SDP Approach for Solving LQ Control Problem Subject to Implicit System

Muhafzan

Abstract. This paper deals with the linear quadratic (LQ) control problem subject to implicit systems for which the semidefinite programming (SDP) approach is used to solve it. We propose a new sufficient condition in terms of SDP for existence of the optimal state-control pair of the considered problem. The numerical examples are given to illustrate the results.

Resumen. Este artículo trata sobre el problema de control lineal cuadrático (LQ) sujeto a sistemas implícitos, los cuales se resuelven usando el método de programación semidefinida (SDP). Proponemos una nueva condición suficiente en términos de SDP para la existencia de un par estado-control óptimo del problema considerado. Se dan ejemplos numéricos para ilustrar los resultados.

1 Introduction

The LQ (linear quadratic) control problem subject to implicit systems is one of the most important classes of optimal control problems, in both theory and application. In general, it is a problem to find a controller that minimizes the linear quadratic objective function governing by the implicit systems, either continuous or discrete.

It is well known that the implicit systems have attracted the attention of many researcher in the past years due to the fact that in some cases, the implicit systems describe better the behavior of physical systems than that of standard systems. They can preserve the structure of physical systems and can include non dynamic constraint and impulsive element. This kind of systems have many important applications, e.g., in biological phenomena, in economics as Leontif dynamic model, in electrical and in mechanical model, see [4, 12]. Therefore, it is fair to say that implicit systems give a more complete class of dynamical models than conventional state-space systems. Likewise, the LQ control problem subject to the implicit systems has a great potential for the system modelling.

A great number of results on solving LQ control problem subject to implicit systems have been appeared in literatures, see [3, 5, 7, 8, 10]. However, almost all of these results consider the assumption of the regularity of the implicit system and the positive definiteness of control weighting matrix in the quadratic cost functional. To the best of the author's knowledge, not much work has been done with the nonregular implicit system as a constraint and the control weighting matrix in the quadratic cost being positive semidefinite. In this last case, i.e., the quadratic cost being positive semidefinite, the existing LQ problem theories always involve the impulse distributions, see [5, 10]. Thus it does not provide any answer to a basic question such as when does the LQ control problem subject to implicit system possess an optimal solution in the form of a conventional control, in particular, one that does not involve impulse distribution. However, this issue has discussed in [7], in which they transform the LQ control problem subject to implicit system into a standard LQ control problem in which both are equivalent. Nonetheless, they still remains an open problem, that is, the new standard LQ control problem may be singular and this is not answered yet in [7].

In this paper, we reconsider the problem in [7], and in particular, the open problem, i.e. the singular version of the new standard LQ control problem, is solved. Here, we do not assume that the implicit systems is regular, thus our work is more general than some previous results, see [3, 5, 8, 10]. The method in [7] is still maintained to transform the original problem into the equivalent singular LQ control problem.

In addition, the semidefinite programming (SDP) approach is used to solving this singular LQ problem. A new sufficient condition in terms of SDP for existence of the optimal state-control pair of such problem is proposed. It is well-known that SDP has been one of the most exciting and active research areas in optimization recently. This tremendous activity is spurred by the discovery of important applications in various areas, mainly, in control theory; see [2, 11, 13].

This paper is organized as follows. Section 2 considers brief account of the problem statement. Section 3 presents the process of transformation from the original LQ control problem into an equivalent LQ problem. In Section 4, main result to solve the LQ control problem subject to implicit systems is presented. Numerical examples are given to illustrate the results in Section 5. Section 6 concludes the paper.

Notation. Throughout this paper, the superscript "T" stands for the transpose, $\varnothing$ denotes the empty set, I_n is the identity matrix of n-dimension, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of all $m\times n$ real matrices, $\mathbb{C}_{\mathrm{p}}^{+}[\mathbb{R}^n]$ denotes the n-dimensional piecewise continuous functions space with domain in $[0,\infty]$, $\mathbb{S}_{+}^{p}$ denotes set of all p-dimensional symmetric positive semidefinite matrices, and $\mathbb{C}$ denotes set of complex number.

2 Problem Statement

Let us consider the following continuous time implicit system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t),\ t \geq 0,\ Ex(0) = x_0 \\ y &= Cx(t) + Du(t), \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^r$ denotes the control (input) vector and $y(t) \in \mathbb{R}^q$ denotes the output vector. The matrices $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}, C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times r}$ are constant, with $\text{rank}E \equiv p < n$. This system is denoted by (E, A, B, C, D). The system (E, A, B, C, D) is said to be regular if $\det(sE - A) \neq 0$ for almost all $s \in \mathbb{C}$. Otherwise, it is called nonregular if $\det(sE - A) = 0$ for each $s \in \mathbb{C}$ or if $E, A \in \mathbb{R}^{m\times n}$ with $m \neq n$. In particular, when $m \neq n$, it is called a rectangular implicit system [6].

It is well known that the solution of (2.1) exists and unique if it is regular. Otherwise, it is possible to have many solutions, or no solution at all.

Next, for a given admissible initial state $x_0 \in \mathbb{R}^n$, we consider the following associated objective function (cost functional):

$$J(u(.), x_0) = \int_0^\infty y^T(t)y(t)dt. \tag{2.2}$$

In general, the problem of determining the stabilizing feedback control $u(t) \in \mathbb{R}^r$ which minimizes the cost functional (2.2) and satisfies the dynamic system (2.1) for an admissible initial state $x_0 \in \mathbb{R}^n$, is often called as LQ control problem subject to implicit system. If D^TD is positive semidefinite, it is called a singular LQ control problem subject to implicit system. We denote, for simplicity, this LQ control problem as Ω. Next, we define the set of admissible control-state pairs of problem Ω by:

$$\begin{aligned} \mathbb{A}_{\text{ad}} \equiv \{(u(.), x(.)) \mid u(.) \in \mathbb{C}_{\text{p}}^+[\mathbb{R}^r] \text{ and } &\ x(.) \in \mathbb{C}_{\text{p}}^+[\mathbb{R}^n] \\ &\text{satisfy}(2.1) \text{ and } J(u(.), x_0) < \infty\}. \end{aligned}$$

The optimization problem under consideration is to find the pair $(u^*, x^*) \in \mathbb{A}_{\text{ad}}$ for a given admissible initial condition $x_0 \in \mathbb{R}^n$, such that

$$J(u^*, x_0) = \underset{(u(.),x(.))\in\mathbb{A}_{\text{ad}}}{\text{minimize}}\ J(u(.), x_0), \tag{2.3}$$

under the assumption that (2.1) is solvable, impulse controllable and D^TD is positive semidefinite.

Definition 1 *[7] Two systems (E, A, B, C, D) and $(\bar{E}, \bar{A}, \bar{B}, \bar{C}, D)$ are termed restricted system equivalent (r.s.e.), denoted by $(E, A, B, C, D) \sim (\bar{E}, \bar{A}, \bar{B}, \bar{C}, D)$, if there exists nonsingular matrices $M, N \in \mathbb{R}^{n\times n}$ such that their associated system matrices are related by $MEN = \bar{E}$, $MAN = \bar{A}, MB = \bar{B}$ and $CN = \bar{C}$.*

Remark 1 The operations of r.s.e. correspond to the constant nonsingular transformations of (2.1) itself and of the basis in the space of internal variables x. The behavior of x in the original system may thus be simply recovered from the behavior of any system *r.s.e.* to it. These operations therefore constitute an eminently safe set of transformations that are unlikely to destroy any important properties of the system. Furthermore, such operations suffice to display the detailed structure of the original system.

Definition 2 *[7] Two optimal control problems are said to be equivalent if there exist a bijection between the two sets of admissible control - state pairs of the problems, and the quadratic cost value of any image is equal to that of corresponding preimage.*

Obviously, definition 2 conforms to the reflexivity, symmetry, and transitivity of an equivalent relation, thus the two equivalent optimal control problems will have the same solvability, uniqueness of solution and optimal cost. Thus solving one can be replaced by solving the other.

3 Transformation into an Equivalent LQ problem

Let us utilize the Singular Value Decomposition(SVD) theorem [9] for the matrix E. Since rank$E = p < n$, there exists the nonsingular matrices $M, N \in \mathbb{R}^{n\times n}$ such that

$$MEN = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that we have

$$MAN = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, MB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, CN = \begin{pmatrix} C_1 & C_2 \end{pmatrix},$$

and

$$N^{-1}x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $A_{11} \in \mathbb{R}^{p\times p}$, $A_{12} \in \mathbb{R}^{p\times(n-p)}$, $A_{21} \in \mathbb{R}^{(n-p)\times p}$, $A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$, $B_1 \in \mathbb{R}^{p\times r}$, $B_2 \in \mathbb{R}^{(n-p)\times r}$, $C_1 \in \mathbb{R}^{q\times p}$, $C_2 \in \mathbb{R}^{q\times(n-p)}$, $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^{n-p}$. Therefore, for a given admissible initial state $x_0 \in \mathbb{R}^n$, the system (2.1) is *r.s.e.* to the system

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \ x_1(0) = x_{10} \\ 0 &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t) \end{aligned} \tag{3.1}$$

where $x_{10} = \begin{pmatrix} I_p & 0 \end{pmatrix} Mx_0$.

Theorem 1 *[7] The implicit system* (2.1) *is impulse controllable if and only if the matrix* $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$ *has full row rank.*

Using the expression (3.1), the objective function (2.2) can be changed into

$$J_1(u(.),x_{10}) = \int_0^\infty \begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} C_1^T C_1 & C_1^T C_2 & C_1^T D \\ C_2^T C_1 & C_2^T C_2 & C_2^T D \\ D^T C_1 & D^T C_2 & D^T D \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{pmatrix} dt.$$

Likewise, we have the new LQ control problem which minimizes the objective function $J_1(u(.),x_{10})$ subject to the dynamic system (3.1), and denote this LQ control problem as Ω_1. Further, we define the set of admissible control-state pairs of the problem Ω_1 by

$$\begin{aligned} \mathbb{A}_{\mathrm{ad}}^1 \equiv \{ (u(.),(x_1(.),x_2(.))) \mid u(.) \in \mathbb{C}_{\mathrm{p}}^+[\mathbb{R}^r],\ x_1(.) \in \mathbb{C}_{\mathrm{p}}^+[\mathbb{R}^p] \\ \text{and } x_2(.) \in C_{\mathrm{p}}^+[\mathbb{R}^{n-p}] \text{ satisfy(3.1) and } J_1(u(.),x_{10}) < \infty \}. \end{aligned}$$

By virtue of definition 2, it is easily seen that the LQ control problem Ω_1 is equivalent to Ω.

Furthermore, under assumption that the implicit system (2.1) is impulse controllable implies that $\operatorname{rank}\begin{pmatrix} A_{22} & B_2 \end{pmatrix} = n-p$. Hence, the solution of the second equation of (3.1) can be stated as

$$\begin{pmatrix} x_2(t) \\ u(t) \end{pmatrix} = -\hat{A}^+ A_{21} x_1(t) + W v(t), \tag{3.2}$$

for some $v(t) \in \mathbb{R}^r$ and for some full column rank matrix $W \in \mathbb{R}^{(n-p+r)\times r}$ with $W \in \ker \begin{pmatrix} A_{22} & B_2 \end{pmatrix}$, and

$$\hat{A}^+ = \begin{pmatrix} A_{22} & B_2 \end{pmatrix}^T \left[\begin{pmatrix} A_{22} & B_2 \end{pmatrix} \begin{pmatrix} A_{22} & B_2 \end{pmatrix}^T \right]^{-1}$$

is the generalized inverse of the matrix $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$.

Using expression (3.2), we can further create the following transformation

$$\begin{pmatrix} x_1(t) \\ \hline x_2(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ -\hat{A}^+ A_{21} & W \end{pmatrix} \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix}. \tag{3.3}$$

By substituting (3.3) into Ω_1, we obtain a new LQ control problem as follows:

$$\begin{aligned} &\underset{(v,x_1)}{\text{minimize}} \quad J_2(v(.),x_{10}) \\ &\text{subject to} \quad \dot{x}_1(t) = \bar{A} x_1(t) + \bar{B} v(t),\ x_1(0) = x_{10}) \\ &\qquad\qquad\quad\ \ y(t) = \bar{C} x_1(t) + \bar{D} v(t) \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} J_2(v(.),x_{10}) &= \int_0^\infty \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix} dt \qquad (3.5) \\ \bar{A} &= A_{11} - \begin{pmatrix} A_{12} & B_1 \end{pmatrix} \hat{A}^+ A_{21}, \\ \bar{B} &= \begin{pmatrix} A_{12} & B_1 \end{pmatrix} W, \\ \bar{C} &= C_1 - \begin{pmatrix} C_2 & D \end{pmatrix} \hat{A}^+ A_{21}, \\ \bar{D} &= \begin{pmatrix} C_2 & D \end{pmatrix} W, \\ Q_{11} &= \bar{C}^T\bar{C}, Q_{12} = \bar{C}^T\bar{D} \quad \text{and} \quad Q_{22} = \bar{D}^T\bar{D}, \end{aligned}$$

and denote this LQ control problem as Ω_2. Further, we define the set of admissible control-state pairs of problem Ω_2 by

$$\mathbb{A}^2_{\mathrm{ad}} \equiv \{(v(.),x_1(.)) \mid v(.) \in \mathbb{C}^+_{\mathrm{p}}[\mathbb{R}^r] \text{ and } x_1(.) \in \mathbb{C}^+_{\mathrm{p}}[\mathbb{R}^p] \\ \text{satisfy(3.4) and } J_2(v(.),x_{10}) < \infty\}.$$

It is obvious that the system (3.4) is a standard state space system with the state x_1, the control v and the output y, so Ω_2 is a standard LQ control problem.

It is easy to show that the transformation defined by (3.3) is a bijection from $\mathbb{A}^2_{\mathrm{ad}}$ to $\mathbb{A}^1_{\mathrm{ad}}$, and thus the problem Ω_2 is equivalent to the problem Ω_1. It follows that Ω_2 is equivalent to the problem Ω as well. Therefore, in order to solve the problem Ω, it suffices to consider the problem Ω_2 only.

4 Solving the LQ Control Problem

It is well known that the solution of Ω_2 hinges on the behavior of input weighting matrix Q_{22} in the cost functional (3.5), whether it is positive definite or positive semidefinite.

In the case where Q_{22} is positive definite, one can use the classical theory of LQ control that asserts that Ω has a unique optimal control-state pair if the pair $(Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T, \bar{A} - \bar{B}Q_{22}^{-1}Q_{12}^T)$ is detectable and the pair $(\bar{A},\bar{B})$ is asymptotically stabilizable [1]. In this case, the optimal control v^* is given by

$$v^* = -Lx_1^* \qquad (4.1)$$

where the state x_1^* is the solution of differential equation

$$\dot{x}_1(t) = (\bar{A} - \bar{B}L)x_1(t), \quad x_1(0) = x_{10} \qquad (4.2)$$

with $L = Q_{22}^{-1}(Q_{12}^T + \bar{B}^TP)$ and P is the unique positive semidefinite solution of the following algebraic Riccati equation:

$$\bar{A}^TP + P\bar{A} + Q_{11} - (P\bar{B} + Q_{12})Q_{22}^{-1}(P\bar{B} + Q_{12})^T = 0, \qquad (4.3)$$

where every eigenvalue λ of $\bar{A}-\bar{B}L$ satisfies $\mathrm{Re}\lambda<0$. Thus, in this case the optimal control-state pair of the problem Ω is given by

$$\begin{pmatrix} x^* \\ u^* \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} I_p \\ \Lambda_1 - W_1 Q_{22}^{-1}(P^*\bar{B}+Q_{12})^T \\ \hdashline \Lambda_2 - W_2 Q_{22}^{-1}(P^*\bar{B}+Q_{12})^T \end{pmatrix} x_1^* \tag{4.4}$$

where

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} \equiv -\hat{A}A_{21}, \quad \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \equiv W,$$

with $\Lambda_1 \in \mathbb{R}^{(n-p)\times p}$, $\Lambda_2 \in \mathbb{R}^{r\times p}$, $W_1 \in \mathbb{R}^{(n-p)\times r}$, $W_2 \in \mathbb{R}^{r\times r}$.

On the other hand, when the matrix Q_{22} is positive semidefinite ($Q_{22}\geq 0$), the algebraic Riccati equation (4.3) seems to be meaningless, and therefore this result can no longer be used to handle the singular LQ control problem Ω.

A natural extension is to generalize the algebraic Riccati equation (4.3) by replacing the matrix Q_{22} with the matrix Q_{22}^{+}, such that the equation (4.3) is replaced by

$$\mathcal{F}(P) \equiv \bar{A}^T P + P\bar{A} + Q_{11} - (P\bar{B}+Q_{12})Q_{22}^{+}(P\bar{B}+Q_{12})^T = 0 \tag{4.5}$$

where Q_{22}^{+} stands for the generalized inverse of Q_{22}. Corresponding to this generalized algebraic Riccati equation, let us consider an affine transformation of the matrix P as follows:

$$\mathcal{L}(P) \equiv \begin{pmatrix} Q_{22} & (P\bar{B}+Q_{12})^T \\ P\bar{B}+Q_{12} & Q_{11}+\bar{A}^T P + P\bar{A} \end{pmatrix},$$

where $\mathcal{L}:\mathbb{S}_+^p \to \mathbb{R}^{(p+r)\times(p+r)}$. By using the extended Schur's Lemma [11], we have the following lemma which shows that $\mathcal{F}(P)\geq 0$ and $\mathcal{L}(P)\geq 0$ are closely related.

Lemma 1 *$\mathcal{L}(P)\geq 0$ if and only if $\mathcal{F}(P)\geq 0$ and $(I_r - Q_{22}Q_{22}^{+})(P\bar{B}+Q_{12})^T = 0$.*

Now, let us consider the following primal SDP:

$$\boxed{\begin{array}{ll} \text{maximize} & \langle I_p, P\rangle \\ \text{subject to} & P\in\mathcal{P} \end{array}} \tag{P}$$

where

$$\mathcal{P} \equiv \left\{ P\in\mathbb{S}_+^p \mid \mathcal{L}(P)\geq 0 \right\}$$

is the set of feasible solution of primal SDP problem (P). It is easy to show that $\mathcal{P}$ is a convex set, and it may be empty which in particular implies that there is no solution to the primal SDP. Moreover, it is easy to show that the objective function of the problem (P) is convex. Since the objective function and $\mathcal{P}$ satisfy the convexity properties then the above primal SDP is a convex optimization problem.

Corresponding to the above primal SDP, we have the following dual problem:

$$\boxed{\begin{array}{ll} \text{minimize} & \langle Q_{22}, Z_b\rangle + 2\left\langle Q_{12}^T, Z_u\right\rangle + \langle Q_{11}, Z_p\rangle \\ \text{subject to} & Z_u^T\bar{B}^T + \bar{B}Z_u + Z_p\bar{A}^T + \bar{A}Z_p + I_p = 0 \\ & Z \equiv \left[\begin{array}{cc} Z_b & Z_u \\ Z_u^T & Z_p \end{array}\right] \geq 0. \end{array}} \tag{D}$$

where Z denotes the dual variable associated with the primal constraint $\mathcal{L}(P) \geq 0$ with Z_b, Z_u and Z_p being a block partitioning of Z of appropriate dimensions.

Remark 2 Semidefinite programming are known to be special forms of conic optimization problems, for which there exists a well-developed duality theory, see, e. g.,[2], [11], [13] for the exhaustive theory of SDP. Key points of the theory can be highlighted as follows.
1. The weak duality always holds, i.e., any feasible solution to the primal problem always possesses an objective value that is greater than the dual objective value of any dual feasible solution. contrast, the strong duality needs not always hold.
2. A sufficient condition for the strong duality is that there exist a pair of complementary optimal solution, i.e., both the primal and dual SDP problems have attainable optimal solutions, and that these solutions are complementary to each other. This means that the optimal solution P^* and the dual optimal solution Z^* both exists and satisfy $\mathcal{L}(P^*)Z^* = 0$.
3. If both (P) and (D) satisfy the strict feasibility, namely, there exists primal and dual feasible solution P_0 and Z_0 such that $\mathcal{L}(P_0) > 0$ and $Z_0 > 0$, then the complementary solutions exist.

In the following we present the condition for stability of the singular LQ control problem Ω_2.

Theorem 2 *The singular* LQ *control problem* Ω_2 *has a stabilizing feedback control if and only if the dual problem* (D) *is strictly feasible.*

Proof. ($\Rightarrow$) First assume that the system (3.4) is stabilizable by some feedback control $v(t) = Lx_1(t)$. Then all the eigenvalues of the matrix $\bar{A} + \bar{B}L$ have negative real parts. Consequently, using the Lyapunov's theorem [2], there exists positive definite matrix Y such that

$$(\bar{A}+\bar{B}L)Y + Y(\bar{A}+\bar{B}L)^T = -I_p.$$

By setting $Z_p = Y$ and $Z_u = LZ_p$, then this relation can be rewritten as

$$Z_u^T\bar{B}^T + \bar{B}Z_u + Z_p\bar{A}^T + \bar{A}Z_p + I_p = 0.$$

Now choose

$$Z_b = \epsilon I_r + Z_u(Z_p)^{-1}Z_u^T.$$

Then by Schur's lemma, Z is strictly feasible to (D).
($\Leftarrow$) If the dual problem (D) is strictly feasible, then $Z_p > 0$ by Schur's lemma. Putting

$$L = Z_u(Z_p)^{-1},$$

then Z satisfying the equality constraint of (D) yields

$$(\bar{A}+\bar{B}L)Z_p + Z_p(\bar{A}+\bar{B}L)^T = -I_p.$$

By constructing a quadratic Lyapunov function $x_1^T Z_p x_1$, it is easily verified that the system in (3.4) is stabilizable. ∎

Theorem 3 *If* (P) *and* (D) *satisfy the complementary slackness condition, then the optimal solution of* (P) *satisfies the generalized algebraic Riccati equation* $\mathcal{F}(P) = 0$.

Proof. Let P^* and Z^* denote the optimal solution of (P) and (D), respectively. Since P^* is optimal then it is also feasible and satisfies $\mathcal{L}(P^*) \geq 0$. By lemma 1, we have

$$(I_r - Q_{22}Q_{22}^+)(P^*\bar{B}+Q_{12})^T = 0.$$

Thus, the following decomposition is true:

$$\mathcal{L}(P^*) = \begin{pmatrix} I_r & 0 \\ (P^*\bar{B}+Q_{12})Q_{22}^+ & I_p \end{pmatrix}\begin{pmatrix} Q_{22} & 0 \\ 0 & \mathcal{F}(P^*) \end{pmatrix}\begin{pmatrix} I_r & Q_{22}^+(P^*\bar{B}+Q_{12})^T \\ 0 & I_p \end{pmatrix}.$$

From the relation $\mathcal{L}(P^*)Z^* = 0$, we have

$$\mathcal{L}(P^*)Z^* = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\mathcal{L}_{11} &= Q_{22}(Z_b^* + Q_{22}^+(P^*\bar{B}+Q_{12})^T(Z_u^*)^T) \\
\mathcal{L}_{12} &= Q_{22}(Z_u^* + Q_{22}^+(P^*\bar{B}+Q_{12})^T Z_p^*) \\
\mathcal{L}_{21} &= \mathcal{F}(P^*)(Z_u^*)^T \\
\mathcal{L}_{22} &= \mathcal{F}(P^*)Z_p^*.
\end{aligned}$$

Therefore

$$\mathcal{F}(P^*)(Z_u^*)^T = 0 \quad \text{and} \quad \mathcal{F}(P^*)Z_p^* = 0,$$

and hence

$$Z_u^*\mathcal{F}(P^*) = 0 \quad \text{and} \quad Z_p^*\mathcal{F}(P^*) = 0.$$

Since Z^* is dual feasible then it also satisfies

$$Z_u^T\bar{B}^T + \bar{B}Z_u + Z_p\bar{A}^T + \bar{A}Z_p + I_p = 0.$$

Multiplying $\mathcal{F}(P^*)$ on both sides of the above equation, i.e.,

$$\mathcal{F}(P^*)(Z_u^T\bar{B}^T + \bar{B}Z_u + Z_p\bar{A}^T + \bar{A}Z_p + I_p)\mathcal{F}(P^*),$$

yields $\mathcal{F}(P^*)^2 = 0$, and implies $\mathcal{F}(P^*) = 0$. ■

Now, let us consider the subset $\mathcal{P}_{\text{bound}}$ of $\mathcal{P}$ as follows:

$$\mathcal{P}_{\mathbf{bound}} \equiv \left\{P \in \mathbb{S}_+^p \mid \mathcal{L}(P) \geq 0 \text{ and } \mathcal{F}(P) = 0\right\}.$$

Note that $\mathcal{P}_{\mathbf{bound}}$ may be empty, which in particular implies that there is no solution to the generalized algebraic Riccati equation (4.5).

In the following, we present our main results, where the LQ control problem is explicitly constructed in terms of the solution to the primal and dual SDP.

Theorem 4 *If* $\mathcal{P}_{\mathbf{bound}} \neq \varnothing$ *and*

$$v^*(t) = -Q_{22}^+(P^*\bar{B} + Q_{12})^T x_1(t) \tag{4.6}$$

is a stabilizing control for some $P^* \in \mathcal{P}_{\mathbf{bound}}$, *where* $x_1(t)$ *satisfies the differential equation*

$$\dot{x}_1(t) = (\bar{A} - \bar{B}_2Q_{22}^+(P^*\bar{B} + Q_{12})^T)x_1(t),\ x_1(0) = x_{10},$$

then (P) *and* (D) *satisfy the complementary slackness property . Moreover,* $v^*(t)$ *is optimal control for* LQ *control problem* Ω_2.

Proof. Let $P^* \in \mathcal{P}_{\mathbf{bound}}$ and $L = -Q_{22}^+(P^*\bar{B} + Q_{12})^T$. Since the control $v^*(t) = Lx_1(t)$ is stabilizing, then Lyapunov equation

$$(\bar{A} + \bar{B}L)Y + Y(\bar{A} + \bar{B}L)^T + I_p = 0$$

has a positive definite solution, let it be $Y^* > 0$. Let

$$Z_p^* = Y^*,\ Z_u^* = LY^*, Z_b^* = LY^*L^T.$$

By this construction, we can easily verify that

$$\begin{pmatrix} Z_b^* & Z_u^* \\ (Z_u^*)^T & Z_p^* \end{pmatrix} = \begin{pmatrix} I_r & L \\ 0 & I_p \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & Z_p^* \end{pmatrix}\begin{pmatrix} I_r & 0 \\ L^T & I_p \end{pmatrix} \geq 0,$$

and

$$I_p + (Z_u^*)^T \bar{B}^T + \bar{B} Z_u^* + Z_p^* \bar{A}^T + \bar{A} Z_p^* = 0.$$

Therefore,

$$Z^* = \begin{pmatrix} Z_b^* & Z_u^* \\ (Z_u^*)^T & Z_p^* \end{pmatrix}$$

is a feasible solution of (D). Since $\mathcal{L}(P) \geq 0$, by lemma 1, we have

$$(I_r - Q_{22}Q_{22}^+)(P\bar{B} + Q_{12})^T = 0.$$

It follows that the following identity

$$\begin{aligned}\mathcal{L}(P) \quad = \quad & \begin{pmatrix} I_r & 0 \\ (P\bar{B} + Q_{12})Q_{22}^+ & I_p \end{pmatrix} \begin{pmatrix} Q_{22} & 0 \\ 0 & \mathcal{F}(P) \end{pmatrix} \times \\ & \begin{pmatrix} I_r & Q_{22}^+(P\bar{B} + Q_{12})^T \\ 0 & I_p \end{pmatrix}\end{aligned}$$

is valid. Moreover, we can verify that

$$\begin{aligned}\mathcal{L}(P^*)Z^* &= \begin{pmatrix} I_r & 0 \\ -L^T & I_p \end{pmatrix} \begin{pmatrix} Q_{22} & 0 \\ 0 & \mathcal{F}(P^*) \end{pmatrix} \times \\ & \qquad \begin{pmatrix} I_r & -L \\ 0 & I_p \end{pmatrix} \begin{pmatrix} Z_b^* & Z_u^* \\ (Z_u^*)^T & Z_p^* \end{pmatrix} \\ &= \begin{pmatrix} I_r & 0 \\ -L^T & I_p \end{pmatrix} \begin{pmatrix} Q_{22}(Z_b^* - L(Z_u^*)^T) & Q_{22}(Z_u^* - LZ_p^*) \\ \mathcal{F}(P^*)(Z_u^*)^T & \mathcal{F}(P^*)Z_p^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

that is the problem (P) and (D) satisfy the complementary slackness property . Now, we prove that

$$v^*(t) = -Q_{22}^+(P^*\bar{B} + Q_{12})^T x_1(t)$$

is the optimal control for LQ problem Ω_2. Firstly, consider any $P \in \mathcal{P}$ and any admissible stabilizing control $v(.) \in \mathbb{C}_{\mathrm{p}}^+[\mathbb{R}^r]$. We have,

$$\begin{aligned}\frac{d}{dt}(x_1^T(t)Px_1(t)) \quad &= \quad (\bar{A}x_1(t) + \bar{B}v(t))^T Px_1(t) + \; x_1^T(t)P(\bar{A}x_1(t) + \bar{B}v(t)) \\ &= \quad x_1^T(t)(\bar{A}^T P + P\bar{A})x_1(t) + 2v^T(t)\bar{B}^T Px_1(t).\end{aligned}$$

Integrating over $[0, \infty)$ and making use of the fact that

$$\lim_{t \to 0} x_1^T(t)Px_1(t) = \infty,$$

we have

$$0 = x_{10}^T P x_{10} + \int_0^\infty \left[x_1^T(t)\left(\bar{A}^T P + PA\right)x_1(t) + 2v^T(t)\bar{B}^T P x_1(t)\right] dt.$$

Therefore,

$$\begin{aligned} J_2(v(.), x_{10}) &= \int_0^\infty \left[x_1^T(t) Q_{11} x_1(t) + 2v^T(t) Q_{12}^T x_1(t) + v^T(t) Q_{22} v(t)\right] dt \\ &= x_{10}^T P x_{10} + \int_0^\infty \Big[x_1^T(t)\left(\bar{A}^T P + PA + Q_{11}\right)x_1(t) + \\ &\qquad 2v^T(t)\left(P\bar{B} + Q_{12}\right)^T x_1(t) + v^T(t) Q_{22} v(t)\Big] dt \\ &= x_{10}^T P x_{10} + \int_0^\infty \Big[\Big(v(t) + Q_{22}^+\left(P\bar{B} + Q_{12}\right)^T x_1(t)\Big)^T Q_{22} \times \\ &\qquad \Big(v(t) + Q_{22}^+\left(P\bar{B} + Q_{12}\right)^T x_1(t)\Big) + x_1^T(t)\mathcal{F}(P)x_1(t)\Big] dt. \end{aligned}$$

Since $P \in \mathcal{P}$, we have $\mathcal{F}(P) \geq 0$. This means that

$$J_2(v(.), x_{10}) \geq x_{10}^T P x_{10}, \tag{4.7}$$

for each $P \in \mathcal{P}$ and each admissible stabilizing control $v(.) \in \mathbb{C}_{\mathrm{p}}^+[\mathbb{R}^r]$. On the other hand, under the feedback control

$$v^*(t) = -Q_{22}^+(P^*\bar{B} + Q_{12})^T x_1(t),$$

and if we take into account $P^* \in \mathcal{P}_{\mathbf{bound}}$ then we have

$$\begin{aligned}
0 \leq\ & J_2(v^*(.), x_{10}) \\
=\ & \int_0^\infty [x_1^T(t)Q_{11}x_1(t) + 2{v^*}^T(t)Q_{12}^T x_1(t) + {v^*}^T(t)Q_{22}v^*(t)]\ dt \\
=\ & \lim_{t\to\infty}\int_0^t [x_1^T(\tau)Q_{11}x_1(\tau) + 2{v^*}^T(\tau)Q_{12}^T x_1(\tau) + {v^*}^T(\tau)Q_{22}v^*(\tau)]d\tau \\
=\ & \lim_{t\to\infty}\{\ x_{10}^T P^* x_{10} - x_1^T(t)P^*x_1(t) + \\
& \int_0^t [x_1^T(\tau)(\bar{A}^T P^* + P^*A + Q_{11})x_1(\tau) + \\
& 2{v^*}^T(\tau)(P^*\bar{B} + Q_{12})^T x_1(\tau) + {v^*}^T(\tau)Q_{22}v^*(\tau)]\ d\tau\} \\
\leq\ & x_{10}^T P^* x_{10} + \lim_{t\to\infty}\int_0^t [(v^*(\tau) + Q_{22}^+(P^*\bar{B} + Q_{12})^T x_1(\tau))^T Q_{22} \times \\
& (v^*(\tau) + Q_{22}^+(P^*\bar{B} + Q_{12})^T x_1(\tau)) + x_1^T(\tau)\mathcal{F}(P^*)x_1(\tau)]d\tau \\
=\ & x_{10}^T P^* x_{10}.
\end{aligned}$$

It follows that

$$J_2(v^*(.), x_{10}) \leq x_{10}^T P^* x_{10}. \tag{4.8}$$

The facts (4.7) and (4.8) lead us to conclude that the LQ control problem Ω_2 has an attainable optimal feedback control which is given by (4.6) with the cost is $x_{10}^T P^* x_{10}$. ■

The significance of theorem 4 is that, one can solve LQ control problem for standard state space system by simply solving a corresponding SDP problem. Consequently, since there exists the equivalent relationship between the LQ control problem subject to implicit system and the standard LQ control problem, one can also solve the LQ control problem subject to implicit system via such corresponding SDP approach.

By reconsidering the transformation (3.3), it follows that

$$\begin{aligned}
\begin{pmatrix} x_1^* \\ x_2^* \\ u^* \end{pmatrix} &= \begin{pmatrix} I_p & 0 \\ -\hat{A}^+A_{21} & W \end{pmatrix}\begin{pmatrix} I_p \\ -Q_{22}^+(P^*\bar{B} + Q_{12})^T \end{pmatrix} x_1^* \\
&= \begin{pmatrix} I_p & 0 \\ \Lambda_1 & W_1 \\ \Lambda_2 & W_2 \end{pmatrix}\begin{pmatrix} I_p \\ -Q_{22}^+(P^*\bar{B} + Q_{12})^T \end{pmatrix} x_1^* \\
&= \begin{pmatrix} I_p \\ \Lambda_1 - W_1Q_{22}^+(P^*\bar{B} + Q_{12})^T \\ \Lambda_2 - W_2Q_{22}^+(P^*\bar{B} + Q_{12})^T \end{pmatrix} x_1^*.
\end{aligned}$$

Ultimately, the optimal control-state pair (u^*, x^*) of Ω is given by

$$\begin{aligned} x^* &= N \begin{pmatrix} I_p \\ \Lambda_1 - W_1 Q_{22}^{+}(P^*\bar{B} + Q_{12})^T \end{pmatrix} x_1^*, \\ u^* &= \left(\Lambda_2 - W_2 Q_{22}^{+}(P^*\bar{B} + Q_{12})^T\right) x_1^* \end{aligned}$$

We end this section by presenting the sufficient condition for the existence of the optimal control of the LQ control problem subject to implicit system.

Corollary 1 *Assume that the implicit system* (2.1) *is impulse controllable and the* LQ *control problem* Ω *is equivalent to* Ω_2 *where matrix* Q_{22} *is positive semidefinite. If* $\mathcal{P}_{\mathbf{bound}} \neq \varnothing$ *and*

$$v^*(t) = -Q_{22}^{+}(P^*\bar{B} + Q_{12})^T x_1(t)$$

is a stabilizing control for some $P^* \in \mathcal{P}_{\mathbf{bound}}$, *where* $x_1(t)$ *satisfies the differential equation*

$$\dot{x}_1(t) = (\bar{A} - \bar{B}_2 Q_{22}^{+}(P^*\bar{B} + Q_{12})^T)x_1(t), \qquad x_1(0) = x_{10},$$

then

$$u^*(t) = \left(\Lambda_2 - W_2 Q_{22}^{+}(P^*\bar{B} + Q_{12})^T\right) x_1(t)$$

is optimal control for LQ *control problem* Ω.

5 Numerical Examples

Example 1 *The following is an example of the LQ control problem subject to the nonregular descriptor system, where the matrices* E, A, B, C *and* D *are as follows:*

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

with the initial state is $x_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T$.

By taking the matrices $M = I_4$ and

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

it is easy to check that the matrix $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$ has full row rank, thus the non-regular implicit system is impulse controllable. By choosing $W \in \ker \begin{pmatrix} A_{22} & B_2 \end{pmatrix}$, for example,

$$W = \begin{pmatrix} 3 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

the problem Ω can be equivalently changed into the following standard LQ control problem:

$$\begin{aligned} \underset{(v,x_1)}{\text{minimize}} & \int_0^\infty \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix} dt \\ \text{subject to } & \dot{x}_1(t) = \begin{pmatrix} -1 & 0 \\ -1 & -2 \end{pmatrix} x_1(t) + \begin{pmatrix} 3 & 4 \\ -3 & -4 \end{pmatrix} v(t), \\ & y(t) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} v(t) \end{aligned}$$

with initial condition $x_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where $x_1, v \in \mathbb{R}^2$,

$$Q_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, Q_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, Q_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

To identify a positive semidefinite feasible solution P^* to the primal SDP that satisfies the generalized Riccati equation $\mathcal{F}(P^*) = 0$, we first consider the constraint

$$(I_2 - Q_{22}Q_{22}^+)(P^*\bar{B} + Q_{12})^T = 0$$

as stipulated by lemma 2, i.e.

$$\begin{aligned} & \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix} \right] \times \\ & \qquad \left[\begin{pmatrix} p & q \\ q & r \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -3 & -4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right]^T = \begin{pmatrix} -p+q & -q+r \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This gives rise

$$P^* = \begin{pmatrix} p & p \\ p & p \end{pmatrix}$$

for some p. By substituting P^* into the generalized algebraic Riccati equation (4.5), we have

$$\begin{pmatrix} -4p + 0.5 & -4p + 0.5 \\ -4p + 0.5 & -4p + 0.5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and solving p, gives to $p = 0.125$. It follows that

$$\bar{A} - \bar{B}Q_{22}^{+}(P^{*}\bar{B} + Q_{12})^{T} = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix},$$

which has eigenvalues of -2 and -1, and these are stable. Hence the control

$$v^{*}(t) = -Q_{22}^{+}(P^{*}\bar{B} + Q_{12})^{T}x_1^{*}(t),$$

where $x_1^{*}(t)$ is the solution of the following differential equation

$$\dot{x}_1(t) = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix} x_1(t),\ x_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

is stabilizing. It is easy to verify that

$$x_1^{*}(t) = \begin{pmatrix} 4e^{-2t} - 3e^{-t} \\ -2e^{-2t} + 3e^{-t} \end{pmatrix}.$$

Thus, according to the theorem 5, the control $v^{*}(t)$ must be optimal to the LQ control problem Ω_2. Thereby, according to the corollary 6, the optimal state-control are as follows:

$$x^{*}(t) = \begin{pmatrix} 4e^{-2t} - 3e^{-t} \\ -2e^{-2t} \\ -4e^{-2t} \\ 0 \end{pmatrix},\ u^{*}(t) = \begin{pmatrix} 2e^{-2t} \\ -e^{-2t} \end{pmatrix}.$$

with the optimal cost $J_{\text{opt}} = 0.5$. The trajectories for the optimal state-control of Ω are given in the figures 1.a and 1.b below.

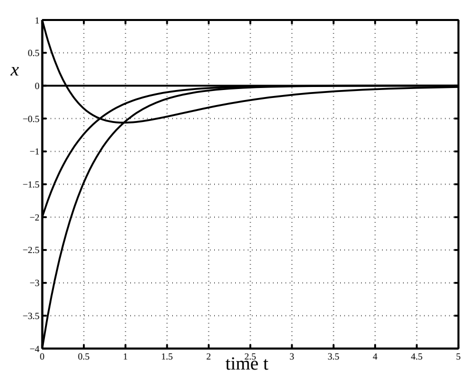

Figure 1.a. State Trajectories

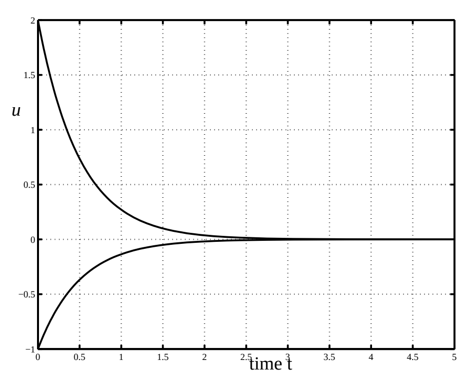

Figure 1.b. Control Trajectories

Example 2 *The following is an example of the LQ control problem subject to the regular implicit system, where the matrices E and B are the same as in example 1, with*

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & -2 & 0 & -1 \\ -1 & 2 & 0 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

and the initial state $x_0 = \begin{pmatrix} 2 & 1 & 0 & 0 \end{pmatrix}^T$.

By taking the matrices M and N as in example 1, it is easy to check that the matrix $\begin{pmatrix} A_{22} & B_2 \end{pmatrix}$ has full row rank, thus the regular implicit system is impulse controllable. By choosing $W \in \ker \begin{pmatrix} A_{22} & B_2 \end{pmatrix}$, for example,

$$W = \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 2 \end{pmatrix},$$

the problem Ω can be equivalently changed into the following standard LQ control problem:

$$\begin{aligned} \underset{(v,x_1)}{\text{minimize}} & \int_0^\infty \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ v(t) \end{pmatrix} dt \\ \text{subject to } & \dot{x}_1(t) = \begin{pmatrix} -1 & 0 \\ -1 & -2 \end{pmatrix} x_1(t) + \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix} v(t) \\ & y(t) = \begin{pmatrix} 0 & 0 \\ 4 & 2 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} v(t) \end{aligned}$$

with initial condition $x_1(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where $x_1, v \in \mathbb{R}^2$,

$$Q_{11} = \begin{pmatrix} 16 & 8 \\ 8 & 4 \end{pmatrix}, Q_{12} = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}, \; Q_{22} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

To identify a positive semidefinite feasible solution P^* to the primal SDP that satisfies the generalized Riccati equation $\mathcal{F}(P^*) = 0$, we first consider the constraint

$$(I_2 - Q_{22}Q_{22}^+)(P^*\bar{B} + Q_{12})^T = 0$$

as stipulated by lemma 2, so that we have

$$\begin{pmatrix} 2p - 2q & 2p - 2q \\ -p + q & -q + r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is satisfied only by $p = q = r$, i.e.,

$$P^* = \begin{pmatrix} p & p \\ p & p \end{pmatrix}.$$

In fact, this matrix P^* satisfies the generalized algebraic Riccati equation (4.5). It follows that

$$\bar{A} - \bar{B}Q_{22}^{+}(P^*\bar{B} + Q_{12})^T = \begin{pmatrix} -5 & -2 \\ 3 & 0 \end{pmatrix},$$

which has eigenvalues of -3 and -2, and these are stable. Hence the control

$$v^*(t) = -Q_{22}^{+}(P^*\bar{B} + Q_{12})^T x_1^*(t),$$

where $x_1^*(t)$ is the solution of the following differential equation

$$\dot{x}_1(t) = \begin{pmatrix} -5 & -2 \\ 3 & 0 \end{pmatrix} x_1(t),\ x_1(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

is stabilizing. It is easy to verify that

$$x_1^*(t) = \begin{pmatrix} 8e^{-3t} - 6e^{-2t} \\ -8e^{-3t} + 9e^{-2t} \end{pmatrix}.$$

Thus, according to the theorem 5, the control $v^*(t)$ must be optimal to the LQ control problem Ω_2. Thereby, according to the corollary 6, the optimal state-control of the LQ control problem Ω are as follows:

$$\begin{aligned} x^*(t) &= \begin{pmatrix} 8e^{-3t} - 6e^{-2t} \\ -3e^{-2t} \\ -16e^{-3t} + 6e^{-2t} \\ 8e^{-3t} + 6e^{-2t} \end{pmatrix}, \\ u^*(t) &= \begin{pmatrix} 8e^{-3t} \\ -16e^{-3t} + 6e^{-2t} \end{pmatrix}, \end{aligned}$$

with the optimal cost $J_{\text{opt}} = 0$. Moreover, the optimal control can be synthesized as

$$u^* = \begin{pmatrix} 7 & 4 \\ 3 & 0 \end{pmatrix} x_1^* + \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} x_2^*.$$

The trajectories for the optimal state-control are given below.

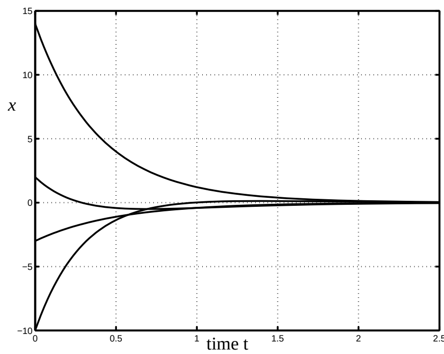

Figure 2.a. State Trajectories

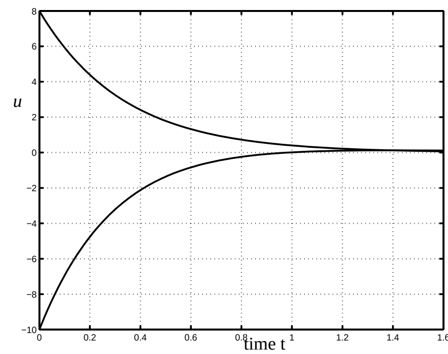

Figure 2.b. Control Trajectories

6 Conclusion

We have solved the LQ control problem subject to implicit system using the SDP approach. We have also proposed a new sufficient condition in terms of a semidefinite programming (SDP) for existence of the optimal state-control pair of the considered problem. The results show that the optimal control-state pair is free of the impulse distribution, i.e. it is smooth functions.

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Muhafzan
Department of Mathematics, Andalas University,
Kampus UNAND Limau Manis, Padang, Indonesia, 2516