* H*-system whenever*

i) each $H(A)$ *takes at most two values with positive probability,*
ii) the σ*-algebra* $\mathcal{F}_A := \{H(B) \,|\, B \subset A\}$ *consists of exactly* $2^{|A|}$ *atoms*
iii) $E[H(C) \,|\, \mathcal{F}_A] = 0$ *for* $C \neq A$.

Definition 7.2 *The complete orthonormal system* $\{H(A) \,|\, A \subset [1, N]\}$ *is an extended* H*-system whenever for any* $X \in L_2(\Omega, \mathcal{F}, P)$ *and any* $A \subset [1, N]$ *we have*

$$E[X \,|\, \mathcal{F}_A] = \sum_B X(B)H(B)$$

where the $X(B)$ *are the Fourier coefficients of* X *appearing in (8).*

Proposition 7.1 *With the notations introduced above, we have* $\{H(A) \,|\, A \subset [1, N]\}$ *is an extended* H*-system according to Definition 7.1 if and only if it is an extended* H*-system according to Definition 7.2.*

Proof We repeat Gundy's proof almost verbatim. Assume $\{H(A)\}$ is an extended H-system according to the first definition, that X is a random variable and $A \subset [1, N]$. Then $E[X \,|\, \mathcal{F}] = \sum_i x_i I_{B_i}$ where B_i are the $2^{|A|}$ blocks generating $\mathcal{F}_A$. Since $\mathcal{V}_A := \{\sum_{B \subset A} c(B)H(B)\}$ is a $2^{|A|}$ dimensional vector space, $E[X \,|\, \mathcal{F}_A]$ must coincide with its orthogonal projection on that space, and therefore Definition 7.2 is satisfied.

Let us assume now that $\{H(A)\}$ is an extended H-system according to Definition 7.2. For $A = \emptyset$ we agree to set $H(\emptyset) = 1$ and then $E[X \,|\, \mathcal{F}_\emptyset] = E[X]H(\emptyset)$ and $\mathcal{F}_\emptyset$ is generated by one block. Thus (i)-(iii) hold in this case. Assume that we have verified the conditions for any A with $|A| \leq n$. Let $|A| = n$ and $a \notin A$ and put $A' = A \cup \{a\}$. We know that $0 = E[H(A')\,\mathcal{F}_A] = \sum b_B(A')H(B)$, or $b_B(A') = 0$ by hypothesis. Therefore, $E[H(A')H(B)] = 0$ for any block B that generates $\mathcal{F}_A$, and thus $P(\{H(A)' > 0\} \cap B) > 0$ and $P(\{H(A)' < 0\} \cap B) > 0$. Since the vector space $\mathcal{V}'_A = \{\sum_{B \subset A'} c(B)H(B)\}$ is $2^{|A|+1}$-dimensional, the orthogonal projection $E[X\,\mathcal{F}_{A'}]$ must coincide with the orthogonal projection on $\mathcal{V}_{A'}$, or (ii) and (iii) of Definition 7.1 hold. This, and what we observed before, means that every block of $\mathcal{F}_A$ splits into two blocks, on which $H(A')$ must be constant. Thus (i) holds as well.

Definition 7.3 *Let $\mathcal{I}$ be a partially-ordered index set and $\{\mathcal{F}_i \mid i \in \mathcal{I}\}$ a collection of sub-σ-algebras of a probability space $(\Omega, \mathcal{F}, P)$. We shall say that a collection $\{Y_i\}$ of integrable and $\mathcal{F}_i$-adapted random variables, is a collection of martingale differences whenever $E[Y_j \mid \mathcal{F}_i] = 0$ for $j \succ i$.*

Comment. According to lemma 2.2, the collection $\{H(A) \mid A \subset [1,N]\}$ is a collection of martingale differences with respect to $\{\mathcal{F}_A \mid A \subset [1,N]\}$. The following variation on the theme of a result in [Gu] adds a bit to the probabilistic nature of the Walsh functions explored in [HLOU-1] and [L-M].

Proposition 7.2 *Assume that $\{H(A) \mid A \subset [1,N]\}$ is a complete orthonormal system and a collection of martingale differences with respect to $\{\mathcal{F}_A \mid A \subset [1,N]\}$. Then it also is an extended H-system.*

Proof Let X be a random variable, necessarily in $L_2(P)$ in our finite setting. By assumption $X = \sum_A X(A)H(A)$. Define the random variables $Z_A = E[X \mid \mathcal{F}_A] - \sum_{B \subset A} X(B)H(B)$. From the assumptions, we obtain that the Z_A vanish identically for all $A \subset [1,N]$. Therefore, the condition in Definition 7.2 is satisfied and $\{H(A) \mid A \subset [1,N]\}$ is an extended H-system.
Comment. Even though the connection between Haar, Rademacher, Walsh, and other families of functions has been well studied in many places, take a look at [TMD] for example, the probabilistic nature of these connections seems to be missing.

8 Simple applications

8.1 An application in mathematical finance

We saw above that discrete exponential martingales are true exponential martingales when multiplication is performed with the Wick product. Now we shall examine a situation in which discrete exponential martingales are related to standard exponentials. We assume to be given a sequence of i.i.d. random variables $\{\xi_n \mid n \geq 1\}$ that take values in a finite set S, and we shall need the polynomials $\{\phi(s) \mid j = 0, 1, \ldots, d-1 \quad s \in S\}$, orthogonal with respect to $p_s = P(\xi_1 = s)$ introduced in Section 2.

Let $\{c_{n,j} \mid n \geq 1;\ j = 1, \ldots, d-1\}$ be any sequence of real numbers such that $\sum_{n \geq 1} \sum_{j=1}^{d-1} |c_{n,j}|^2 < \infty$ and let us again set

$$\Delta Z_n = \sum_{j=1}^{d-1} c_{n,j} \phi^j(\xi_n) \ \text{ for } n \geq 1,\ Z_0 = 0, \quad \text{and} \quad Z_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} \Delta Z_k \ .$$

That is, $\{Z_n \,|\, n \geq 0\}$ is a martingale with respect to the filtration generated by the $\{\xi_n\}$. We should add that obviously, given that our setup is finite dimensional in all senses, when considering a finite time horizon, there is no real need of the square integrability condition. But some statements can be easily extended to other setups, so let us keep the condition.

Consider now the stochastic difference equation

$$S_n - S_{n-1} = aS_{n-1} + \sigma S_{n-1} \Delta Z_n \tag{9}$$

where $1 + a > 0$ and $\sigma > 0$ are given numbers. Such an equation could perhaps be used to model time changing volatilities. In this regard check work by Bender and Elliott in [BE]. Now multiply both sides of (9) by $(1+a)^{-n}$ and rewrite it as

$$(1+a)^{-n} S_n - (1+a)^{-(n-1)} S_{n-1} = \frac{\sigma}{(1+a)} (1+a)^{-(n-1)} S_{n-1} \Delta Z_n.$$

Therefore, the solution to the stochastic difference equation is given by the discrete exponential

$$S_n = (1+a)^n \prod_{k=1}^{n} (1 + \frac{\sigma}{1+a} \Delta Z_k)$$

and clearly, if $a = 0$, S_n would be a discrete exponential martingale. An interesting question studied by many authors is:
Given an r such that $1 + r > 0$, can we change our measure in such a way that $S_n^* \stackrel{\text{def}}{=} \dfrac{S_n}{(1+r)^n}$ is a martingale?
More precisely:
Does there exist a measure $Q \sim P$ on the canonical sample space $(\Omega, \mathcal{F}_{n \leq N})$ on which the $\{\xi_n\}$ are given, such that $\{S_n^* \,|\, n \leq N\}$ is an $(\Omega, \mathcal{F}_{n \leq N}, Q)$-martingale?

This amounts to requiring that $E_Q[S_{n+1}^* \,|\, \mathcal{F}_n^*] = S_n^*$, or equivalently that

$$E_Q[(1 + \frac{\sigma}{1+a} \Delta Z_{n+1}) \,|\, \mathcal{F}_n] = \frac{1+r}{1+a} = 1 + \frac{r-a}{1+a}$$

or to $E_Q[\Delta Z_{n+1}] = \dfrac{r-a}{\sigma}$.

In this finite dimensional setup, $Q = \rho P$ with

$$\rho = \frac{\rho}{E[\rho \,|\, \mathcal{F}_N]} \frac{E[\rho \,|\, \mathcal{F}_N}{E[\rho \,|\, \mathcal{F}_{N-1}]} \cdots E[\rho \,|\, \mathcal{F}_1] \stackrel{\text{def}}{=} \rho_N \rho_{N-1} \ldots \rho_1$$

and, as can be easily checked, $\rho^{(n)} \stackrel{\text{def}}{=} \prod_{k=1}^{n} \rho_k$ is an $\{\mathcal{F}_n\}$-martingale. Actually the maximum entropy candidate for ρ_n is, (see [Gz] for example)

$$\rho_n = \frac{e^{-\lambda \Delta Z_n}}{E[e^{-\lambda \Delta Z_n}]}$$

where the Lagrange multiplier λ is chosen so that

$$\frac{E[e^{-\lambda\Delta Z_n}\Delta Z_n]}{E[e^{-\lambda\Delta Z_n}]} = \frac{r-a}{\sigma}.$$

Whenever r, a, σ are such that $(r-a)/\sigma$ fall in the interior of the convex interval spanned by the values of ΔZ_n, such λ exists. What is of interest for us here is that once λ is found, we can make use of the fact that the $\phi^j(s)$ are an orthogonal system to write

$$\begin{aligned} e^{-\lambda\Delta Z_n} &= \sum_{s,j} e^{-\lambda\zeta(s)} p_s \phi^j(s)\phi^j(\xi_n) \\ &= \sum_s e^{-\lambda\zeta(s)} p_s + \sum_s \sum_{j=1}^{d-1} e^{-\lambda\zeta(s)} p_s \phi^j(s)\phi^j(\xi_n) \\ &= E[e^{-\lambda\Delta Z_n}] + \sum_s \sum_{j=1}^{d-1} e^{-\lambda\zeta(s)} p_s \phi^j(s)\phi^j(\xi_n) \end{aligned}$$

where by $\zeta_n(s)$ we denote the different values that ΔZ_n can assume. The last identity can be rewritten as

$$\frac{E[e^{-\lambda\Delta Z_n}\Delta Z_n]}{E[e^{-\lambda\Delta Z_n}]} = 1 + \sum_{j=1}^{d-1} E_Q[\phi^0(\xi_n)\phi^j(\xi_n)]\phi^j(\xi_n) = 1 + \Delta\hat{Z}_n$$

and we thus exhibit $\rho^{(n)}$ as a discrete exponential martingale, to wit

$$\rho^{(n)} = \prod_{k=1}^{n}(1+\Delta\hat{Z}_k).$$

Now we could bring the results of section 2 to have an explicit expansion of the density $\rho = \dfrac{dQ}{dP}$ in terms of the discrete chaos determined by the $\{\xi_n\}$.

8.2 Discrete random evolution equations

Let us examine two very simple evolution equations. The gist of the example is to compare ordinary and Wick products. Consider for example the random evolution equation

$$y(n) = \hat{K}y(n-1), \qquad y(0) \text{ given} \tag{10}$$

where $K = L_2(\Omega, \mathcal{F}, P)$ and K admits an expansion in Wiener-like chaos. To make it really simple, assume that $K = \sum_A K(A)\chi_A$ as in section 2. The solution to (10) is given by

$$y(n) = \sum_A y(n,A)\chi_A = \hat{K}^n y(0) = y(0)\sum_A E[K^n(A)\chi_A]\chi_A$$

and obviously $y(n,A) = E[K^n \chi_A]$.

Another stochastic evolution equation is

$$y(n) = K \diamondsuit y(n-1), \qquad y(0) \text{ given} \tag{11}$$

the solution to which is given by

$$y(n) = \sum_A y(n,A)\chi_A = K^{\diamondsuit n} y(0) = y(0) \sum_{\{A_1,\ldots,A_n\}} K(A_1)\ldots K(A_n)\chi_{\bigcup A_j}$$

from which $y(n,A) = \sum_{\{A_1,\ldots,A_n\}} K(A_1)\ldots K(A_n)$ where the sum is over all partitions of the set A.

Actually, another simple case, which is a particular case of the computations presented in section 3 of [HLOU-2], corresponds to the following discrete random evolution equation

$$y(n) = (I+G)y(n-1) + c\,y(n-1)\diamondsuit\zeta_n$$

where the ζ_n is short for any of the martingale increments described in section 2. In this equation $y(n)$ can be assumed to be a vector in some finite dimensional space $\mathbb{R}^K$ and G to be a $K \times K$-matrix representing some discretized operator. Anyway, while we carry out the following computations we assume that $(I + G)^{-1}$ is defined. From what we proved in section 2, if $y(n)$ is adapted to the filtration $\mathcal{F}_n$, then the Wick product becomes an ordinary product and the equation can be recast as $y(n) = (I + G + c\,\zeta_n)y(n-1)$, which after iteration and factorization, can be written as

$$y(n) = (I+G)^n \prod_{j=1}^{n} (I + c\,\zeta_j (I+G)^{-1}) y(0)$$

and expanding the sum we obtain

$$y(n) = \sum_{k=1}^{n} c^k \left(\sum_{1 \le j_1 < \ldots < j_k \le n} \prod_{i=1}^{k} \zeta_{j_i} \right) (I+G)^{n-k} y(0)$$

which exhibits the expansion of $y(n)$ in terms of the chaos associated to ζ.

We direct the reader to [XK] for more on the theme of this subsection and for further references to these techniques.

8.3 A randomly driven binomial model

In this section we shall consider a simple random evolution describing the price of an asset which increases or decreases depending on the state of an environment described by a simple two-state Markov chain.

Let X describe a time homogeneous Markov chain with state space $\{-1,1\}$ with $p_{s,s'} = p(s,s') = P(X_1 = s' \mid X_0 = s) > 0$ for all s, s'. Let now $\phi_s^0(s') = 1$ and $\phi_s^1(s') = \dfrac{s' - E^s[X_1]}{2\sigma(s)}$ be the orthogonal basis determined by the transition matrix as described in section 1. In this basis the function $I_s(X_k)$ can be expanded as

$$I_{\{s\}}(X_k) = p(X_{k-1}, s) + p(X_{k-1}, s)\phi^1_{X_{k-1}}(X_k) \tag{12}$$

with respect to $P(\,\cdot \mid \mathcal{F}_{k-1})$.

Denote by $\{g_n \mid n \geq 1\}$ and $\{b_n \mid n \geq 1\}$ two independent sequences of i.i.d. Bernoulli random variables such that $1+g_n > 0$ and $1+b_n > 0$ for all n. Define now the price process of the asset to be

$$S(n) = \prod_{k=1}^{n} (1+g_k)^{I_{\{+\}}(X_k)}(1+b_k)^{I_{\{-\}}(X_k)})S(0).$$

Clearly at each time n the price change occurs according to whether during the preceding time interval the environment was good or bad. Notice that $a^{I_{\{+\}}(X_k)} = 1 + aI_{\{+\}}(X_k)$, therefore after some calculations, $S(n)$ can be written as

$$S(n) = \prod_{k=1}^{n} (1 + g_k I_{\{+\}}(X_k) + b_k I_{\{+\}}(X_k))S(0).$$

If we now bring in (12) and introduce the symbol

$$E^{X_{k-1}}[\Delta S_k] \stackrel{\text{def}}{=} g_k p(X_{k-1}, +) + b_k p(X_{k-1}, -)$$

(which denotes the price change averaged over the possible environmental changes), we can rewrite the price at time n, given the history of the environment up to n, as

$$S(n) = \langle S(n) \rangle \prod_{k=1}^{n} (1 + \delta_k \phi^1_{X_{k-1}}(X_k)) \tag{13}$$

where $\langle S(n) \rangle \stackrel{\text{def}}{=} \prod_{k=1}^{n} E^{X_{k-1}}[\Delta S_k]$ and $\delta_k \stackrel{\text{def}}{=} \dfrac{E^{X_{k-1}}[\Delta S_k]}{1 + E^{X_{k-1}}[\Delta S_k])}$. Certainly (13) is the starting point for the expansion of the price process in terms of the chaos associated to X.

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