Parallelism in diagram geometry

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Dedicated to J. A. Thas on his fiftieth birthday

Abstract

We introduce the concept of parallelism in diagram geometry, we apply it to a new gluing concept that provides geometries of higher rank, we combine it with another recent extension procedure for geometries and collect many examples solving existence questions for geometries over specified diagrams.

1 Introduction

An extensive survey and history of parallelism in incidence geometry is missing in the literature. In particular, the forthcoming Handbook of Incidence Geometry [7] is giving only a brief sketch of the subject with some references. The central part of the subject, namely affine geometry is of course better known. However the apparently stable affine geometry has undergone several important evolutions as we can see for instance from the work of Schmidt [38] where a deep synthesis of various approaches to affine ring geometry is covered, and in the work of André (see for instance [1]).

The present work arose from existence questions about geometries with specified diagrams and properties in the spirit of [7] (chapter 22 by Buekenhout and Pasini). Such questions lead us to a rather general construction that we call gluing. The natural context for gluing appears to be the concept of a diagram incidence geometry endowed with a convenient parallelism.

Starting with two such geometries Γ and Γ' whose "geometries at infinity" are isomorphic, we show that Γ and Γ' can be "glued" along their geometry at infinity, providing a geometry of higher rank in which Γ and Γ' appear as proper residues.

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The first observations of this procedure were made by the third author who glued affine planes of the same order, getting a geometry of type $Af.Af^*$.

Parallelism in affine spaces has an old tradition of "expansions": if S is a set of points at infinity of the affine space A together with some subspaces at infinity, then the set of points of A equipped with those affine subspaces whose subspace at infinity "belongs" to S, provides an affine expansion of S. This goes back at least to the space-time of special relativity. In the context of incidence geometry, it is described in Pasini [30]. Recently, Buekenhout, Dehon and Deschutter [8] gave a broad setting to affine expansion, providing many new examples of geometries with a specific diagram. We observe that their expansions bear a parallelism and so they can be submitted further to gluing. We observe further that their procedure can be generalized from affine spaces to geometries with a parallelism and we produce new interesting examples from this.

2 Geometry with parallelism

2.1 Preliminaries

We shall recall some basic facts on incidence geometries taken for instance from Buekenhout [7] or Pasini [30].

Let I be a set whose elements (and subsets) are called *types*. An *incidence geometry* or more simply, a *geometry* Γ over I, is a triple $(X, *, t)$ where X is a set whose members are called *elements* of Γ, where $*$ is a binary reflexive relation defined on X, called the *incidence relation*, and t is a mapping of X onto I, called the *type function*; these data are submitted to the conditions:

(1) $x * y$ and $t(x) = t(y)$ implies $x = y$;

(2) any maximal flag is of type I, where a *flag* is a set of pairwise incident elements of Γ and its *type* is its image by t.

The cardinality of I is called the *rank of* Γ. The pair $(X, *)$ is a graph, called the *incidence graph* of Γ. We call *i-elements* the elements of Γ of type i. The set of i-elements is denoted by X_i. The set of i-elements incident with a flag F is denoted by $\sigma_i(F)$; it is called the *i-shadow* of F.

2.1.1 Subgeometries, residues and truncations

Let $\Gamma = (X, *, t)$ be a geometry over I. Given a nonempty subset X' of X, let $*'$ and t' be the restrictions of $*$ to $X' \times X'$ and of t to X' respectively. If $\Gamma' = (X', *', t')$ is a geometry over $I' = t'(X')$, then we call Γ' a *subgeometry* of Γ.

Let F be a non-maximal flag of Γ. The *residue of* F, denoted by Γ_F, is the subgeometry of Γ over $I \backslash t(F)$ whose elements are the elements of $\Gamma \backslash F$ that are incident with all the elements of F. We say that Γ is *residually connected* if for any flag F whose residue is of rank at least two, the incidence graph of Γ_F is connected.

Let J be a nonempty subset of the type set I of Γ. The *J-truncation* of Γ is the subgeometry ${}^J\Gamma = (t^{-1}(J), *', t')$ of Γ over J.

2.1.2 Diagrams

Let I be a set of types. A *diagram* Δ *over* I consists of a map Δ defined from $\{\{i,j\}\}_{i,j\in I, i\neq j}$ which assigns to every pair $\{i,j\}$ some class $\Delta(i,j) = \Delta(j,i)$ of rank 2 geometries over $\{i,j\}$. A geometry Γ over I *belongs to the diagram* Δ over I if for every pair of distinct types $i, j \in I$ and every flag F of Γ such that Γ_F is of type $\{i,j\}$ one has $\Gamma_F \in \Delta(i,j)$.

A diagram $\Delta = (\Delta(i,j))_{i,j\in I, i\neq j}$ is usually depicted as a graph on I, drawing an edge between two types i, j if and only if $\Delta(i,j)$ is not a class of generalized digons, and labelling an edge $\{i,j\}$ by some symbol denoting the class $\Delta(i,j)$, with some additional conventions to make the picture easier to draw; for instance, putting no label on $\{i,j\}$ if $\Delta(i,j)$ is the class of projective planes, or putting two strokes with no label between i and j if $\Delta(i,j)$ is the class of generalized quadrangles. Some symbols have come to form a small "vocabulary" of labels for edges of diagrams (see [7], chapters 3 and 22; also [30], chapter 3). We will freely use them in this paper.

Since a diagram can be viewed as a graph, we can speak of paths in it, of its connected components, and so on. Thus, we can state the following definitions.

Let Δ be a diagram over a finite set of types I, let $0 \in I$ and let $X, Y \subseteq I$. Following Tits [39], we shall say that X *separates* 0 *from* Y *in* Δ if there is no path in $\Delta\backslash X$ joining 0 to some element of Y.

Let Δ' be a diagram over $J \cup \{0\}$, let $K \subseteq J$ and let Δ be the diagram over J obtained by removing 0 from Δ'. We say that Δ' is a $(0, K, \Delta)$-*diagram* if K separates 0 from $J\backslash\{K\}$ in Δ'.

We freely use the direct sum theorem whose statement is as follows. *Let* Γ *be a residually connected geometry of finite rank over* I. *Let* i, j *be elements of* I *which are contained in distinct connected components of the diagram of* Γ. *Then every* i*-element of* Γ *is incident with every* j*-element of* Γ.

2.1.3 Isomorphisms and automorphisms

Let $\Gamma = (X, *, t)$ and $\Gamma' = (X', *', t')$ be geometries over I. An *isomorphism* of Γ onto Γ' is a bijection α of X onto X' such that for all x, y in X, $x * y$ implies $\alpha(x) *' \alpha(y')$ and $t(x) = t(y)$ implies $t'(\alpha(x)) = t'(\alpha(y))$. In the particular case where $\Gamma = \Gamma'$, α is called an *automorphism* of Γ. An isomorphism (resp. automorphism) is said to be *type-preserving* if $t(x) = t'(\alpha(x))$ for any $x \in X$. We denote by $\mathrm{Aut}(\Gamma)$ the group of all type-preserving automorphisms of a geometry Γ. A *duality* is a non type-preserving automorphism of Γ such that $t(\alpha^2(x)) = t(x)$ for all $x \in \Gamma$. In this case, we call $\alpha(\Gamma)$ the *dual* of Γ and we denote it by Γ^*.

From now on, except in §3.5, isomorphisms and automorphisms are always assumed to be type-preserving.

2.1.4 Orders, thinness, firmness and thickness

Let Γ be a geometry over I and let $i, j \in I$ with $i \neq j$. We denote by N_i the cardinality of X_i. If the number of j-elements in the residue of x is independent of the choice of x in X_i, then we denote this number by $N_{i,j}$. If there is a number q_i such that each flag F of type $I\backslash\{i\}$ is incident with $q_i + 1$ elements of type i, then q_i is called the *i-order* of Γ. If q_i exists for any $i \in I$, then we say that Γ *has orders* $(q_i)_{i\in I}$. If $q_i = 1$ for every $i \in I$, then Γ is said to be *thin*.

Γ is said to be *firm* (resp. *thick*) if any non-maximal flag of Γ is contained in at least two (resp. three) maximal flags.

2.2 Parallelism

2.2.1 Definition

Let $\Gamma = (X, *, t)$ be a geometry over the type set I, with $|I| \geq 2$. We need to distinguish an element $0 \in I$ and we decide to call *points* the 0-elements of Γ. Next we require a binary equivalence relation $\parallel$ on $X\backslash X_0$ with the following properties:

(P1) $x \parallel y$ implies $t(x) = t(y)$ for all $x, y \in X\backslash X_0$;

(P2) for any points p, p' and elements $x, y, x', y' \in X\backslash X_0$, if $p * x * y * p$, $x' * p' * y'$, $x \parallel x'$ and $y \parallel y'$, then $x' * y'$.

We call $\parallel$ a *partial parallelism*. Note that for every element $x \in X\backslash X_0$ and every point p, there is at most one element $y \in \Gamma_p$ such that $x \parallel y$. Indeed, if $y \parallel x \parallel y'$ for $y, y' \in \Gamma_p$, then $y * y'$ by (P2) (with $x' = x$), hence $y = y'$ by (P1) and (1) of §2.1.

We call $\parallel$ a *parallelism* if, for every element $x \in X\backslash X_0$ and every point p, there is one element $y \in \Gamma_p$ such that $x \parallel y$. Then we call $(\Gamma, 0, \parallel)$ a *geometry with parallelism*. We also say that Γ is a *geometry with a 0-parallelism*.

2.2.2 Examples and comments

1. A classical affine geometry is obviously a geometry with parallelism.

2. Another well developed general context for parallelism goes as follows. Let (P, B) be a *block space* (also called *hypergraph*) namely a set of points P together with a family B of proper subsets of P called *blocks*, We define a *parallelism* on it as an equivalence relation $\parallel$ on B such that each equivalence class partitions P.

This subject is briefly surveyed in Buekenhout [7], chapter 3, §5.1 where references can be found. A block space with parallelism can be seen as a rank 2 geometry with parallelism. The converse holds true provided that the rank 2 geometry of points and blocks is such that no two blocks are incident with the same set of points and each block is incident with fewer than two points.

3. The block spaces with all blocks of size 2 are precisely graphs. A partial parallelism of a graph is called an *edge colouring* in graph theory. Similarly, parallelisms

of graphs correspond to 1-*factorizations* in graph theory. A rich literature exists on this topic and some important results are known, some since a long time.

4. Given any geometry Γ over the set of types I and $0 \in I$, we may wonder whether it admits a 0-parallelism. In §2.4 we give some necessary conditions in order that Γ admits a 0-parallelism and we show that they are not always satisfied. On the other hand, in section 5 we observe that there are many geometries with parallelism. There are also some cases where we do not know whether Γ admits a parallelism (see §5.4 for instance).

5. We have defined partial parallelism mainly with a thought for affine polar spaces (see Pasini [32]) but we shall not work further with partial parallelisms.

We do not seriously try to integrate buildings of affine type to the present context but it may be worth the effort in some future work.

2.2.3 The geometry at infinity

Let $(\Gamma, 0, \|)$ be a geometry with parallelism over the type set I, with $\Gamma = (X, *, t)$. Let x be an element of some type $i \neq 0$. The equivalence class of $\|$ containing x is denoted by $\infty(x)$ and we call it the *direction of* x or the *element at infinity* of x. We also call it an i-*direction*.

We define the *geometry at infinity* Γ^∞ of $(\Gamma, 0, \|)$ as follows. The set of types is $I \backslash \{0\}$. For $i \in I \backslash \{0\}$, the elements of type i are the i-directions. Incidence is defined by the following rule: if $x * y$, then $\infty(x) * \infty(y)$.

Theorem 2.1 *If $(\Gamma, 0, \|)$ is a geometry with parallelism over the type set I, then for any point p of Γ, the residue Γ_p is isomorphic to the geometry at infinity Γ^∞. In particular, Γ^∞ is a geometry.*

Proof. Straightforward. □

It will be clear from section 4 that every firm geometry can be viewed as the geometry at infinity of some geometry with parallelism. This is also implicit in a construction of Buekenhout, Dehon and De Schutter ([8], §3 example 2).

2.2.4 Trivial parallelism

Given a geometry Γ with 0-parallelism $\|$ and a type $i \neq 0$, we say that $\|$ is *trivial at* i if for any two elements x, y of type i, we have $x \| y$ only if $x = y$. A parallelism $\|$ is said to be *trivial* if it is trivial at every type $i \neq 0$.

Let Δ_0 be the connected component of 0 in a diagram Δ of Γ. Then for any $i \notin \Delta_0$, $\|$ is trivial at i. The same holds if every i-element of Γ is incident with every 0-element of Γ. In particular, if Γ is a geometry over the diagram Δ and if 0 is an isolated node of Δ, then Γ admits a unique 0-parallelism which is the trivial one. On the other hand, if Γ is a residually connected geometry over a finite diagram Δ, with a 0-parallelism which is trivial at i, then the types i and 0 are not joined in Δ. In particular, if $\|$ is the trivial parallelism on Γ, then 0 is an isolated node of Δ.

2.3 Truncations, residues and subgeometries

Let $(\Gamma, 0, \|)$ be a geometry with parallelism over the set of types I. The following lemmas are straightforward.

Lemma 2.2 *For any subset J of I with $0 \in J$ and $|J| \geq 2$, the J-truncation ${}^J\Gamma$ of Γ is a geometry with parallelism for the parallelism inherited from $\|$.*

Lemma 2.3 *For a flag F of Γ, with $0 \notin t(F)$ and $|I \backslash t(F)| \geq 2$, the residue Γ_F of F is a geometry with parallelism for the parallelism inherited from $\|$.*

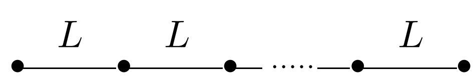
Lemma 2.4 *Let $\Gamma' = (X', *', t')$ be a subgeometry of Γ of rank ≥ 2 with $0 \in I' = t'(X')$. Assume that for every choice of $p, x \in X'$ with $t'(p) = 0$ and every $y \in \Gamma_p$, if $y \parallel x$ then $y \in X'$. Then Γ' is a geometry with parallelism.*

2.4 Some conditions for the existence of parallelisms

Let Γ be a geometry over I. In this section we introduce a few conditions that Γ must verify in order to admit a 0-parallelism.

First $\Gamma_p \cong \Gamma_q$ for any two points p, q. This is an obvious consequence of Theorem 2.1. Assume now that Γ is a finite geometry having orders. In this case, the number $N_{i,0}$ divides N_0 for any $i \in I \backslash \{0\}$, because the members of an i-direction partition the pointset of Γ. From this easy remark we deduce that a linear space Γ of orders (s, t) where s is the 0-order and $s+1$ does not divide t, cannot admit a 0-parallelism. This is the case for the linear spaces of orders $(1, q)$ with q odd and for every even dimensional projective geometry.

Another easy observation is that for any $i \in I \backslash \{0\}$, there must exist $N_i/N_{0,i}$ pairwise disjoint i-elements of Γ. For instance, let Γ belong to the following diagram

(L_n) 

Then Γ can admit a parallelism only if 0 is the left end node of the diagram.

By Lemma 2.2, the above properties must be verified in every residue containing 0-elements. Consequently, any geometry with a residual projective plane over $\{0, i\}$ for some i cannot admit a 0-parallelism, and so the A_n-, D_n-, E_n-, $\widetilde{A}_n$-geometries cannot admit any parallelism, as well as the truncations of rank n projective geometries on the subspaces of dimension $\leq j$, with $2 \leq j < n$.

2.5 Parallel-preserving isomorphisms

We recall that, according to the convention of §2.1.3, all isomorphisms and automorphisms are assumed to be type-preserving.

Let $(\Gamma, 0, \|)$ and $(\Gamma', 0, \|')$ be two geometries with 0-parallelism and with the same geometry at infinity. Each isomorphism α from Γ to Γ' maps $\|$ onto a 0-parallelism of Γ'. Indeed the relation $\|_\alpha$ defined on the elements of $X' \backslash X'_0$ by $x' \|_\alpha y'$ if and only if $\alpha^{-1}(x') \| \alpha^{-1}(y')$, is a 0-parallelism of Γ'.

An isomorphism $\alpha : \Gamma \longrightarrow \Gamma'$ is said to be *parallel-preserving* if the relations $\|_\alpha$ and $\|'$ coincide. In particular, an automorphism α of Γ is *parallel-preserving* if $\|$ and $\|_\alpha$ coincide.

It is straightforward to see that an isomorphism $\alpha : \Gamma \longrightarrow \Gamma'$ is parallel-preserving if and only if $x \parallel y$ implies $\alpha(x) \parallel \alpha(y)$. If α is parallel preserving, we also say that α is an *isomorphism* of $(\Gamma, 0, \|)$ onto $(\Gamma', 0, \|')$.

We denote by $\mathrm{Aut}(\Gamma, \|)$ the group of all parallel-preserving automorphisms of Γ.

A fundamental observation is that $\|$ may be sometimes built-in Γ already, a fact that we formalize by the property that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma, \|)$. In this case we call $(\Gamma, 0, \|)$ a *geometry with rigid parallelism.* This is of course the case for every geometry admitting a unique parallelism, and so in particular for any affine geometry.

The group $A = \mathrm{Aut}(\Gamma, \|)$ acts as an automorphism group on Γ^∞ where it induces a group A^∞ which can be the full automorphism group $\mathrm{Aut}(\Gamma^\infty)$. In the latter case Γ^∞ is called *complete*.

For instance, if Γ is an affine geometry of dimension $d \geq 3$, then Γ^∞ is complete. A typical situation where Γ^∞ is not complete is provided by the case where Γ is an affine plane and more generally, by the case where Γ is of rank 2. However, there are also non-complete geometries of higher rank (see §7.4.1).

The action on Γ^∞ of a parallel-preserving automorphism α will be denoted by α^∞. The kernel K^∞ of the homomorphism of A onto A^∞ is the group of *dilatations*, namely those parallel-preserving automorphisms that fix each element at infinity.

Clearly, each orbit of K^∞ on the set of elements of Γ not of type 0 is contained in one class of $\|$.

Theorem 2.5 *If K^∞ is transitive on the set of points of Γ, then its orbits on the set of elements of Γ not of type 0 are just the classes of $\|$.*

Proof. Given elements x, y of Γ with $x \parallel y$, let p, q be points incident with x and y respectively. If K^∞ is point-transitive, then there is an element β of K^∞ mapping p onto q. We have $\beta(x) \parallel x$. Hence $\beta(x) = y$, since both $\beta(x)$ and y are incident with the point q. □

The following is an easy consequence of Theorem 2.5:

Corollary 2.6 *Let K^∞ be point-transitive on Γ. Then A is the normalizer of K^∞ in Aut(Γ).*

Given a point p of Γ, let A_p be its stabilizer in A and let A_p^∞ be the image of A_p by the homomorphism of A onto A^∞.

Lemma 2.7 *If K^∞ is point-transitive on Γ, then $A_p^\infty = A^\infty$.*

Proof. Let K^∞ be point-transitive on Γ. Then, given any $\alpha \in A$, we can always find an element β of K^∞ such that $\beta\alpha \in A_p$. Clearly, $(\beta\alpha)^\infty = \alpha^\infty$. Hence $A_p^\infty = A^\infty$. □

2.6 J-parallelism

In this section, we introduce the concept of J-parallelism that will be useful in section 3. To define a J-parallelism, we replace 0-elements by flags of a fixed type J in the definition of 0-parallelism. More precisely, let $\Gamma = (X, *, t)$ be a geometry over the type set $I = J \cup K$, where J and K are disjoint and nonempty. The *J-pointed geometry associated to* Γ is the geometry Γ' over $\{0\} \cup K$ constructed from Γ by taking as 0-elements the flags of type J of Γ and as k-elements, $k \in K$, the k-elements of Γ, with the incidence inherited from Γ. Denote by X_K the set of elements of Γ whose type is in K. A *J-parallelism* on Γ is a binary equivalence relation $\|$ on X_K that defines a 0-parallelism of Γ'. The concepts of J-parallelism and 0-parallelism are very close, the only difference being about the objects we decide to take as points. Thus, we use the same notation and the same terminology for J-parallelism as for 0-parallelism (the symbol $(\Gamma, J, \|)$, the expressions "geometry at infinity", "parallel-preserving", etc. ...).

3 Gluing

As mentioned in the introduction, the idea of gluing is to start with a family of at least two geometries with parallelism whose geometries at infinity have been isomorphically identified and to construct a new geometry from these data.

3.1 The construction

Let I be a set of types of size at least 2 and let $0 \in I$. Let $\mathcal{G}$ be a family of geometries over I with a 0-parallelism, say $\mathcal{G} = \{(\Gamma_j, 0, \|_j)\}_{j \in J}$ where J is a finite set of n elements, $2 \leq n$. We assume that all geometries at infinity Γ_j^∞ are isomorphic to some given geometry Γ^∞ and we fix a family $\mathcal{A} = \{\alpha_j\}_{j \in J}$ of isomorphisms $\alpha_j : \Gamma_j^\infty \longrightarrow \Gamma^\infty$, which we call *matching isomorphisms*.

We now define a *glued geometry* or *gluing* $\Gamma = \Gamma(\mathcal{G}, \mathcal{A})$ over the set of types $(I \backslash \{0\}) \cup O_J$, where $O_J = \{0_j\}_{j \in J}$. For $j \in J$, the elements of Γ of type 0_j are the elements of Γ_j of type 0. As elements of type $i \in I \backslash \{0\}$, we take the n-tuples $(x_j)_{j \in J}$, where x_j is an element of Γ_j of type i and $\alpha_j(\infty(x_j)) = \alpha_h(\infty(x_h))$ for any $j, h \in J$. We decide that any two elements x, y of respective types 0_j, 0_k with $j \neq k$, are incident. Also, we decide that an element $(x_j)_{j \in J}$ of type $i \in I \backslash \{0\}$ and an element y of type 0_j are incident precisely when $y * x_j$ in Γ_j. Finally, we put $(x_j)_{j \in J} * (y_j)_{j \in J}$ if and only if $x_j * y_j$ in Γ_j, for all $j \in J$.

For the rest of this section we develope a theory of gluings, postponing the discussion of examples to section 6.

3.2 A natural parallelism in glued geometries

Let Γ be a glued geometry, with set of types $(I \backslash \{0\}) \cup O_J$. Given $i \in I \backslash \{0\}$, let $x = (x_j)_{j \in J}$ be an i-element of Γ. We have $\alpha_j(\infty(x_j)) = \alpha_k(\infty((x_k))$ for $j, k \in J$ by the definition of glued geometries. Thus, we set $\infty(x) = \alpha_j(\infty(x_j))$.

This observation allows us to define a O_J-parallelism $\|_J$ on Γ by stating that, for any two elements x, y of Γ of the same type $i \in I\backslash\{0\}$, we have $x \|_J y$ if and only if $\infty(x) = \infty(y)$. Clearly, the geometry at infinity of $(\Gamma, O_J, \|_J)$ is isomorphic to Γ^∞.

From now on, we will use the expression "parallel-preserving" when dealing with the geometries $(\Gamma_j, 0, \|_j)$. When speaking of $(\Gamma, O_J, \|_J)$, we will use the expression "$\|_J$-preserving".

3.3 Residues and diagrams

Given $\mathcal{G}$ and $\mathcal{A}$ as in 3.1 and $j \in J$, we set $\mathcal{G}_j = \mathcal{G}\backslash\{\Gamma_j\}$ and $\mathcal{A}_j = \mathcal{A}\backslash\{\alpha_j\}$. Provided that $n > 2$, there is an obvious gluing $\Gamma(\mathcal{G}_j, \mathcal{A}_j)$. We extend this notation to the case $n = 2$ by the convention that $\Gamma(\mathcal{G}_j, \mathcal{A}_j)$ is $\mathcal{G}_j$. The following is straightforward

Theorem 3.1 *Let $\Gamma = \Gamma(\mathcal{G}, \mathcal{A})$ be a glued geometry and let p be an element of type 0_j in Γ. Then $\Gamma_p \cong \Gamma(\mathcal{G}_j, \mathcal{A}_j)$.*

We shall now describe residues of elements of type $i \notin O_J$ in a glued geometry $\Gamma = \Gamma(\mathcal{G}, \mathcal{A})$. If $|I|= 2$, then the residues of the elements of Γ of the unique type of $I\backslash\{0\}$ are just direct sums of geometries of rank 1.

Assume $|I|> 2$. For every $j \in J$, let $\Gamma_{j,x}$ be the residue of x_j in Γ_j with the parallelism inherited from $\|_j$ (Lemma 2.3) and let $\alpha_{j,x} : \Gamma_{j,x} \longrightarrow \Gamma^\infty_{\infty(x)}$ be the restriction of α_j to $\Gamma_{j,x}$. Put $\mathcal{G}_x = \{\Gamma_{j,x}\}_{j\in J}$ and $\mathcal{A}_x = \{\alpha_{j,x}\}_{j\in J}$. Clearly, there is a gluing $\Gamma(\mathcal{G}_x, \mathcal{A}_x)$. It is straightforward to prove that $\Gamma_x \cong \Gamma(\mathcal{G}_x, \mathcal{A}_x)$.

Theorem 3.1 and the construction of the glued geometry Γ allow to derive a diagram for Γ, from diagrams Δ_j for the Γ_j, $j \in J$ in which the same diagram is induced on $I\backslash\{0\}$. Applying Theorem 3.1 inductively over $j \in J$, we see that a diagram for Γ is obtained by pasting the diagrams Δ_j, $j \in J$, over $I\backslash\{0\}$.

3.4 Isomorphisms and automorphisms

We recall that, according to the convention of §2.1.3, all isomorphisms and automorphisms are assumed to be type-preserving.

3.4.1 Gluing families of isomorphisms

Let $\mathcal{G} = \{(\Gamma_j, 0, \|_j)\}_{j\in J}$ and $\mathcal{G}' = \{(\Gamma'_j, 0, \|'_j)\}_{j\in J}$ be two families of geometries with parallelism, over the same set of types I, with the same selected type 0 and the same geometry at infinity Γ^∞. Let $\mathcal{A} = \{\alpha_j\}_{j\in J}$ and $\mathcal{A}' = \{\alpha'_j\}_{j\in J}$ be families of matching isomorphisms $\alpha_j : \Gamma^\infty_j \longrightarrow \Gamma^\infty$ and $\alpha'_j : (\Gamma'_j)^\infty \longrightarrow \Gamma^\infty$, and let $\mathcal{F} = \{\varphi_j\}_{j\in J}$ be a family of parallel-preserving isomorphisms

$$\varphi_j : (\Gamma_j, 0, \|_j) \longrightarrow (\Gamma'_j, 0, \|'_j).$$

For every $j \in J$, φ_j induces an isomorphism φ^∞_j from Γ^∞_j to $(\Gamma'_j)^\infty$, uniquely determined by the following clause: $\infty(\varphi_j(x)) = \varphi^\infty_j(\infty(x))$ for every element x of Γ_j of type $i \neq 0$.

If $\alpha'_j\varphi^\infty_j\alpha^{-1}_j$ is independent on the choice of $j \in J$, we put $\varphi = \alpha'_j\varphi^\infty_j\alpha^{-1}_j$ and we say that $\mathcal{F}$ *has a unique action on* Γ^∞. In this case we can define an isomorphism

$\Phi_{\mathcal{F}}$ from the glued geometry $\Gamma = \Gamma(\mathcal{G}, \mathcal{A})$ to the glued geometry $\Gamma' = \Gamma(\mathcal{G}', \mathcal{A}')$ as follows. If x is an element of Γ of type 0_j, then we put $\Phi_{\mathcal{F}}(x) = \varphi_j(x)$. Given an element $x = (x_j)_{j\in J}$ of Γ of type $i \notin O_J$, we set $\Phi_{\mathcal{F}}(x) = (\varphi_j(x_j))_{j\in J}$. It is straightforward to check that $\Phi_{\mathcal{F}}$ is indeed an isomorphism. We call it the *gluing* of $\mathcal{F}$, also a *glued isomorphism.*

Notice that starting from a glued isomorphism $\Phi : \Gamma \longrightarrow \Gamma'$ we can uniquely reconstruct the family $\mathcal{F}$ of which Φ is the gluing. Furthermore, if $\Phi : \Gamma \longrightarrow \Gamma'$ is a glued isomorphism, then $\infty(x) = \infty(y)$ implies $\infty(\Phi(x)) = \infty(\Phi(y))$ for any two elements x, y of Γ of type $i \notin O_J$, in other words, Φ is $\|_J$-preserving.

Lemma 3.2 *An isomorphism Φ from Γ to Γ' is a glued isomorphism if and only if it is $\|_J$-preserving.*

Proof. We have already observed that the "only if" claim is true. Let us prove the "if" statement. Let $\Phi : \Gamma \longrightarrow \Gamma'$ be a $\|_J$-preserving isomorphism. We shall define a family $\mathcal{F} = \{\varphi_j\}_{j\in J}$ of parallel-preserving isomorphisms

$$\varphi_j : (\Gamma_j, 0, \|_j) \longrightarrow (\Gamma'_j, 0, \|'_j)$$

such that $\Phi = \Phi_{\mathcal{F}}$. Every element x of Γ_j of type 0 can be viewed as an element of Γ of type 0_j. We set $\varphi_j(x) = \Phi(x)$ for every such element.

Given $k \in J$ and an element z of Γ_k of type $i \in I\backslash\{0\}$, we choose an element $x = (x_j)_{j\in J}$ of Γ such that $x_k = z$. Let $(x'_j)_{j\in J} = \Phi(x)$. We set $\varphi_k(z) = x'_k$. This clause defines a function. Indeed, let $y = (y_j)_{j\in J}$ be another element of Γ of type i with $y_k = z$ and let $(y'_j)_{j\in J} = \Phi(y)$. As Φ is $\|_J$-preserving and $x_k = y_k = z$, we have $\infty(x'_k) = \infty(y'_k)$. On the other hand, if p is an element of Γ_k of type 0 incident with z, then $x * p * y$ in Γ. Hence $\Phi(x) * \Phi(p) * \Phi(y)$. That is, $x'_k * \varphi_k(p) * y'_k$. Therefore $x'_k = y'_k$, since these elements are parallel and incident with the same point of Γ'_k. Thus φ_k is well-defined.

It is not difficult to check that φ_k is in fact an isomorphism from $(\Gamma_k, 0, \|_k)$ to $(\Gamma'_k, 0, \|'_k)$. Moreover, $\alpha'_j\varphi_j^\infty\alpha_j^{-1}(\infty(x)) = \infty(\Phi(x))$ for any i-element x with $i \in I\backslash\{0\}$. Consequently, we can define the glued isomorphism $\Phi_{\mathcal{F}}$ where $\mathcal{F} = \{\varphi_j\}_{j\in J}$. It is easy to see that $\Phi = \Phi_{\mathcal{F}}$. □

3.4.2 Isomorphisms and the property (O)

Let (O) denote the following axiom:

(O) distinct elements of the same type have different 0-shadows.

Let $\Gamma = \Gamma(\mathcal{G}, \mathcal{A})$ and $\Gamma' = \Gamma(\mathcal{G}', \mathcal{A}')$ as in the previous paragraph §3.4.1.

Theorem 3.3 *If all members of $\mathcal{G}'$ verify axiom (O), then all isomorphisms from Γ to Γ' are glued isomorphisms.*

Proof. Let (O) hold in Γ'_j for every $j \in J$ and let Φ be an isomorphism from Γ to Γ'. We shall show that Φ is $\|_J$-preserving. Then the conclusion will follow from Lemma 3.2. Given $x = (x_j)_{j\in J}$ and $y = (y_j)_{j\in J}$ with $\infty(y) = \infty(x)$, let $(x'_j)_{j\in J} = \Phi(x)$ and $(y'_j)_{j\in J} = \Phi(y)$. We firstly suppose $x_k = y_k$ for some $k \in J$. We have $\sigma_0(x'_k) = \Phi(\sigma_0(x_k))$ and $\sigma_0(y'_k) = \Phi(\sigma_0(y_k))$ because Φ is an isomorphism and the elements of Γ_k and Γ'_k of type 0 can be viewed as elements of type 0_k of Γ and Γ' respectively. Hence x'_k and y'_k have the same 0-shadow. Therefore $x'_k = y'_k$ by (O). Thus, $\infty(\Phi(x)) = \alpha'_k(\infty(x'_k)) = \alpha'_k(\infty(y'_k)) = \infty(\Phi(y))$.

Let now $x_j \neq y_j$ for every $j \in J$. Choose $k \in J$ and set $z_j = x_j$ if $j \neq k$ and $z_k = y_k$. Then $z = (z_j)_{j\in J}$ is an element of Γ, because $\alpha_j(\infty(x_j)) = \alpha_k(\infty(y_k)) = \infty(x) = \infty(y)$. Clearly, $\infty(z) = \infty(x) = \infty(y)$. The previous argument applied to x and z and to z and y now yields $\infty(\Phi(x)) = \infty(\Phi(z)) = \infty(\Phi(y))$. □

3.4.3 Isomorphism classes of gluings

In this paragraph we consider one family $\mathcal{G} = \{(\Gamma_j, 0, \|_j)\}_{j\in J}$ of geometries with parallelism and two families $\mathcal{A} = \{\alpha_j\}_{j\in J}$ and $\mathcal{B} = \{\beta_j\}_{j\in J}$ of matching isomorphisms. Thus $\Gamma = \Gamma(\mathcal{G}, \mathcal{A})$ and $\Gamma' = \Gamma(\mathcal{G}, \mathcal{B})$ are two gluings of the same family of geometries. As in §2.5, we denote by A_j^∞ the action at infinity of $A_j = \mathrm{Aut}(\Gamma_j, \|_j)$. We set $\alpha_j(A_j^\infty) = \{\alpha_j\beta\alpha_j^{-1} \mid \beta \in A_j^\infty\}$. Clearly, $\alpha_j(A_j)$ is a subgroup of the group $\mathrm{Aut}(\Gamma^\infty)$ of automorphisms of Γ^∞.

Observe that for each automorphism ϵ of Γ^∞, $\Gamma = \Gamma(\mathcal{G}, \mathcal{A}) \cong \Gamma' = \Gamma(\mathcal{G}, \mathcal{A}')$, where $\mathcal{A}' = \{\epsilon\alpha_j\}_{j\in J}$. Indeed, it is straightforward to construct a glued isomorphism from Γ onto Γ' (intuitively, the construction of $\Gamma(\mathcal{G}, \mathcal{A})$ is independent of the choice of a geometry in the isomorphism class of Γ^∞).

Lemma 3.4 *Let all members of $\mathcal{G}$ satisfy property (O). Then $\Gamma \cong \Gamma'$ if and only if $\alpha_j\beta_j^{-1}\beta_k\alpha_k^{-1} \in \alpha_j(A_j^\infty)\alpha_k(A_k^\infty)$ for any two distinct indices $j, k \in J$.*

Proof. By (O) and Theorem 3.3, $\Gamma \cong \Gamma'$ if and only if there are automorphisms $\varphi_j^\infty \in A_j^\infty$ (for $j \in J$) such that $\beta_k\varphi_k^\infty\alpha_k^{-1} = \beta_j\varphi_j^\infty\alpha_j^{-1}$ for a given $k \in J$ and for each $j \in J$. These conditions are equivalent to the following: $\beta_j^{-1}\beta_k \in A_j^\infty\alpha_j^{-1}\alpha_k A_k^\infty$ for every $j \in J$. The statement is now evident. □

We say that $\mathcal{G}$ *admits a unique gluing* if, for any to families $\mathcal{A}$, $\mathcal{B}$ of matching isomorphisms, there is an isomorphism between $\Gamma(\mathcal{G}, \mathcal{A})$ and $\Gamma(\mathcal{G}, \mathcal{B})$.

We say that a group G *admits factorization* over a family $\{G_j\}_{j\in J}$ of its subgroups if $G = G_k . \bigcap_{j\in J\setminus\{k\}} G_j$ for every $k \in J$.

Theorem 3.5 *Let (O) hold in all members of $\mathcal{G}$. Then the following are equivalent:*

(i) $\mathcal{G}$ admits a unique gluing;

(ii) the group Aut(Γ^∞) admits factorization over $\{\alpha_j(A_j^\infty)\}_{j\in J}$ for some family $\{\alpha_j\}_{j\in J}$ of matching isomorphisms;

(iii) the group Aut(Γ^∞) admits factorization over $\{\alpha_j(A_j^\infty)\}_{j\in J}$ for every family $\{\alpha_j\}_{j\in J}$ of matching isomorphisms.

Proof. Clearly, (iii) implies (ii). Let us prove that (ii) implies (i). We use induction on $n = |J|$. If $n = 1$ there is nothing to prove.

Let $n > 1$ and let $\mathcal{A} = \{\alpha_j\}_{j\in J}$ be as in (ii). Let $\mathcal{B} = \{\beta_j\}_{j\in J}$ be any other family of matching isomorphisms and choose $k \in J$. We shall prove that $\Gamma(\mathcal{G},\mathcal{A}) \cong \Gamma(\mathcal{G},\mathcal{B})$.

By induction, $\mathcal{G}_k = \mathcal{G}\backslash\{\Gamma_k\}$ admits a unique gluing. By Theorem 3.3, there are elements φ_j of A_j (for $j \in J\backslash\{k\}$) and $\varphi \in \mathrm{Aut}(\Gamma^\infty)$ such that

$$(1)\quad \beta_j\varphi_j^\infty = \varphi\alpha_j$$

for every $j \in J\backslash\{k\}$. Put $\gamma = \beta_k\alpha^{-1}\varphi^{-1}$. As $\mathrm{Aut}(\Gamma^\infty)$ admits factorization over $\mathcal{A}$, there are elements ψ_j of A_j and $\theta \in \mathrm{Aut}(\Gamma^\infty)$ such that

$$(2)\quad \varphi\alpha_j\psi_j^\infty\alpha_j^{-1}\varphi^{-1} = \theta$$

for every $j \in J\backslash\{k\}$ and

$$(3)\quad \theta\varphi\alpha_k\varphi_k^\infty\alpha_k^{-1}\varphi^{-1} = \gamma.$$

By (1) and (2) we get

$$(4)\quad \theta\varphi\alpha_j = \beta_j\varphi_j^\infty\psi_j^\infty$$

whereas (3) gives us

$$(5)\quad \theta\varphi\alpha_k = \gamma\varphi\alpha_k(\psi_k^\infty)^{-1} = \beta_k(\psi_k^\infty)^{-1}.$$

(4) and (5) show that we can glue the automorphisms ψ_k^{-1} and $\varphi_j\psi_j$ (with $j \in J\backslash\{k\}$) thus obtaining a glued isomorphism $\Phi : \Gamma(\mathcal{G},\mathcal{A}) \longrightarrow \Gamma(\mathcal{G},\mathcal{B})$ with $\theta\varphi$ as its "action at infinity". Thus, (i) is proved.

Finally, let $\mathcal{G}$ admit a unique gluing. Let $\mathcal{A} = \{\alpha_j\}_{j\in J}$ be any family of matching isomorphisms. Given an automorphism ψ of Γ^∞ and an index $k \in J$, we consider the family $\mathcal{A}_k = \{\psi\alpha_k\} \cup \{\alpha_j\}_{j\in J\backslash\{k\}}$. Since $\mathcal{G}$ admits a unique gluing, $\Gamma(\mathcal{G},\mathcal{A}) \cong \Gamma(\mathcal{G},\mathcal{A}_k)$. By Theorem 3.3 there are elements $\varphi_j \in A_j$ and an automorphism φ of Γ^∞ such that $\varphi\alpha_j = \alpha_j\varphi_j^\infty$ for $j \in J\backslash\{k\}$ and $\varphi\psi\alpha_k = \alpha_k\varphi_k^\infty$. Therefore $\varphi = \alpha_j\varphi_j^\infty\alpha_j^{-1}$ for every $j \in J\backslash\{k\}$. Furthermore $\psi = \varphi^{-1}\alpha_k\varphi_k^\infty\alpha_k^{-1}$. Hence $\varphi \in \bigcap_{j\in J\backslash\{k\}}\alpha_j(A_j^\infty)$. Therefore

$$\psi \;\in\; \Big(\bigcap_{j\in J\backslash\{k\}}\alpha_j(A_j^\infty)\Big)\cdot\alpha_k(A_k^\infty).$$

Since ψ and k are arbitrary elements of $\mathrm{Aut}(\Gamma^\infty)$ and J, $\mathrm{Aut}(\Gamma^\infty)$ admits factorization over $\{\alpha_j(A_j^\infty)\}_{j\in J}$. □

3.4.4 Automorphism groups of glued geometries

Again, let $\mathcal{G} = \{(\Gamma_j, 0, \|_j)\}_{j\in J}$ be a family of geometries with parallelism, $\mathcal{A} = \{\alpha_j\}_{j\in J}$ a family of matching isomorphisms and let $\Gamma = \Gamma(\mathcal{G},\mathcal{A})$.

Note that, by Theorem 3.3, if (O) holds in all members of $\mathcal{G}$, then Γ has a rigid parallelism (i.e. $\mathrm{Aut}(\Gamma, \|_J) = \mathrm{Aut}(\Gamma)$).

Let A_j^∞ and $\alpha_j(A_j^\infty)$ be as in §3.4.4 and let K_j^∞ be the group of dilatations of $(\Gamma_j, 0, \|_j)$ (see §2.5). We set $A^* = \bigcap_{j\in J}\alpha_j(A_j^\infty)$. The following is a straightforward consequence of Lemma 3.2

Theorem 3.6 *$Aut(\Gamma, \|_J) = (\prod_{j\in J}K_j^\infty).A^*$, where A^* acts on the direct product $\prod_{j\in J}K_j^\infty$ stabilizing each of its factors. Furthermore, A^* acts as $\alpha_j^{-1}(A^*)$ on K_j^∞, for every $j \in J$.*

Corollary 3.7 *Assume that, for some $k \in J$, the subgroup $K_k^\infty.\alpha_k^{-1}(A^*)$ of A_k is flag-transitive on Γ_j for some $k \in J$ and K_j^∞ is point-transitive on Γ_j for every $j \in J\backslash\{k\}$. Then $Aut(\Gamma, \|_J)$ is flag-transitive on Γ.*

Proof. Easy, by Theorem 3.8. □

Corollary 3.8 *Let K_j^∞ be transitive on the set of points of Γ_j for every $j \in J$. Then $Aut(\Gamma, \|_J)$ is flag-transitive on Γ if and only if A^* is flag-transitive on Γ^∞.*

Proof. Easy, by Lemma 2.7. □

3.4.5 Gluing two copies of a geometry with parallelism

In this section, we consider the particular case where $\mathcal{G} = \{(\Gamma_j, 0, \|_j)_{j=1,2}$ and where $(\Gamma_1, 0, \|_1)$ and $(\Gamma_2, 0, \|_2)$ are two copies of a geometry with parallelism $(\Gamma, 0, \|)$. Clearly, we can assume without loss of generality that $\Gamma_1^\infty = \Gamma_2^\infty = \Gamma^\infty$ and $A_1^\infty = A_2^\infty = A^\infty$, where A^∞ is the action of $\mathrm{Aut}(\Gamma, \|)$ on Γ^∞, as in §2.5. Thus, the matching isomorphisms α_j are just automorphisms of Γ^∞. Consequently, using an observation made at the beginning of §3.4.3, every gluing $\Gamma(\mathcal{G}, \mathcal{A})$ is isomorphic to a gluing $\Gamma_\alpha = \Gamma(\mathcal{G}, \{id, \alpha\})$ where *id* is the identity automorphism of Γ^∞ and α is an automorphism of Γ^∞.

Theorem 3.9 *The isomorphism classes of gluings of two copies of $(\Gamma, 0, \|)$ bijectively correspond to the double cosets $A^\infty \alpha A^\infty$, $\alpha \in Aut(\Gamma^\infty)$.*

Proof. Let Γ_α and Γ_β be two gluings. By Lemma 3.4, $\Gamma_\alpha \cong \Gamma_\beta$ if and only if $\beta \in A^\infty \alpha A^\infty$. □

Corollary 3.10 *Assume that (O) holds in Γ. There is a unique way to glue two copies of $(\Gamma, 0, \|)$ if and only if Γ^∞ is complete.*

Proof. Easy, by Theorem 3.9. □

By Theorem 3.6 we have $\mathrm{Aut}(\Gamma_\alpha, \|_J) = (K^\infty \times K^\infty).(A^\infty \cap \alpha(A^\infty))$. Thus, $\mathrm{Aut}(\Gamma_\alpha, \|_J)$ is "maximal" (that is, as large as possible) precisely when α normalizes A^∞. In particular, if $\alpha \in A^\infty$, then $\mathrm{Aut}(\Gamma_\alpha, \|_J)$ is "maximal".

It follows from Theorem 3.9 that $\alpha \in A^\infty$ if and only if $\Gamma_\alpha \cong \Gamma_{id}$. We call Γ_{id} the *canonical* gluing of two copies of $(\Gamma, 0, \|)$.

3.5 Twisted gluings

We have assumed that matching isomorphisms are type-preserving in the previous sections. However, we can also drop that requirement in the definition of gluings.

Thus, given I, 0, $\mathcal{G}$ and Γ^∞ as in §3.1, we consider a family $\mathcal{A} = \{\alpha_j\}_{j \in J}$ of possibly non type-preserving isomorphisms $\alpha_j : \Gamma_j^\infty \longrightarrow \Gamma^\infty$, which we still call *matching isomorphisms*. Denoting by τ_j the permutation induced by α_j on $I\backslash\{0\}$, we set $\mathcal{T} = \{\tau_j\}_{j \in J}$ and we call $\mathcal{T}$ the set of *type-permutations* induced by $\mathcal{A}$.

We generalize the definition of §3.1 as follows. When defining an element $x = (x_j)_{j\in J}$ of type $i \in I\backslash\{0\}$, we assume that $\tau_j(t(x_j)) = i$ for every $j \in J$ instead of $t(x_j) = i$. All the rest is as in §3.1.

We still use the symbol $\Gamma(\mathcal{G}, \mathcal{A})$ for the geometry obtained by this construction. We call $\Gamma(\mathcal{G}, \mathcal{A})$ a $\mathcal{T}$-*glued* geometry. Gluings as defined in §3.1 will be called *plain*, when the context will not suffice to make it clear that we are speaking of them.

Clearly, we can turn every non-plain gluing $\Gamma(\mathcal{G}, \mathcal{A})$ into a plain one by applying τ_j to the type set of Γ_j^∞, for every $j \in J$. Consequently, as long as we do not want to identify the members of $\mathcal{G}$, we can always assume that $\Gamma(\mathcal{G}, \mathcal{A})$ is a plain gluing, and so everything we said on plain gluings in §§3.2-3.4 holds for non-plain gluings (modulo some obvious changes), except in §3.4.5, where some identification is assumed between the geometries to glue. In contexts like this we really need to distinguish between non-plain and plain gluings.

Let $\Gamma(\mathcal{G}, \mathcal{A})$ be a $\mathcal{T}$-glued geometry. If there are distinct types j, k in J such that $\tau_j \neq \tau_k$, then $\Gamma(\mathcal{G}, \mathcal{A})$ is called a *twisted* gluing.

We now adapt statement 3.9 for twisted gluings. From now on, $(\Gamma, 0, \|)$ is a geometry with parallelism over the type set I. We assume that Γ verifies (O). An argument as in the proof of Lemma 3.4 yields the following:

Theorem 3.11 *If σ and τ are distinct permutations of $I\backslash\{0\}$ induced by automorphisms of Γ^∞, then the isomorphism classes of $\{\sigma,\tau\}$-gluings of two copies of $(\Gamma, 0, \|)$ bijectively correspond to the double cosets $A^\infty\alpha A^\infty$, where α ranges over a right coset $\varphi.Aut(\Gamma^\infty)$ of the group $Aut(\Gamma^\infty)$ of type-preserving automorphisms of Γ^∞, with φ an automorphism of Γ^∞ inducing $\sigma^{-1}\tau$ on $I\backslash\{0\}$.*

Permuting the types in $I\backslash\{0\}$ if necessary, we can always assume that $\sigma = id$. Namely, every $\{\sigma, \tau\}$-gluing with $\sigma \neq id$ is isomorphic to a $\{id, \sigma^{-1}\tau\}$-gluing via some non type-preserving isomorphism. By this remark and by Theorem 3.11 we obtain the following generalization of Theorem 3.9 for (possibly twisted) gluings:

Corollary 3.12 *The number of non-isomorphic (possibly twisted) ways of gluing two copies of $(\Gamma, 0, \|)$ equals the number of double cosets $A^\infty\alpha A^\infty$ in the group of all (possibly non type-preserving) automorphisms of Γ^∞.*

3.6 Parallelisms in glued geometries

In §3.2, we defined a O_J-parallelism in every glued geometry $\Gamma = \Gamma(\mathcal{G}, \mathcal{A})$ over $(I\backslash\{0\}) \cup O_J$. By definition, the O_J-pointed geometry associated to Γ is a geometry over I with a 0-parallelism. Consequently, starting from a family $\mathcal{G} = \{(\Gamma_j, 0, \|_j)\}_{j\in J}$ of geometries over I with a 0-parallelism and whose geometries at infinity are isomorphic to Γ^∞, we can construct a new geometry Γ' over I with a 0-parallelism and such that Γ'^∞ is isomorphic to Γ^∞.

We can also define 0_k-parallelisms $\|_k$ on Γ as follows. Let $k \in J$. Given two elements $x = (x_j)_{j\in J}$ and $y = (y_j)_{j\in J}$ of Γ of type $i \in I\backslash\{0\}$ we set $x \|_k y$ if $x_j = y_j$ for every $j \in J\backslash\{k\}$. We decide that $\|_k$ is trivial at every type 0_j, $j \in J\backslash\{k\}$ (see §2.6). It is easy to check that the relation $\|_k$ defined in this way is actually a parallelism of Γ.

Notice that the geometry at infinity of $(\Gamma, 0_k, \|_k)$ is isomorphic to $\Gamma(\mathcal{G}_k, \mathcal{A}_k)$. (This follows from theorems 2.1 and 3.1.)

4 Parallel expansion

4.1 The setting

In [8] a very general construction of geometries is given, starting from an affine space A, some geometry Γ and an injective mapping from the set of "points" of Γ into the set of points at infinity of A. This method provides many interesting diagram geometries. A classical ancestor of this method is to start rather with a subgeometry Γ of the projective space at infinity of A. Then the *affine expansion* of Γ is described and illustrated in [30] (2.3). The idea of the construction is to consider all affine subspaces of A whose subspace at infinity is a member of Γ.

In this section we generalize the construction of [8] as we replace A by any geometry with parallelism. We do also slightly modify the construction of [8] and so, formally speaking, it is a variation of [8] even in the affine case.

4.2 The initial data

Let $(A, 0, \|)$ be a firm geometry with parallelism over a set of types I with $0 \in I$ and $1 \in I \backslash \{0\}$. This is the geometry in which our process of expansion will occurr. We shall need its "points at infinity". This is the reason to distinguish a second type 1 in I.

The next data is the geometry we want to expand in A. Let Γ be a firm geometry over some set of types $J \cup K$, with $J \cap K = \emptyset$ and $K \neq \emptyset$. The k-elements of Γ with $k \in K$ are the elements we shall relate to the 1-elements of A^∞. We assume that $0 \notin J \cup K$.

For each $k \in K$, let α_k be an injective mapping of the set X_k of k-elements of Γ into the set of 1-elements of A^∞. These are the mappings relating the k-elements of Γ with $k \in K$ to the 1-elements of A^∞. We assume that for each $k \neq k'$ in K, $\alpha_k(X_k) \cap \alpha_{k'}(X_{k'}) = \emptyset$.

Given a flag F of Γ, we set $B^\infty(F) = \bigcup_{k \in K} \alpha_k(\sigma_k(F))$.

4.3 Flats and dense sets

The following definitions are needed in view of the construction we shall describe in §4.4.

Given any set B^∞ of elements of A^∞, every 0-element a of A determines a *cone with vertex a and basis* B^∞, say $A(a, B^\infty)$, consisting of all elements y incident with a and such that $\infty(y) \in B^\infty$.

A set S of 0-elements of A is called B^∞*-closed* if, for any $a \in S$ and any $y \in A(a, B^\infty)$, all 0-element of A incident with y belong to S. Intersections of B^∞-closed sets are B^∞-closed. Hence any set of 0-elements has a B^∞*-closure*. We call the B^∞-closure of a 0-element a B^∞*-flat*. A B^∞-flat is the B^∞-closure of any of its elements.

Hence, given a set B^∞ of elements of A^∞, every 0-element of A belongs to precisely one B^∞-flat. Thus, it is natural to say that the B^∞-flats are mutually *parallel*.

$A(a, B^\infty)$ is a geometry if and only if B^∞ is a geometry. If this is the case, a B^∞-flat S is always *connected*, in this sense: the set $A(S) = \bigcup_{a \in S} A(a, B^\infty)$ with the incidence inherited from A is a connected geometry. Let $A(S)$ be a geometry. Then the 0-parallelism of A induces a 0-parallelism on $A(S)$ and $A(S)^\infty = B^\infty$.

B^∞ is called *dense* if the set of all 0-elements of A is one B^∞-flat (hence it is the unique B^∞-flat in A).

For example if A is an affine plane, then any subset of A^∞ of cardinality ≥ 2 is dense. If A is the point-line system of $\mathrm{AG}(n, K)$, then A^∞ is the set of points of the geometry at infinity $\mathrm{PG}(n-1, K)$ of $\mathrm{AG}(n, K)$ and the dense subsets of A^∞ are those spanning A^∞ in $\mathrm{PG}(n-1, K)$.

4.4 The construction

Starting with the triple $((A, 0, \|), \Gamma, \{\alpha_k\}_{k \in K})$ as in §4.2, we want to define a geometry with parallelism $(\overline{\Gamma}, 0, \|')$ over $J \cup K \cup \{0\}$.

We define the 0-elements of $\overline{\Gamma}$ as the 0-elements of A. For $k \in K$, the k-elements of $\overline{\Gamma}$ are the $\alpha_k(x_k)$-flats where x_k is a k-element of Γ. In other words, the k-elements of $\overline{\Gamma}$ are the 1-elements L of A such that $\infty(L) \in \alpha_k(X_k)$. We denote an $\alpha_k(x_k)$-flat by $a(x_k)$ or $a'(x_k)$ or ... For $j \in J$ the j-elements of $\overline{\Gamma}$ are the pairs $a(x_j) = (b(x_j), x_j)$ where x_j is a j-element of Γ and $b(x_j)$ is a $B^\infty(x_j)$-flat. The type function of $\overline{\Gamma}$ will be denoted by t, as the one of Γ.

For $h \in K$ (resp. $h \in J$), a h-element $a(x_h)$ is declared to be incident with all its 0-elements (resp. all the 0-elements of $b(x_h)$). Let $h, h' \in J \cup K$. We declare $a(x_h)$ and $a(y_{h'})$ to be incident in $\overline{\Gamma}$ if and only if $a(x_h)$ and $a(y_{h'})$ have a common incident 0-element and $x_h * y_{h'}$ in Γ. We say that $a(x_h) \|' y_{h'}$ if and only if $x_h = y_{h'}$.

As in [8] we can have trouble with $\overline{\Gamma}$ of rank ≥ 3 in the sense that there might be maximal flags with no 0-element. Moreover, if $K \geq 2$ there could be flags of type K incident with exactly one 0-element. We restrict the data of §4.2 in order to avoid these situations and we call $(\overline{\Gamma}, 0, \|')$ the *parallel expansion of* Γ *in* $(A, 0, \|)$ *via* $\{\alpha_k\}_{k \in K}$ provided that

(i) for every flag F of $\overline{\Gamma}$ there is a 0-element incident with all members of F;

(ii) every flag of $\overline{\Gamma}$ of type K is incident with at least two 0-elements.

If $|K|= 1$, condition (ii) always holds because A is assumed to be firm.

4.5 Some properties of parallel expansions

In [8] a theory is developed in order to ensure that $\overline{\Gamma}$ is a parallel expansion under various suitable conditions. Here, we are not trying to extend that theory to the present construction although it would be a valuable task. We only state the following.

Theorem 4.1 *Let $(\overline{\Gamma}, 0, \|')$ be the parallel expansion of a geometry Γ in $(A, 0, \|)$ via a family $\{\alpha_k\}_{k\in K})$ of injections as in 4.4. Then the following hold:*

(1) *$\overline{\Gamma}$ is a geometry over $J \cup K \cup \{0\}$ and $\overline{\Gamma}_p \cong \Gamma$ for any 0-element p of $\overline{\Gamma}$;*
(2) *$\overline{\Gamma}$ is firm;*
(3) *$(\overline{\Gamma}, 0, \|')$ is a geometry with parallelism and $\overline{\Gamma}^\infty \cong \Gamma$;*
(4) *$\overline{\Gamma}$ belongs to a $(0, K, \Delta)$-diagram, with Δ a diagram for Γ;*

(5) *assuming that $J \cup K$ is finite, $\overline{\Gamma}$ is residually connected if and only if the following are realized:*
(5.1) *Γ is residually connected;*
(5.2) *$\bigcup_{k\in K} \alpha_k(X_k)$ is dense in A^∞;*
(5.3) *for every flag F of Γ such that $t(F) \cap K = \emptyset$ and for every point p of A, the $B^\infty(F)$-flat containing p is the intersection of the $B^\infty(f)$-flats containing p for $f \in F$.*

Proof. (1) In view of (i) of §4.4, if M is a maximal flag of $\overline{\Gamma}$, M includes a 0-element p and so $M\backslash\{p\}$ is a maximal flag of the residue $\overline{\Gamma}_p$ of p. Therefore M is of type $J \cup K \cup \{0\}$ and $\overline{\Gamma}$ is a geometry. Clearly $\overline{\Gamma}_p \cong \Gamma$.

(2) Thanks to (ii) of §4.4 and the fact that $\overline{\Gamma}_p \cong \Gamma$ is a firm geometry, $\overline{\Gamma}$ is firm.

(3) Thanks to an observation made in §4.3 it is easy to see that $\|'$ is a parallelism. $\overline{\Gamma}^\infty \cong \Gamma$ follows from the isomorphism $\overline{\Gamma}_p \cong \Gamma$, stated in (2), and from Theorem 2.1.

(4) is straightforward.

(5) Let $\mathcal{F}$ be the set of flags $\overline{F}$ of $\overline{\Gamma}$ such that $t(\overline{F}) \cap (\{0\} \cup K) = \emptyset$. First $\overline{\Gamma}$ is residually connected if and only if (5.1) holds and for each $\overline{F} \in \mathcal{F}$, $\overline{\Gamma}_{\overline{F}}$ is connected.

It is easy to show that $\overline{\Gamma}$ is connected for each $\overline{F} \in \mathcal{F}$ if and only if the $(\{0\}\cup K)$-truncation of $\overline{\Gamma}_{\overline{F}}$ is connected for each $\overline{F} \in \mathcal{F}$. Let us denote that truncation by $T_{\overline{F}}$. If $\overline{F} = \emptyset$ then $T_{\overline{F}}$ is connected if and only if $B^\infty(X_k)$ is dense in A^∞. Assume that $\overline{F} \neq \emptyset$ and denote by $\overline{X}_0$ the set of 0-elements of $\overline{\Gamma}_{\overline{F}}$. Let $p \in \overline{X}_0$ and let $\mathcal{I}$ be the $\bigcap_{\bar{f}\in\overline{F}} B^\infty(\infty(\bar{f}))$-flat containing p. As $\mathcal{I}$ is included in $\overline{X}_0$, we have that $T_{\overline{F}}$ is connected if and only if $\overline{X}_0 = \mathcal{I}$. Consequently, $\overline{\Gamma}$ is residually connected if and only if (5.1), (5.2) and (5.3) hold. □

4.6 Automorphisms

We state a useful set of sufficent conditions in order that $\mathrm{Aut}(\overline{\Gamma}, \|')$ be flag-transitive in the particular case where K is a singleton, say $K = \{1\}$.

Theorem 4.1 *Let $\overline{\Gamma}$ be the parallel expansion of Γ in $(A, 0, \|)$ via one injective mapping α from the set of 1-elements of Γ into the set of 1-elements of A^∞. Assume that the following conditions hold:*

(1) *the group of dilatations of A is point-transitive;*
(2) *$Aut(\Gamma)$ is flag-transitive;*
(3) *α_1 maps $Aut(\Gamma)$ to a subgroup G^∞ of $Aut(A^\infty)$ restricted to $\alpha_1(\sigma_1(\Gamma))$;*
(4) *every element of G^∞ extends to an automorphism of A.*

Then $Aut(\overline{\Gamma}, \|')$ is flag-transitive.

Proof. Straightforward. □

4.7 Parallel expansions and affine expansions

From now on, we use the expression "affine expansion" instead of "parallel expansion" when the geometry A in which the expansion is made is an affine space $\mathrm{AG}(n, K)$ and when the geometry Γ we want to expand is a subgeometry of $A^\infty = \mathrm{PG}(n-1, K)$.

We insist on the fact that our affine expansions are more general than their ancestors in [30]. Indeed, our affine expansions use S^∞-flats, where S^∞ is a set of points of $\mathrm{PG}(n-1, K)$, whereas their ancestors need affine subspaces and these two notions coincide only if S^∞ is a projective subspace of $\mathrm{PG}(n-1, K)$.

5 Examples of geometries with parallelism

5.1 A reminder

Various examples and means to construct more of them were explicitly mentioned in earlier sections. Let us refer to §2.2.2 (affine geometries, block spaces with parallelism, graphs and their factorizations), to §2.3 (truncations, residues, subgeometries) and let us recall that gluing (see §3.6) and affine expansions (previous section) provide further constructions. In this section we shall expand on some of the preceding examples and we shall provide further ones.

5.2 Affine geometries

This is the central class of examples. We use the expression for all affine geometries over a division ring, including those whose dimension is infinite, and it covers also the non-desarguesian affine planes. These spaces can be submitted to rather different approaches that are essentially equivalent but that provide also channels for generalizations which are no longer equivalent. Let us underline here the recent comparison and complete coherence of such approaches made at the level of geometries over general rings by Schmidt [38].

5.2.1 The vector space or coset approach

Here, we start with a vector space V over a division ring. The affine geometry derived from V consists of points, namely the elements of V, affine subspaces, namely the cosets of all proper non-trivial vector subspaces of V, together with the obvious parallelism (to be cosets of the same subspace of V) and inclusion relation. This construction extends only partially to non-desarguesian planes.

5.2.2 The projective space or hyperplane approach

Here the initial structure is a projective space P together with a distinguished hyperplane H. The affine geometry derived from these data consists of points (those of

$P \backslash H$), subspaces (the sets $X\backslash(X\cap H)$ with X a proper subspace of P not contained in H) and the parallelism determined by declaring $X\backslash(X\cap H)$ and $Y\backslash(Y\cap H)$ to be parallel when $X\cap H = Y\cap H$.

5.2.3 The permutation group approach

This works for affine spaces over division rings. Here we think of a permutation group defined on the set of affine points and of the group consisting of all dilatations: the translations and the homoteties.

5.2.4 The axiomatic approach

We have no need here to enter into the details of this approach.

5.3 Nets and cartesian (or Hamming) spaces

5.3.1 Nets

Consider an affine plane A. Let Π be a nonempty set of parallel classes of lines of A. Delete all lines that do not belong to a member of Π and keep all points as well all other lines. What is left is a net. More generally, a *net* is a rank 2 geometry with parallelism in which any two lines that are not parallel have exactly one common point.

The simplest (connected) case of a net has exactly two parallel classes. These objects are often called *grids*. They coincide essentially with any cartesian product $X \times Y$ of two sets X and Y. This leads us to another situation.

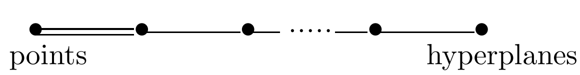
5.3.2 Cartesian (or Hamming) spaces

Let X_1, X_2,..., X_n be (non necessarily distinct) nonempty sets and let X be the cartesian product $\prod_{i=1}^{n} X_i$. We get an obvious rank n geometry $\Gamma(X)$ with parallelism, called a *cartesian* (or *Hamming*) *space of dimension* n. Its points are the elements of X. Incidence is symmetrized inclusion. The hyperplanes or "maximal subspaces" are the sets

$$X_1 \times X_2 \times ... \times X_{i-1} \times \{a\} \times X_{i+1} \times ... \times X_n$$

with $a \in X_i$. All other elements of $\Gamma(X)$ are intersections of hyperplanes. $\Gamma(X)^\infty$ is the thin projective geometry of rank $n-1$ and $\Gamma(X)$ belongs to the following diagram

(C_n)



with order 1 at all nodes except possibly the first one. If all X_i have the same size $q+1$, then $\Gamma(X)$ admits order q at the first node of the above diagram and it is flag-transitive. In any case, $\Gamma(X)$ is just the dual of a thin-lined polar space of rank n (see [30], Chapter 1).

Cartesian spaces generalize grids. It is conceivable to add further directions of subspaces to them, as in the case of nets. The typical prototype is an affine space in which some directions of subspaces are deleted.

5.4 Packings of projective spaces

Historically, projective spaces came as objects extending affine (actually euclidean) spaces so as to make rid of parallelism. Also, it is obvious that a projective plane cannot be equipped with a parallelism. It came as a rather surprising fact, first observed by Clifford in 1882 (see Veblen-Young [41]) that the projective space of dimension 3 over the reals can be equipped with a parallelism on its lines thus giving us a rank 2 geometry of points and lines with parallelism. There is a rich literature on this matter, especially in the finite case (see [14],[4],[2]).

Let P be a projective space of order q and dimension $d \geq 2$. Consider the geometry Γ of points and lines of P. As observed in 2.4, if Γ admits a parallelism then d is odd. For $d = 3$, there is always a parallelism (see [14]). The same holds for d odd if either $d = 2^i - 1$ with $i \geq 2$ or $q = 2$ (see [4],[2], [14]). The other cases for d odd seem to be unsolved.

5.5 Parallelism in linear spaces

Since affine spaces and projective spaces, namely the main representatives of linear spaces, go along so well with parallelism it is natural to look for further examples involving linear spaces. We only consider 0-parallelisms where the 0-elements are the points of the linear spaces because we noticed in §2.4 that there cannot exist other parallelisms.

5.5.1 Witt-Bose-Shrikhande spaces

Let $\mathcal{O}$ be a hyperoval of a finite projective plane Π of even order $q \geq 4$. It is well known that the lines and the points of Π external to $\mathcal{O}$ with the incidence relation inherited from Π form a linear space $\mathcal{W}$ with orders $q/2 - 1$ and q, which is a called a *Witt-Bose-Shrikhande space* (the points and the lines of $\mathcal{W}$ are respectively lines and points of Π). Chosen a point $p \in \mathcal{O}$, we set $a \parallel b$ for two lines a, b of $\mathcal{W}$ if the points a, b of Π are collinear with p. It is easy to see that $\parallel$ is a parallelism of $\mathcal{W}$.

5.5.2 Hermitian Unitals

Let H be a hermitian unital in $\Pi = \mathrm{PG}(2, q^2)$. It is well known that the lines of Π that are secants of H form a linear space $\mathcal{H}$ with orders $(q, q^2 - 1)$ and H as the set of points.

Choose a point $p \in H$ and let L be the line tangent to H at p. For every line X of Γ, set $\infty(X) = X \cap L$ if $p \notin X$ and $\infty(X) = X^\perp \in L$ if $p \in X$ (with $\perp$ denoting the polarity of Π associated to H). Finally, we set $X \parallel Y$ if $\infty(X) = \infty(Y)$. It is straightforward to check that $\parallel$ is a parallelism of $\mathcal{H}$.

Note that, if $q = 2$, then $\mathcal{H} = \mathrm{AG}(2, 3)$ and $\parallel$ is the unique parallelism of $\mathrm{AG}(2, 3)$.

5.5.3 Ree Unitals

Let $\mathcal{U}$ be a Ree unital with orders q and q^2-1 (see Buekenhout-Delandtsheer-Doyen [9]). A basic property is that each line l of $\mathcal{U}$ is the set of fixed points of a unique involution $i(l)$. As a matter of fact, $i(l)$ has a set of invariant lines other than l which provide a partition of the set of points not on l. Using this, we can imitate the construction made for hermitian unitals.

Choose a point $p \in \mathcal{U}$. For every line l on p we decide that the parallels to l are all lines invariant by $i(l)$. Given a line m not on p, how do we find its parallel l on p ? Well, $i(m)$ maps p onto a point $p' \neq p$ and l is the line pp'.

5.5.4 Linear spaces from spreads

Let A be an affine space, P its projective space at infinity and S a *spread of subspaces* of P, namely a family of projective subspaces, not necessarily of the same dimension, that partitions the set of points of P. We derive a linear space with parallelism whose points are those of A and whose lines are all affine subspaces X of A such that $\infty(X) \in S$. The parallelism is inherited from A.

Famous examples are the translation planes and the two flag-transitive Hering spaces on 3^6 points with lines of 3^2 points.

5.5.5 Projective hyperplanes

Here we apply inspiration from §5.2.2. Let L be a linear space and let H be a *projective* (or *geometric*) *hyperplane* of it, namely a proper subspace of L such that every line of L has at least one point in H. Deleting all points and all lines of L in H we get a linear space with parallelism provided that any line not in H has at least three points.

Conversely, from any linear space S with parallelism we get on the set of points $S \cup S^\infty$ a structure of linear space admitting S^∞ as projective hyperplane with many possible choices as to the lines contained in S^∞.

5.5.6 Linear spaces over ternary rings

There are constructions of “affine spaces” over more or less restricted ternary rings providing linear spaces with parallelism (see Nizette [28]).

5.5.7 Complete graphs

Let L be a finite linear space all of whose lines have exactly two points. That is, L is a complete graph. A parallelism on L is the same concept as a 1-factorization of the complete graph (§2.2.2, example 3), a subject extensively studied by Kőnig (see Harary [15]; also [3]). Now L has at least one parallelism if and only if its number of points is even. To prove the existence of a parallelism we describe a construction of Kőnig [22]. Fix an element x of L. Denote by 0, 1,..., $2n-2$ the elements of $L\backslash\{x\}$ and provide $L\backslash\{x\}$ with the addition modulo $2n-1$. For each $i \in L\backslash\{x\}$, the parallel class of $\{x,i\}$ is $\{x,i\} \cup \{\{i-j, i+j\}\}_{1 \leq j \leq n-1}$.

According to a notation popular in graph theory, we denote L by $\mathcal{K}_n$, where n is the number of points of L.

The 1-factorization of $\mathcal{K}_4$ is obviously unique. The 1-factorization $\|$ of $\mathcal{K}_6$ is unique up to isomorphisms [23]. Let $A = \mathrm{Aut}(\mathcal{K}_6, \|)$. It is straightforward to check that the identity automorphism is the unique dilatation of A, namely $A = A^\infty$, and that $A^\infty = \mathrm{Sym}(5)$. Therefore the geometry at infinity $\mathcal{K}_6^\infty$, being a set of size 5, is complete (see §2.5).

It is quite remarkable that the flag-transitive parallelisms of $\mathcal{K}_n$ have been recently classified in [12]. Apart from the point-line system of $\mathrm{AG}(n, 2)$ with its natural parallelism, there are only 3 more examples with $\mathrm{Aut}(\mathcal{K}_2, \|)$ flag-transitive. These exceptional examples arise with $n = 6$, 12 and 28 and can be realized as the 6 points of a hyperoval of $\mathrm{PG}(2, 4)$, as the 12 points of a non-degenerate conic of $\mathrm{PG}(2, 11)$ and the 28 of the smallest Ree unital, respectively.

5.6 Graphs

We could produce many examples of parallelism in *semilinear spaces*, namely rank 2 geometries in which any two points are incident with at most one line. The nets (see 5.3.1) are a particular case. The same holds true for graphs. Here all lines have two points. On a graph, a parallelism is often called a 1-factorization. We provide some explicit examples.

5.6.1 The octahedron

The octahedron with 6 vertices and 12 edges (namely, the complete 3-partite graph with classes of size 2) has a parallelism, which is unique up to isomorphisms. This can be obtained from the parallelism on the complete graph of 6 vertices by deleting the three lines of some parallel class. That method applies as well to any hyperoctahedron, viewed as a complete n-partite graph with all classes of size 2. We warn that this parallelism cannot be extended to the faces of the hyperoctahedron of dimension ≥ 2, by a remark made in §2.4.

5.6.2 Trees and dual grids

A 1-factorization can be defined on every tree of valency k. Let S be a set of k colours. Start with a vertex x and give each of the k edges at x a colour of S such that no two edges have the same colour. Then, do the same thing for every vertex y adjacent to x, keeping in mind that the edge $\{x, y\}$ has already got a colour. Repeat this process for ever. At the end of ever, every edge will have got its own colour, in such a way that distinct edges attached to the same vertex never share the same colour. That is, we have defined a 1-factorization of that tree.

A 1-factorization can also be defined on every complete bi-partite graph Γ with both classes of size n (that is a dual grid of order $(1, n-1)$). Denote by $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ the two classes of Γ. We can take any sharply 1-transitive set X of permutations on I as line at infinity (for instance, X might be a cyclic group of order n). If ξ is the unique element of X mapping i onto j, then the line $\{x_i, y_j\}$ will be given ξ as point at infinity.

5.6.3 Cayley graphs

A graph with a parallelism can be seen as follows. It consists of a set S (of vertices or points) and a family I of involutions of S having no fixed points and such that any two distinct points are permuted by at most one member of I. The edges of the graphs are the orbits of those involutions, two edges being parallel if they are orbits of the same involution.

The Cayley graphs corresponding to a group G generated by a set I of elements of order 2 constitute an important particular case.

5.7 Parallelism in generalized polygons

Every thin $2n$-gon admits an obvious unique parallelism with two points at infinity.

A generalized quadrangle admits a 0-parallelism (resp. 1-parallelism) if and only if we can partition its set of lines (resp. points) in spreads (resp. ovoids). We saw in 5.3.1 and 5.6.2 that grids and dual grids with order admit parallelisms. Besides grids and dual grids, the only known finite examples are the generalized quadrangles of type $T_2^*(O)$, their duals and those of type $AS(q)$ ([34] and [35], chapter 3; also §7.4.2 of this paper).

In spite of this, not so many results are known stating that certain classes of finite generalized quadrangles do not admit any partition of the set of lines (resp. points) into spreads (resp. ovoids). The reader can see [35] (1.8.3, 1.8.5, 3.4.1, 3.4.2, 3.4.4) for some negative results of this kind.

How about generalized hexagons and octagons ?

5.8 Affine-like coset geometries

Let us consider a generalization of the vector space approach to affine geometry.

Let $\Gamma = (X, *, t)$ be a geometry over a set of types I. This corresponds to the projective geometry at infinity in the above case. Next, let G be a group (replacing the vector space V) and $\{G_x\}_{x \in X}$ a collection of subgroups of G (replacing the subspaces of V). We assume that I has no element called 0 and we create a new set of types $\overline{I} = I \cup \{0\}$.

From the preceding data, we derive an *affine coset geometry with parallelism* $\overline{\Gamma}$ over $\overline{I}$. For $i \in I$, the i-elements of $\overline{\Gamma}$ are the cosets gG_x with $g \in G$, $x \in X$ and $t(x) = i$. We put $gG_x \parallel hG_y$ if and only if $x = y$. Finally, in a quite natural way, we decide that $g * hG_y$ if and only if $g \in hG_y$ and that $gG_x * hG_y$ if and only if $x * y$ (in Γ) and $gG_x \cap hG_y \neq \emptyset$.

It is easy to check that $\overline{\Gamma}$ is indeed a geometry with parallelism. Moreover $\overline{\Gamma}^\infty \cong \Gamma$. Also, the group G acts as an automorphism group of $\overline{\Gamma}$ by left translation and this action fixes each element at infinity.

5.8.1 The normal case

The preceding data G, $\{G_x\}_{x\in X}$, Γ may be called *normal* if, for every inner automorphism α of G, there is an automorphism $\overline{\alpha} \in \mathrm{Aut}(\Gamma)$ such that $\alpha(G_x) = G_{\overline{\alpha}(x)}$ for every $x \in X$. The automorphism group of $\overline{\Gamma}$ has a point stabilizer containing the group of inner automorphisms of G (modulo the kernel of this action). In particular, if this action is flag-transitive in Γ, then $\mathrm{Aut}(\overline{\Gamma})$ is also flag-transitive.

5.8.2 The case where $\overline{\Gamma}$ is a linear space

Here Γ is a geometry of rank one and we assume that $\{G_x\}_{x\in X}$ is a *partition* of the set of elements of G other than 1.

The subject of group partitions has received much attention (see for instance [7], Chapter 3, §5.4). Here is a class of examples. Consider a Frobenius group G and its Frobenius kernel N, consisting of 1 and all elements having no fixed point. Then N and all point-stabilizers G_a, a any point, constitute a normal partition of G.

5.9 Finite primitive permutation groups

By the theorem of O'Nan-Scott, finite primitive groups fall into five disjoint families (see [5]), namely:

1) the almost simple type;
2) the affine type;
3) the biregular type;
4) the cartesian semi-simple type;
5) the diagonal type.

Geometries with parallelism are present in several of these families. For a group of affine type (resp. cartesian type) we get of course an invariant affine geometry (resp. cartesian, i.e. Hamming geometry). The diagonal case offers special interest.

5.9.1 The diagonal type

Here is a construction of these groups. Let S be a nonabelian finite simple group and $m \geq 3$ an integer.

Consider the group $N = S_1 \times S_2 \times ... \times S_m$ where each S_i is an isomorphic copy of S. Let N_0 be a diagonal subgroup of N which is isomorphic to S, i.e. the projection of N_0 to each S_i is an isomorphism. Consider the action of N on the set E of all left cosets of N_0 by left translation. Then $|E|{=}|S^{m-1}|$. If G is a permutation group on E normalizing N, then G is called of *diagonal type*.

Each subgroup $N_i = S_1 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_m$ is transitive on E because N_i and N_0 generate N. Therefore N_i acts regularly on E. Also, S_i centralizes N_i and so it acts semi-regularly on E. Let a *line* be any orbit of any S_i on E and call two lines *parallel* if they are orbits of the same S_i. This gives us a rank 2 geometry with parallelism. Moreover, if Π is any class of parallel lines and if these lines are deleted, the remaining subgeometry is a cartesian (Hamming) space of dimension

$m-1$. In the *diagonal space* obtained in this way, N fixes each point at infinity. Note that for $m = 3$ this geometry is a net.

5.9.2 Example

Take $S = \mathrm{Alt}(5)$, $m = 3$, $G = (\mathrm{Alt}(5))^3{:}\mathrm{Sym}(3)$, where Sym(3) acts transitively on the three copies of Alt(5). This gives us a diagonal space with 3600 points, lines of 60 points and each point is on three lines.

5.10 Parallelism in affine grassmannians

All examples of rank ≥ 3 previously described in this section provide geometries with a string diagram Δ endowed with a 0-parallelism $\parallel$ where 0 is an end node of Δ. However, this is not always the case as the following example will show (see also §7.3).

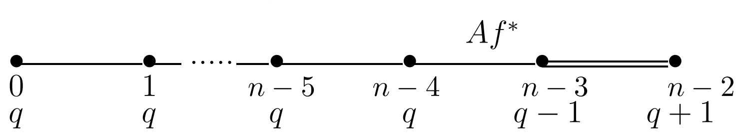
5.10.1 A class of affine grassmannians

Given a subspace S of $\mathrm{PG}(n, K)$ ($n \geq 3$) of positive dimension $d < n-1$, we can form a geometry Γ^S as follows. If $0 \leq i < n-d$, we take as elements of type i the i-dimensional subspaces of $\mathrm{PG}(n+1, K)$ that do not intersect S. If $n-d-1 < i \leq n-1$, then we take as elements of type i the i-dimensional subspaces that joined with S span all of $\mathrm{PG}(n+1, K)$. The incidence relation is symmetrized inclusion. According to [13], we say that Γ^S is an *affine grassmannian.*

An $(n-d-1)$-parallelism $\parallel$ can be defined on Γ^S. Let X, Y be elements of Γ^S of type $i \neq n-d-1$. If $i < n-d-1$, then $X \parallel Y$ means that $X \cup S$ and $Y \cup S$ span the same subspace of $\mathrm{PG}(n+1, K)$. If $i > n-d-1$, then $X \parallel Y$ means that $X \cap S = Y \cap S$.

5.10.2 A subgeometry of two affine grassmannians

Let now $K = GF(q)$, q even. Given a plane S of $\mathrm{PG}(n, q)$, let O be a hyperoval of S and L a line of S external to O. Let Γ^O be the subgeometry of Γ^S consisting of all elements of Γ^S of type $i < n-2$ and all elements X of type $n-2$ such that the point $X \cap S$ belongs to O. If $n = 3$ then Γ^O is just the generalized quadrangle of type $T_2^*(O)$. If $n > 3$ then Γ^O belongs to the following diagram of rank $n-1$ (where 0, 1,..., $n-2$ are the types and q, q,..., q, $q-1$, $q+1$ are orders):



By Lemma 2.4, the parallelism of Γ^S induces on Γ^O an $(n-3)$-parallelism. However, Γ^O is also a subgeometry of Γ^L. Thus, by Lemma 2.4, Γ^O also admits an $(n-2)$-parallelism (compare 7.4.2).

5.11 Chamber systems

A chamber system of rank n (see [37], [40], [21]) with all panels of size at least two and no two panels intersecting in more than one chamber, is just the same thing as a semilinear space with parallelism (see §5.6) with n lines on every point. Chambers and panels play the role of points and lines respectively, two panels being called parallel if they have the same type.

We can do more. Given a chamber system $\mathcal{C}$ of rank n with the above properties, the cells of $\mathcal{C}$, with symmetrized inclusion as incidence relation, form a geometry $\Gamma_{\mathcal{C}}$ of rank n. The relation "having the same type" between cells of $\mathcal{C}$ naturally defines a parallelism $\|$ on $\Gamma_{\mathcal{C}}$.

The geometry $\Gamma_{\mathcal{C}}$ is in fact the parallel expansion of the thin projective geometry P of rank $n-1$ in the semilinear space of chambers and panels of $\mathcal{C}$, via any bijection of the set of points of P onto the set of types of $\mathcal{C}$, which are the points at infinity of that semilinear space.

Properties of $\mathcal{C}$ can be revisited as properties of $\Gamma_{\mathcal{C}}$, sometimes with some profit (see [31], for instance; also §7.5 of this paper).

6 Applications of gluing

A fantastic variety of geometries and diagrams can be produced by the gluing procedure. We will only discuss a sample of meaningful examples. We are particularly interested in gluings leading to diagram geometries close to Coxeter diagrams in the spirit of [7], chapter 22. We shall restrict explicit gluing to two geometries but it is clear that we can glue any number of geometries provided they can be pairwise glued.

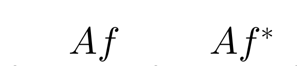
6.1 Gluing two copies of an affine geometry

The investigation of quotients of bi-affine geometries [17] was the source of the gluing construction. We recall that a *bi-affine* geometry (*affine-dual-affine* geometry in [7], chapter 22) is the geometry Γ obtained from a projective geometry $\mathrm{PG}(n+1, K)$ ($n \geq 2$) by deleting the residues of a hyperplane S and of a point p. It has the following diagram:

$(Af.A_{n-1}.Af^*)$

In particular, when $n = 2$ we have

$(Af.Af^*)$



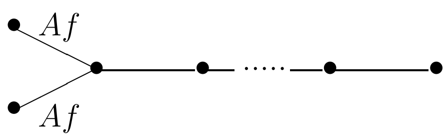
We say that Γ is of *flag-type* if $p \in S$. If K is commutative and Γ is of flag-type, then Γ can be factorized by the group H of all elations of $\mathrm{PG}(n+1, K)$ with axis S and center p. The quotient Γ/H is flag-transitive. It is described in [17]. It is clear from that description that Γ/H is in fact a twisted gluing of two copies of $\mathrm{AG}(n, K)$.

6.1.1 The case where $n > 2$

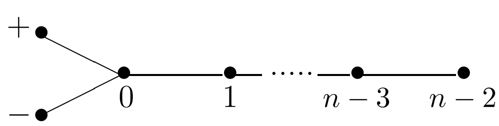
Let $n > 2$. Then the geometry at infinity $\mathrm{PG}(n-1, K)$ of $\mathrm{AG}(n, K)$ is complete. If K is commutative, then $\mathrm{PG}(n-1, K)$ also admits a correlation, which is unique modulo multiplication with collineations of $\mathrm{PG}(n-1, K)$. Therefore, when K is commutative, all twisted gluings of two copies of $\mathrm{AG}(n, K)$ are isomorphic to Γ/H, by Theorem 3.11.

In any case, there is just one plain gluing of two copies of $\mathrm{AG}(n, K)$, by Corollary 3.10. This gluing is flag-transitive (by Corollary 3.8) and it belongs to the following diagram

$(2Af.A_{n-2})$



When K is commutative, this glued geometry can also be obtained by the following construction. Let $\overline{\Gamma}$ be the building of type D_{n+1} over K and let us take $+$, $-$, 0, 1,..., $n-2$ as types, as follows



Let $\overline{\Gamma}^{\varepsilon}$ be the point-line system of $\overline{\Gamma}$ with respect to a type $\varepsilon = +$ or $-$ (see [7], chapter 12 by Cohen). For every element x of $\overline{\Gamma}$, let $\sigma^{\varepsilon}(x)$ be the set of elements of $\overline{\Gamma}$ of type ε incident with x. Let a^+, a^- be incident elements of $\overline{\Gamma}$ of type $+$ and $-$ respectively. For $\varepsilon \in \{+, -\}$, we define a hyperplane S^{ε} of $\overline{\Gamma}^{\varepsilon}$ ([7], chapter 12) as follows.

If n is even, then S^+ is the set of elements of $\overline{\Gamma}$ of type $+$ having distance $< n/2$ from some element of $\sigma^+(a^-)$ in the collinearity graph of $\overline{\Gamma}^+$. If n is odd, then S^+ is the set of elements of type $+$ having distance $< (n+1)/2$ from a^+ in the collinearity graph of $\overline{\Gamma}^+$ (it is not difficult to prove that S^+ is in fact a hyperplane). S^- is defined in the same way, interchanging $+$ with $-$.

Let Ξ be the set of flags F of $\overline{\Gamma}$ such that $\sigma^{\varepsilon}(F) \not\subseteq S^{\varepsilon}$ for $\varepsilon = +$ and $-$ and let X be the set of elements of $\overline{\Gamma}$ belonging to Ξ. We can now define a geometry Γ with X as set of elements by stating that two elements $x, y \in X$ are incident in Γ if they are incident in $\overline{\Gamma}$ and $\{x, y\} \in \Xi$. It is straightforward to check that Γ belongs to the above diagram $2Af.A_{n-2}$.

Let G be the stabilizer of a^+ and a^- in $\mathrm{Aut}(\overline{\Gamma})$ and let H be the elementwise stabilizer of $S^+ \cup S^-$ in G. Then H defines a quotient Γ/H of Γ, which is in fact the (unique) plain gluing of two copies of $\mathrm{AG}(n, K)$ (see [29]).

6.1.2 The case where $n = 2$

We can also assume $n = 2$ in the above construction. Then $\overline{\Gamma} = \mathrm{PG}(3, K)$, $\{a^+, a^-\}$ is a point-plane flag of $\mathrm{PG}(3, K)$, Γ is a bi-affine geometry of flag-type and Γ/H is the quotient considered at the beginning of §6.1. It is the canonical gluing of two copies of $\mathrm{AG}(2, K)$.

If $|K| \leq 4$, then the canonical gluing is the only gluing of two copies of $\mathrm{AG}(2, K)$, by Corollary 3.10. On the other hand, if $|K| > 4$, then non-canonical gluings exist (Theorem 3.9) and some of them are even flag-transitive (an example with $K = GF(7)$ is given in [17]). However, the canonical gluing is characterized by having the largest automorphism group [29].

Clearly, any two (or more) affine planes of the same order can be glued, and the resulting glued geometry might be flag-transitive provided each of these planes is already flag-transitive. For instance, given any flag-transitive affine plane Π, the canonical gluing of two copies of Π is flag-transitive.

6.1.3 Gluing two copies of an affine space

Let Γ be the point-line system of $\mathrm{AG}(n, K)$ ($n \geq 3$), with its natural parallelism $\|$. Since Γ has rank 2, its geometry at infinity bears no structure (it is just a set). By Theorem 3.9, there are non-canonical gluings of two copies of $(\Gamma, 0, \|)$. It is proved in [29] that, when $K = GF(q)$, the canonical gluing of two copies of $(\Gamma, 0, \|)$ is characterized by the property of having the largest automorphism group (compare §3.4.5). As we have remarked in the previous subsection, the same property characterizes the canonical gluing of two copies of $\mathrm{AG}(2, q)$.

Clearly, the canonical gluing of two copies of $(\Gamma, 0, \|)$ is a truncation of the (unique) plain gluing of two copies of $\mathrm{AG}(n, K)$.

6.2 Gluing and quotients of Laguerre-like geometries

6.2.1 Dual-affine expansions

The following construction generalizes Laguerre structures. It is a special case of a rather more general construction by Huybrechts [20].

Let Π be a rank 2 subgeometry of $\mathrm{PG}(2, K)$. Given a point p of $\mathrm{PG}(3, K)$, we identify $\mathrm{PG}(2, K)$ with the star of p in $\mathrm{PG}(3, K)$. Thus the points and the lines of Π are lines and planes on p. Let $\mathcal{C}_{\Pi,1}$ (respectively $\mathcal{C}_{\Pi,2}$) be the set of lines (planes) of $\mathrm{PG}(3, K)$ through p corresponding to points (lines) of Π. A geometry $\Gamma^{\Pi,p}$ of rank 3 can be defined as follows. We take $\bigcup_{L \in \mathcal{C}_{\Pi,1}} L \backslash \{p\}$ as set of points. The lines of $\mathrm{PG}(3, K)$ contained in planes of $\mathcal{C}_{\Pi,2}$ but not containing p are the lines of $\Gamma^{\Pi,p}$. The planes of $\Gamma^{\Pi,p}$ are the planes of $\mathrm{PG}(3, q)$ not through p. The incidence relation of $\Gamma^{\Pi,p}$ is the natural one, inherited from $\mathrm{PG}(3, q)$. The residues of the planes of $\Gamma^{\Pi,p}$ are isomorphic to Π. It is not difficult to prove that the residues of the points of $\Gamma^{\Pi,p}$ are nets.

Actually, $\Gamma^{\Pi,p}$ is the dual of a certain affine expansion to be defined in §7.4.1. In view of this, we call $\Gamma^{\Pi,p}$ the *dual-affine expansion* of Π at the *center* p.

6.2.2 Shrinking and gluing

Given p and Π as in the previous paragraph, let S be a plane with $p \in S \notin \mathcal{C}_{\Pi,2}$. Let H_S be the group of all elations of $\mathrm{PG}(3, K)$ with axis S and center p. Then H_S defines a quotient of $\Gamma^{\Pi,p}$. We call it the *shrinking* of $\Gamma^{\Pi,p}$ at S.

Let now $K = GF(q)$ and assume that every point of Π (line of $\mathcal{C}_{\Pi,1}$) is incident with precisely $s+1$ lines of Π (planes of $\mathcal{C}_{\Pi,2}$). We say that S is (Π, p)-*regular* if there is a set $\mathcal{L}$ of $s+1$ lines of S on p such that $\mathcal{L} \cap \mathcal{C}_{\Pi,1} = \emptyset$ and every plane of $\mathcal{C}_{\Pi,2}$ contains one line of $\mathcal{L}$.

Clearly, if S is (Π, p)-regular and $\mathcal{L}$ is a set of $s+1$ lines as above, then for every line $L \in \mathcal{C}_{\Pi,1}$ there is just one line $M \in \mathcal{L}$ such that $L \cup M$ spans a plane of $\mathrm{PG}(3, q)$ belonging to $\mathcal{C}_{\Pi,2}$. Let us set $\infty_S(L) = M$. For any two lines $L, L' \in \mathcal{C}_{\Pi,1}$, we set $L \parallel_S L'$ if and only if $\infty_S(L) = \infty_S(L')$. It is easily seen that $\parallel_S$ is a parallelism on Π with $\mathcal{L}$ as its line at infinity.

Furthermore, the elementwise stabilizer K_S of S in $\mathrm{PGL}_4(q)$ acts transitively on the set of points $\mathrm{PG}(3, q)$ not in S. Hence, given any two points x, y of Γ, there is some element of K_S mapping x onto y. That element might not stabilize Π. However, it maps the residue of x in $\Gamma^{\Pi,p}$ onto the residue of y in $\Gamma^{\Pi,p}$. Indeed, for every point z of $\Gamma^{\Pi,p}$ not in S, the planes of $\mathcal{C}_{\Pi,2}$ on z are precisely those spanned by z and by some of the lines of $\mathcal{L}$, because there are precisely $s+1$ planes of $\mathcal{C}_{\Pi,2}$ on z and each of them meets S in a line of $\mathcal{L}$. Thus, the residues of the points of $\Gamma^{\Pi,p}$ are pairwise isomorphic. As they are nets, they are isomorphic to a given net $\mathcal{N}$. It is clear from the above that $\mathcal{L}$ can also be viewed as the line at infinity of $\mathcal{N}$. The following is now evident.

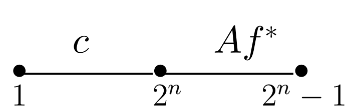
Theorem 6.1 *The shrinking of* $\Gamma^{\Pi,p}$ *at a* (Π, p)*-regular plane is a gluing of a net with* Π *endowed with the parallelism* $\parallel_S$.

6.2.3 Examples

1. The affine plane $\mathrm{AG}(2, K)$ can be viewed as a subgeometry Π of the star of a point p of $\mathrm{PG}(3, K)$. $\Gamma^{\Pi,p}$ is just the bi-affine geometry of flag type and rank 3, obtained from $\mathrm{PG}(3, K)$ by removing the star of p and the plane S on p corresponding to the line at infinity of $\mathrm{AG}(2, K)$. The shrinking of $\Gamma^{\Pi,p}$ at S is the canonical gluing of two copies of $\mathrm{AG}(2, K)$.

2. Let Π be the complete graph on $2^n + 2$ vertices. We can take a hyperoval O of $\mathrm{PG}(2, 2^n)$ as set of points of Π. Viewing $\mathrm{PG}(2, 2^n)$ as the star of a point p of $\mathrm{PG}(3, 2^n)$, we can consider $\Gamma^{\Pi,p}$, which is a *special Laguerre plane* (Heise and Karzel [18]). It belongs to the following diagram:

$(c.Af^*)$



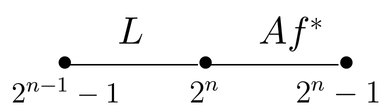
According to the above identification of $\mathrm{PG}(2, 2^n)$ with the star of p, every line S of $\mathrm{PG}(2, 2^n)$ external to O is a (Π, p)-regular plane of $\mathrm{PG}(3, 2^n)$. Furthermore S determines a 1-factorization $\parallel_S$ of Π (Korchmaros [23]): two lines of Π correspond in $\parallel_S$ if and only if they span lines of $\mathrm{PG}(2, 2^n)$ intersecting S in the same point. By Proposition 6.1, the shrinking of $\Gamma^{\Pi,p}$ at S is a gluing of $\mathrm{AG}(2, 2^n)$ with Π endowed with $\parallel_S$.

Let $n = 2$. Then all 1-factorizations of Π are isomorphic to $\parallel_S$ (see §5.5.7) and $\mathrm{Aut}(\Pi, \parallel_S)$ induces $\mathrm{Sym}(5)$ on the five points of Π^∞. Hence the shrinking of $\Gamma^{\Pi,p}$

at S is the unique gluing of Π with $\mathrm{AG}(2,4)$ (Theorem 3.5) and it is flag-transitive (Corollary 3.7) with automorphism group $2^4{:}Z_3.\mathrm{Sym}(5)$ (Theorem 3.6). Of course, this information can also be obtained considering that $\mathrm{Aut}(\Gamma^{\Pi,p}) = 2^6{:}Z_3.\mathrm{Sym}(6)$ (see §7.4.1) and that the shrinking of $\Gamma^{\Pi,p}$ at S is the quotient of $\Gamma^{\Pi,p}$ by the group H of all elations of $\mathrm{PG}(3,4)$ of center p and axis S.

3. Let Π be a Witt-Bose-Shrikhande space realized by choosing a dual hyperoval O^* of $\mathrm{PG}(2,2^n)$ ($n \geq 3$) and taking as points and lines the points of $\mathrm{PG}(2,2^n)$ that do not belong to any of the 2^n+2 lines of O^* and the lines of $\mathrm{PG}(2,2^n)$ that do not belong to O^* (compare §5.5.1). Given a point p of $\mathrm{PG}(3,2^n)$, we can consider $\Gamma^{\Pi,p}$. It belongs to the following diagram:

$(L.Af^*)$



Every line $S \in O^*$ is (Π,p)-regular when viewed as a plane through p. The parallelism $\|_S$ is as in §5.5.1. By Proposition 6.1, the shrinking of $\Gamma^{\Pi,p}$ at S is a gluing of $\mathrm{AG}(2,2^n)$ with Π endowed with $\|_S$.

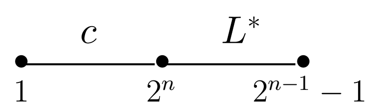
6.2.4 A quotient of a subgeometry of a Laguerre-like geometry

We can sometimes form a subgeometry of a Laguerre-like geometry $\Gamma^{\Pi,p}$ by "intersecting" it with an affine expansion. We only give an example of this construction.

Let (p,S) be a (point,plane)-flag of $\mathrm{PG}(3,2^n)$, $n > 2$. The star of p and the plane S are models of $\mathrm{PG}(2,2^n)$. The affine geometry obtained by removing S from $\mathrm{PG}(3,2^n)$ will be denoted by $\mathrm{PG}(3,2^n)\backslash S$. Let O and O' be hyperovals in the star of p and in S respectively, such that $p \in O'$ but S does not contain any of the lines through p forming O. Let Π be the complete graph with O as set of points. The lines and the points of S external to O' are respectively the points and the lines of a Witt-Bose-Shrikhande space $\mathcal{W}$. We denote the dual of $\mathcal{W}$ by $\mathcal{W}^*$.

The symbol $\Gamma^{\Pi,p}$ has the meaning stated in the previous paragraph whereas $\Gamma_{\mathcal{W}}$ will denote the affine expansion of $\mathcal{W}^*$, having the points of $\mathrm{PG}(3,q)\backslash S$ as points. Let Γ be the set-theoretic intersection of $\Gamma^{\Pi,p}$ and $\Gamma_{\mathcal{W}}$. That is, the elements of Γ are the points, the lines and the planes of $\mathrm{PG}(3,2^n)$ that belong to both $\Gamma^{\Pi,p}$ and $\Gamma_{\mathcal{W}}$, with the incidence relation inherited from $\mathrm{PG}(3,2^n)$. Then Γ is a geometry with diagram and orders as follows:

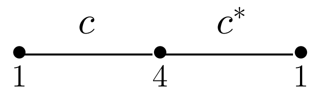
$(c.L^*)$



Let H be the group of elations of $\mathrm{PG}(3,2^n)$ with center p and axis S. Then H defines a quotient Γ/H of Γ. Let $\|_p$ be the parallelism defined by p on $\mathcal{W}$ as in §5.5.1 and let $\|_S$ be the parallelism defined on Π by the plane S as in §5.1.1. Let $\mathcal{L}$ be the bundle of lines of S through p. We can take $\mathcal{L}$ as the line at infinity for both $\|_p$ and $\|_S$. It is clear that Γ/H is a gluing of Π and $\mathcal{W}$ endowed with the parallelisms $\|_S$ and $\|_p$ respectively.

Let $n = 2$. Then both Π and $\mathcal{W}$ are copies of the complete graph on 6 vertices and Γ has diagram and orders as follows:

$(c.c^*)$

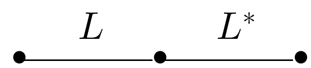


There is just one way to glue two copies of the complete graph Π on 6 points, by the uniqueness of the one-factorization $\|$ of that graph and properties of the group $\mathrm{Aut}(\Pi, \|)$ (compare §5.5.7 and Theorem 3.9). Thus, Γ/H is just that gluing. $\mathrm{Aut}(\Gamma/H) = \mathrm{Sym}(5)$, by Theorem 3.6 and properties of $\mathrm{Aut}(\Pi, \|)$. This group is not flag-transitive.

6.3 Further examples of type $L.L^*$

Many of the previous examples belong to special cases of the following diagram

$(L.L^*)$



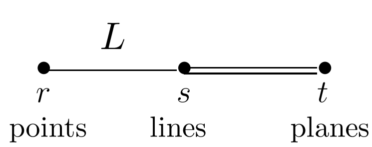
Using gluing and parallelisms described in section 5 we get a series of other examples for this diagram, where the linear spaces are projective spaces of odd dimension (§5.4), unitals (hermitian or Ree) with the same orders (§5.5) or a (hermitian or Ree) unital of orders $(3, 8)$ and an affine plane of order 7, or a unital of orders $(q, q^2 - 1)$ with q odd and a complete graph with $q + 1$ vertices, or...

6.4 Glued geometries of type $L.C_2$

6.4.1 Finite thick examples

Del Fra [16] has proved that a flag-transitive finite thick geometry belonging to the diagram $L.C_2$ (depicted below) has classical generalized quadrangles as point-residues if and only if it is a (possibly improper) "standard" quotient of an affine polar space [32].

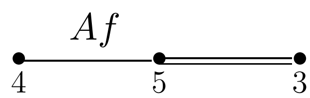
$(L.C_2)$



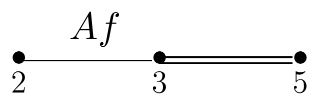
However, there are flag-transitive geometries belonging to this diagram with non-classical point-residues. We can build some of them by the gluing construction.

Let Π_1 be the generalized quadrangle of type $T_2^*(O)$ and order $(3, 5)$ and let Π_2 be its dual. Let $\|_1$ and $\|_2$ be the (unique) 0-parallelism and a 1-parallelism of Π_1, respectively (see §7.4.2). Of course, $\|_2$ is a parallelism of Π_2. Let us set $A_i = \mathrm{Aut}(\Pi_i, \|_i)$, for $i = 1, 2$.

We have $A_1 = \mathrm{Aut}(\Pi_1)$. The stabilizer of a hyperoval O of $\mathrm{PG}(2, 4)$ is the symmetric group $\mathrm{Sym}(6)$. Hence $A_1 = 2^6{:}Z_3.\mathrm{Sym}(6)$ with dilatation group $K_1^\infty = 2^6.Z_3$, point-transitive on Π_1, and $A_1^\infty = \mathrm{Sym}(6)$ acting on the six points of Π_1^∞. Thus, if we glue $\mathrm{AG}(2,5)$ with Π_1 endowed with $\|_1$, we obtain a flag-transitive geometry Γ_1 with $\mathrm{Aut}(\Gamma_1) = (2^6.Z_3 \times 5^2.Z_4)).\mathrm{PGL}_2(5)$ (see Theorem 3.6 and Corollary 3.8). Here Γ_1 belongs to the following special case of $L.C_2$



Turning to $A_2 = \mathrm{Aut}(\Pi_2, \|_2)$, we have $A_2 = 2^6{:}Z_3.\mathrm{Sym}(5)$ with dilatation group $K_2^\infty = 2^4.\mathrm{Alt}(5)$ and $A_2^\infty = 2^2{:}Z_3.2 = \mathrm{Sym}(4)$. Thus, by Theorem 3.6 and Corollary 3.8, if we glue $\mathrm{AG}(2,3)$ with Π_2 endowed with $\|_2$, we get a flag-transitive geometry Γ_2 with $\mathrm{Aut}(\Gamma_2) = (2^4.\mathrm{Alt}(5) \times 3^2.Z_2)).\mathrm{PGL}_2(3)$. Here Γ_1 has diagram and orders as follows



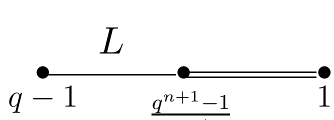
The above are the only flag-transitive finite thick $L.C_2$ geometries with non-classical point-residues that are presently known. Actually, one more non-classical flag-transitive finite thick generalized quadrangle is known, namely the generalized quadrangle Π of type $T_2^*(O)$ and order $(15,17)$ obtained from the Lunelli-Sce hyperoval O of $\mathrm{PG}(2,16)$. If $\|$ is the parallelism that Π inherits from $\mathrm{AG}(3,16)$ (see §7.4.2), then $A = \mathrm{Aut}(\Pi, \|) = \mathrm{Aut}(\Pi)$ with dilatation group $K^\infty = 2^{12}{:}Z_{15}$ (point-transitive on Π) and $A^\infty = Z_2 \times 3^2.Z_8$ (which is the stabilizer of O in $P\Gamma L_2(16)$; see [24]). However, the intersection of A^∞ with any conjugate of $\mathrm{PGL}_2(17)$ in $\mathrm{Sym}(18)$ is not transitive on the 18 points of Π^∞. Hence we cannot hope for any flag-transitive gluing here, by Corollary 3.8.

The generalized quadrangle $\Pi = AS(3)$ is also flag-transitive. In fact, it is classical. It admits a parallelism $\|$, as we noticed in §5.3.7. However $\mathrm{Aut}(\Pi, \|)$ is not flag-transitive. Thus, no flag-transitive gluing can be obtained from it.

6.4.2 Gluing dual grids with affine spaces

Let q be a prime power and let S be a Singer cycle of $\mathrm{PG}(n,q)$. We can build a model of the dual grid Π of order $(1,q)$ by taking $S \times \{0,1\}$ as set of points and representing a line $\{(a,0),(b,1)\}$ of Π by the ordered pair (a,b). We can define a parallelism $\|$ on Π by setting $(a,b) \parallel (c,d)$ when $b^{-1}a = d^{-1}c$ (see §5.6.2). Let $A = \mathrm{Aut}(\Pi, \|)$. It is not difficult to prove that $A = (S.N).2$, with N the normalizer of S in the symmetric group on the $(q^{n+1}-1)/(q-1)$ points of $\mathrm{PG}(n,q)$. We have $K^\infty = S \times Z_2$, point-transitive on Π, and $A^\infty = S$.

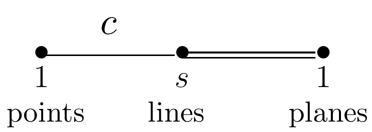
S is a subgroup of $\mathrm{PGL}_{n+1}(q)$. By Corollary 3.8, if we glue the point-line system of $\mathrm{AG}(n+1,q)$ with Π endowed with $\|$, then we get a flag-transitive geometry Γ for the diagram $L.C_2$, with orders $(q-1,\ (q^{n+1}-q)/(q-1),\ 1)$



When $n = 1$ and $q = 2$, Γ is a quotient of an affine polar space (see [26]).

6.4.3 Flat flag-transitive extended grids

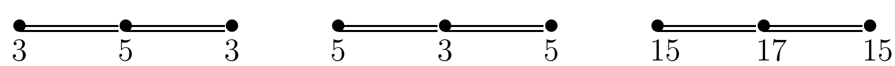
An extended grid is a geometry belonging to the following special case of $L.C_2$



An extended grid Γ is said to be *flat* if every point of Γ is incident with all planes of Γ. Flat flag-transitive finite extended grids have been classified by Meixner and Pasini [25]. All of them can be obtained by gluing a dual grid, endowed with a suitable parallelism, with the point-line system of an affine geometry over $GF(2)$ endowed with its natural parallelism.

6.5 Gluing generalized quadrangles

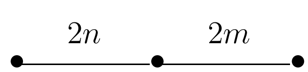
Using the flag-transitive generalized quadrangles with parallelism mentioned earlier we get (probably new) finite flag-transitive GABs of type $\widetilde{C}_2$



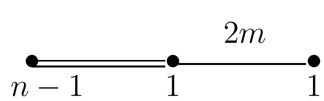
Actually we have three infinite families here, as we can glue any number of copies.

6.6 Gluing polygons, grids and dual grids

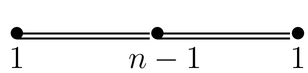
Gluing a $2n$-gon and a $2m$-gon we get a thin geometry over the diagram



It has a flag-transitive automorphism group of order $8mn$. Gluing an $(n \times n)$-grid and a $2m$-gon we get a flag-transitive geometry with diagram and orders as follows



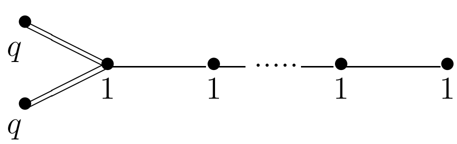
We can also glue two of copies of the dual grid of order $(1, n)$ equipped with a parallelism (§5.6.2). Thus we get a GAB of type $\widetilde{C}_2$ with orders 1, $n - 1$, 1



This GAB is flag-transitive for suitable choices of the parallelisms on the two copies of the dual grid and suitable matchings of the lines at infinity (§3.4.4).

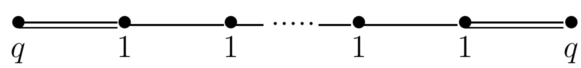
6.6.1 Gluing two copies of a cartesian space

Let Γ be a cartesian space of rank n built on a product $\prod_{i=1}^{n} X_i$ of sets with the same cardinality $q+1$ (see §5.3.2). By Corollary 3.10 there is a unique plain gluing of two copies of Γ (up to isomorphisms). It is flag-transitive by Corollary 3.8 and it belongs to the following diagram



We can also consider the twisted gluing of two copies of Γ, which is unique too. It is flag-transitive and it belongs to the following diagram of affine type

$(\widetilde{C}_n)$



7 Applications of parallel expansion

Some examples of parallel expansions were given in previous sections, sometimes implicitly. For instance, the geometry of cells of a chamber system (§5.11) is a parallel expansion. Bi-affine geometries of flag-type (§6.1) are precisely parallel expansions of dual affine geometries. The dual-affine expansions defined in §6.2.1 are the dual of certain parallel expansions (see §7.4.1). A generalized quadrangle of type $T_2^*(O)$ is a parallel expansion of a geometry of rank 1 (see §7.4.2). We will discuss more examples in this section.

7.1 Affine expansion of buildings

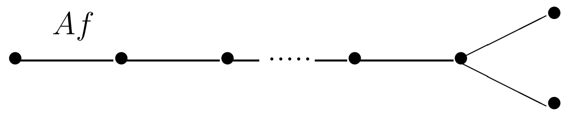
We refer to [7] (Chapter 12 by Cohen, §6.19). Consider the building Γ of a non-twisted group of Lie-Chevalley type and fix a node i in the corresponding Coxeter diagram. The i-shadow space $\sigma_i(\Gamma)$ of Γ has a natural embedding in some projective space in which each line of $\sigma_i(\Gamma)$ is a full projective line. This does extend to some twisted cases for at least one of the end nodes (types ${}^3D_4$, ${}^2F_4$, ${}^2E_6$).

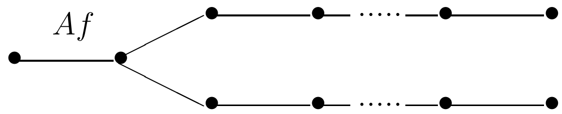
Do those embeddings give rise to parallel expansions? Thanks to the results of [8] about parallel expansion, we know this at least in certain cases. This is giving geometries with parallelism over the following diagrams with buildings as geometries at infinity

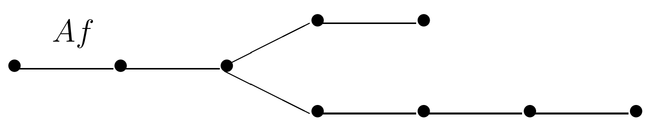
$(Af.A_n)$

$(Af.C_n)$

$(Af.D_n)$





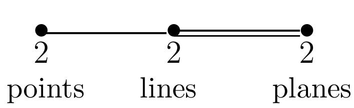


The first diagram $(Af.A_n)$ describes $(n+1)$-dimensional affine geometries, which are the prototypes of affine expansions. Affine expansions of polar spaces of rank n belong to the second diagram $(Af.C_n)$. They are (possibly improper) quotients of affine polar spaces [32] (also [30], Chapter 8, §8.4.7). The third diagram $(Af.D_n)$ describes affine expansions of D_n-buildings. These can be obtained by "unfolding" certain affine polar spaces ([30], Chapter 8, §8.4.7).

The fourth picture describes several diagrams of different kinds, including $Af.A_n$ as a "limit case".

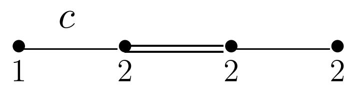
7.2 Affine expansion of the Alt(7)-geometry

Let Γ be the Alt(7)-geometry (Neumaier [27]). Γ is the only known example of a finite thick non-building geometry belonging to a connected Coxeter diagram of spherical type. It has diagram C_3 and uniform order 2



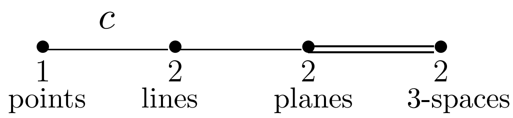
and it can be described as follows [27]. The planes and the lines of Γ are respectively the points and the lines of $\mathrm{PG}(3,2)$. The points of Γ are 7 models of the symplectic generalized quadrangle $W(2)$, transitively permuted by A_7 in its action on $\mathrm{PG}(3,2)$ as a subgroup of $L_4(2) = A_8$. The incidence relation is the natural one (containment).

We can consider the affine expansion $\overline{\Gamma}^*$ in $A = \mathrm{AG}(4,2)$ of the dual Γ^* of Γ, via the identification of the planes of Γ with the points of $A^\infty = \mathrm{PG}(3,2)$ (it is easily seen that (i) of §4.4 holds). $\overline{\Gamma}^*$ has diagram and orders as follows

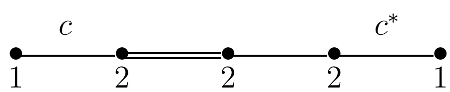


$\overline{\Gamma}^*$ is flag-transitive by Proposition 4.2, with $\mathrm{Aut}(\overline{\Gamma}^*) = 2^4{:}\mathrm{Alt}(7)$. It is the unique flag-transitive geometry with the above diagram involving Γ as a residue [33].

Another geometry Γ' is mentioned in [27] where Γ occurs as a residue. It has diagram and orders as follows

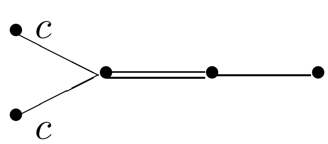


and $\mathrm{Aut}(\Gamma') = \mathrm{Alt}(8)$. It is the unique flag-transitive geometry with this diagram admitting Γ as a residue [33]. There are 15 3-spaces of Γ', as many as the points of $\mathrm{PG}(3,2)$. Hence we can consider the affine expansion $\overline{\Gamma}'^*$ in $\mathrm{AG}(4,2)$ of the dual Γ'^* of Γ', thus obtaining a geometry with diagram and orders as follows

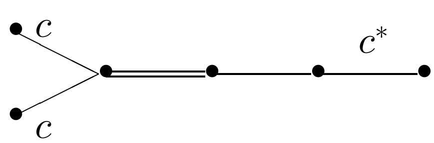


$\overline{\Gamma}'^*$ is flag-transitive by Proposition 4.2, with $\mathrm{Aut}(\overline{\Gamma}'^*) = 2^4{:}\mathrm{Alt}(8)$. It is the unique flag-transitive geometry with the above diagram admitting Γ as a residue [33]. Note that $\overline{\Gamma}^*$ is also a residue of $\overline{\Gamma}'^*$.

Parallel expansions admit parallelisms (see §4.5). Hence we can also glue two copies of $\overline{\Gamma}^*$ or of $\overline{\Gamma}'^*$, thus obtaining geometries belonging to the following diagrams, with the Alt(7)-geometry as a residue



(from $\overline{\Gamma}^*$)

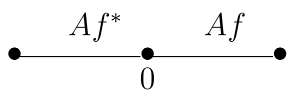


(from $\overline{\Gamma}'^*$)

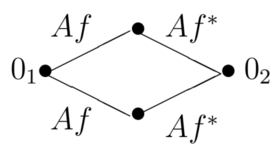
These gluings are flag-transitive (by Corollary 3.7) with automorphism groups $(2^4 \times 2^4){:}\mathrm{Alt}(7)$ and $(2^4 \times 2^4){:}\mathrm{Alt}(8)$ respectively.

7.3 Affine expansions of generalized digons

Let $A = \mathrm{AG}(4,K)$, K commutative. In A^∞ consider a ruled quadric Q and let S_1, S_2 be the two systems of lines that partition the points of Q. Here Γ consists of the lines in $S_1 \cup S_2$ and any two distinct elements are incident provided they have a common point on Q. Thus, Γ is a generalized digon. Here the 0-elements (resp. 1-elements) of A are the points (resp. planes) of $\mathrm{AG}(4,K)$. The parallel expansion $\overline{\Gamma}$ exists (this was observed by Buekenhout [6]). It is flag-transitive and it belongs to the following diagram, where the central node represents the 0-elements



(Actually, $\overline{\Gamma}$ is the unfolding of an affine polar space, by Theorem 7.57 of [30].) By gluing two copies of $\overline{\Gamma}$ we get a flag-transitive geometry belonging to the following diagram:



7.4 Duals of Laguerre-like geometries

Let A and Γ be as in section 4. To define the parallel expansion $\overline{\Gamma}$ of Γ in A we only need the $\{0,1\}$-truncation of A. However, it may be interesting to consider the other elements of A. For instance, for some A and for some choice of Γ among the subgeometries of A^∞, the elements of $\overline{\Gamma}$ may be identified with some elements of A. We will examine some examples of this kind in the next two paragraphs.

The affine expansions we describe here are the dual of some Laguerre-like geometries (see §6.2.1).

7.4.1 Affine expansions of dual hyperovals

Let $A = \mathrm{AG}(3,q)$, with q even and let $\Gamma = O^*$ be the dual of a hyperoval. Let us take $\{0,1,2\}$ as set of types of $\overline{\Gamma}$, with 0 and 1 as in section 4. The 0-elements of $\overline{\Gamma}$ are the points of $\mathrm{AG}(3,q)$, the 1-elements are the lines of $\mathrm{AG}(3,q)$ with point at infinity in Γ and the remaining elements (those of type 2) are the planes of $\mathrm{AG}(3,q)$ with line at infinity in Γ (that is in O^*).

$\overline{\Gamma}$ inherits a 0-parallelism $\|$ from $\mathrm{AG}(3,q)$. It is straightforward to prove that this is the unique 0-parallelism of $\overline{\Gamma}$. Hence $\mathrm{Aut}(\overline{\Gamma}) = \mathrm{Aut}(\overline{\Gamma}, \|)$ (see §2.5).

By an argument of Huybrechts [20], $\mathrm{Aut}(\overline{\Gamma})$ is the subgroup of $A\Gamma L_3(q)$ preserving O^*. Hence $\overline{\Gamma}$ is flag-transitive if and only $q = 2$, 4 or 16, with O^* the dual of the Lunelli-Sce hyperoval when $q = 16$ (see [19], [24]). Furthermore, $\overline{\Gamma}^\infty$ is complete if and only if $q = 2$ or 4.

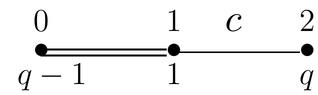
On the other hand, any point p of $A^\infty \backslash O^*$ define a 2-parallelism in the following way : for each point (resp. line) x of $\overline{\Gamma}$, the parallel class of x is the pointset (resp. lineset) in $\overline{\Gamma}$ of the projective subspace $< p, x >$ generated by p and x.

Similar results hold for affine expansions of dual Witt-Bose-Shrikhande spaces (compare §6.2.3, example 3).

7.4.2 A construction for $T_2^*(O)$

Replace O^* by a hyperoval O in §7.4.1 and denote by Γ the complete graph with O as set of vertices. The affine expansion $\overline{\Gamma}$ of Γ in $A = \mathrm{AG}(3,q)$ belongs to the Coxeter diagram $C_2.c$

$(C_2.c)$



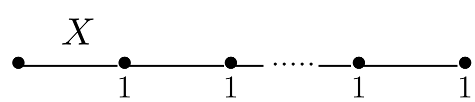
(types are written above and orders below the nodes of the diagram). The 0-elements of $\overline{\Gamma}$ are the points of A, the 1-elements are the lines of A with point at infinity in O and the 2-elements are the planes of A with a secant of O as line at infinity. Actually, $\overline{\Gamma}$ is a truncation of a thin-lined C_n-geometry (see §7.5.2) obtained as an affine expansion of the thin projective geometry having O as set of points (§7.5).

The $\{0,1\}$-truncation of $\overline{\Gamma}$ is the generalized quadrangle $T_2^*(O)$. It is straightforward to check that the parallelism $\|$ inherited from $\mathrm{AG}(3,q)$ is the unique 0-parallelism of $T_2^*(O)$. On the other hand, every line L of A^∞ external to O induces a 1-parallelism $\|_L$ on $T_2^*(O)$. The parallel classes of $\|_L$ are the planes of A with L as line at infinity (compare §6.3.1).

7.5 Expanding thin projective geometries

Given a connected geometry A of rank 2 over the set of types $\{0,1\}$ and with a 0-parallelism $\|$, let B be a finite subset of A^∞ of size $n \geq 2$ and let Γ be the thin projective geometry with B as set of points. Denote by $\Gamma(A,B)$ the parallel expansion of Γ in A. Obviously, $\Gamma(A,B)$ has rank n and it belongs to a diagram of the following form, with order 1 at every node except possibly the first one:

$(X.A_{n-1})$



where X denotes some class of geometries containing the geometries obtained from A by removing all classes of $\|$ but two. Note that all geometries obtained in this way have even gonality.

7.5.1 Cell geometries of chamber systems

Let $\mathcal{C}$ be a chamber system with the properties considered in §5.11. As we noticed in §5.11 , the geometry $\Gamma_{\mathcal{C}}$ of cells of $\mathcal{C}$ is the parallel expansion, in the semilinear space A of chambers and panels of $\mathcal{C}$, of the thin projective geometry Γ having the types $\mathcal{C}$ as points. That is, $\Gamma_{\mathcal{C}} = \Gamma(A,B)$ where $B = A^\infty$ is the set of types of $\mathcal{C}$.

On the other hand, let A be a semilinear space over the set of types $\{0,1\}$, with a 0-parallelism $\|$ and finitely many points at infinity. We can define a chamber system $\mathcal{C}(A,\|)$ by taking the 0-elements of A as chambers and the 1-elements as panels, using A^∞ as set of types. Clearly, if $B = A^\infty$, then $\Gamma(A,B)$ is just the geometry of cells of $\mathcal{C}(A,\|)$.

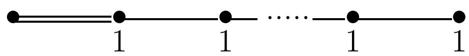
7.5.2 Thin-lined C_n-geometries

Let A be a semilinear space over the set of types $\{0,1\}$, equipped with a 0-parallelism $\|$ satisfying the following property

(P) for any 1-elements L, M, L', M', from $L \parallel L'$, $M \parallel M'$ and $L \cap M \neq \emptyset$ it follows that $L' \cap M' \neq \emptyset$.

Then for every subset B of A^∞ of size $n \geq 2$ the parallel expansion $\Gamma(A, B)$ belongs to the Coxeter diagram C_n, with order 1 at all nodes except possibly the first one

(C_n)



Namely $\Gamma(A, B)$ is a *thin-lined C_n-geometry*, according to a popular terminology.

Deleting some lines of A if necessary, we can always assume that $B = A^\infty$. Thus we can consider the chamber system $\mathcal{C}(A, \|)$ as in §7.5.1 and $\Gamma(A, B)$ is the geometry of cells of $\mathcal{C}(A, \|)$. As (P) holds in $(A, 0, \|)$, the chamber system $\mathcal{C}(A, \|)$ belongs to the diagram with n vertices but no edges. We call this diagram *trivial*.

The following is proved in [31]

Theorem 7.1 *Every thin-lined C_n-geometry is the geometry of cells of a chamber system with trivial diagram.*

That is, every thin-lined C_n geometry is the parallel expansion of the thin projective geometry of rank $n-1$ in some semilinear space with parallelism. Thus, a classification of thin-lined C_n-geometries is equivalent to a classification of chamber systems with trivial diagram of rank n and this is in turn equivalent to a classification of semiliner spaces endowed with a parallelism satisfying (P) with n points at infinity.

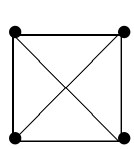
Some classification of finite thin-lined C_3 has been given by S.Rees [36] by means of latin squares.

Hamming spaces of rank n are thin-lined C_n-geometries. In fact they are the duals of thin-lined polar spaces of rank n (§5.3.2). That is, a Hamming space of rank n is the cell geometry of the chamber system of a geometry with trivial diagram of rank n.

7.5.3 An example from the octahedron

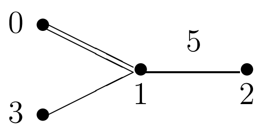
Let A be the graph of the octahedron with its unique parallelism $\|$ (see §5.6.1) and $B = A^\infty$. Then $\Gamma(A, B)$ is a thin geometry for the following Coxeter diagram

The chamber system $\mathcal{C}(A, \|)$ belongs to the following Coxeter diagram



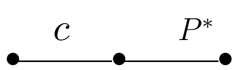
7.6 An example from the Petersen graph

Let Γ be the hemi-icosahedron. The Petersen graph is the system of faces and edges of Γ. The dual of this graph can be embedded in $\mathrm{PG}(n,2)$ with $n = 3,4$ or 5 (see [10]). The vertices of Γ correspond to pentagons of the Petersen graph. Thus, we can realize Γ in $\mathrm{PG}(n,2)$ (n as above), representing the edges and the faces of Γ by suitable points and lines of $\mathrm{PG}(n,2)$ and realizing the vertices of Γ as pentagons. (i) of §4.4 holds. Hence Γ admits an affine expansion $\overline{\Gamma}$ in $\mathrm{AG}(n+1,2)$. It is not difficult to check that $\overline{\Gamma}$ is thin and it belongs to the following Coxeter diagram



where 0, 1, 2 and 3 are types. The elements of type 0, 1 and 2 are respectively points lines and planes of $\mathrm{AG}(n+1,2)$. The residues of the elements of type 0 are isomorphic to Γ, whereas the residues of the elements of type 2 are hemi-cubes. The residues of the elements of type 3 are affine expansions of pentagons. $\overline{\Gamma}$ is flag-transitive by Proposition 4.2 and $\mathrm{Aut}(\overline{\Gamma}) = 2^{n+1}{:}\mathrm{Alt}(5)$.

The $\{0,1,2\}$-truncation $\overline{\Gamma}'$ of $\overline{\Gamma}$ is an affine expansion of the dual Petersen graph (Buekenhout-King [10], section 5) with diagram as follows:



We have $\mathrm{Aut}(\overline{\Gamma}') = 2^{n+1}{:}\mathrm{Sym}(5)$, larger than $\mathrm{Aut}(\overline{\Gamma})$. Indeed there are two isomorphic ways of choosing six pentagons in the Petersen graph to represent the six point of the hemi-icosahedron. Thus, we have two models of $\overline{\Gamma}$ with the same $\{0,1,2\}$-truncation $\overline{\Gamma}'$ and some automorphisms of $\overline{\Gamma}'$ interchange them.

7.7 Alexandrov spaces

A famous theorem of Alexandrov characterizes the space-time of special relativity (see [7], chapter 16 by Lester). In [11], Buekenhout and Masson provide a broad generalization in terms of affine incidence geometry. Their Alexandrov spaces are instances of parallel expansion and the problems left open in that work can be transported to our more general setting.

References

[1] **J. André**. Non-commutative spaces with transitive translation groups. *Geom. Dedicata*, 36, pp. 125–137, 1990.

[2] **R. D. Baker**. Partioning the planes of $\mathrm{AG}_{2^m}(2)$ into 2-designs. *Discrete Math.*, 15, pp. 35–40, 1974.

[3] **C. Berge**. *Graphs and Hypergraphs*. North-Holland, 1973.

[4] **A. Beutelspacher**. On parallelism in finite projective spaces. *Geom. Dedicata*, 3, pp. 35–40, 1974.

[5] **F. Buekenhout**. On a theorem by O'Nan and Scott. *Bull. Soc. Math. Belg. Sér. B*, 40, pp. 1–9, 1988.

[6] **F. Buekenhout**. More geometries for Hering's $3^6{:}SL(2,13)$. In J. W. P. Hirschfeld, D. R. Hughes, and J. A. Thas, editors, *Advances in Finite Geometries and Designs*, pages 57–68. Oxford University Press, 1991.

[7] **F. Buekenhout**. *Handbook of Incidence Geometry.* North-Holland, to appear.

[8] **F. Buekenhout, M. Dehon, and I. De Schutter**. Projective injections of geometries and their affine extensions, 1994. To appear in *J. Geometry.*

[9] **F. Buekenhout, A. Delandtsheer, and J. Doyen**. Finite linear spaces with flag-transitive groups. *J. Combin. Theory Ser. A*, 49, pp. 268–293, 1988.

[10] **F. Buekenhout and O. King**. Geometries with diagram $L.P^*$. In A. Beutelspacher, F. Buekenhout, F. De Clerck, J. Doyen, J. W. P. Hirschfeld, and J. A. Thas, editors, *Finite Geometry and Combinatorics*, pages 73–87. Cambridge University Press, 1993.

[11] **F. Buekenhout and P. Masson**. The Alexandrov theorem of space-time in terms of affine incidence geometry, 1994. In preparation.

[12] **P. Cameron and G. Korchmáros**. One-factorization of complete graphs with a doubly transitive automorphism group. *Bull. London Math. Soc.*, 25, pp. 1–6, 1993.

[13] **H. Cuypers**. Affine grassmannians. To appear in *J. Combin. Theory Ser. A.*

[14] **R. H. F. Denniston**. Some packings of projective spaces. *Rend. Accademia Lincei, Classe Mat. Fis. e Sc. Nat.*, 52, pp. 36–40, 1972.

[15] **F. Harary**. *Graph Theory.* Addison-Wesley, 1969.

[16] **A. Del Fra**. Flag-transitive $Af.C_2$ geometries. To appear.

[17] **A. Del Fra and A. Pasini**. On $Af.Af^*$ geometries. To appear.

[18] **W. Heise and H. Karzel**. Laguerre- und Minkowski-m-Strukturen. *Rend. Ist. Mat. Trieste*, 14, pp. 139–147, 1972.

[19] **J. W. P. Hirschfeld**. *Projective Geometries over Finite Fields.* Clarendon Press, 1979.

[20] **C. Huybrechts**. Some infinite families of locally partial geometries. To appear.

[21] **W. M. Kantor**. Generalized polygons, SCABs and GABs. In L. A. Rosati, editor, *Buildings and the Geometry of Diagrams, Proceedings Como 1984*, volume 1181 of *Lecture Notes in Math.*, pages 79–158. Springer Verlag, 1986.

[22] **D. Kőnig**. *Theorie der Endlichen und Unendlichen Graphen*. Teibner, 1936.

[23] **G. Korchmáros**. Cyclic one-factorization with an invariant one-factor of the complete graph. *Ars. Combinatoria*, 27, pp. 133–138, 1989.

[24] **G. Korchmáros**. Metodi di teoria dei gruppi nello studio delle ovali dei piani proiettivi finiti. *Rendiconti Sem. Mat. Fis. Univ. di Milano*, 60, pp. 93–111, 1990.

[25] **T. Meixner and A. Pasini**. Flat flag-transitive extended grids. In preparation.

[26] **T. Meixner and A. Pasini**. On flag-transitive extended grids. To appear.

[27] **A. Neumaier**. Some sporadic geometries related to PG$(3,2)$. *Arch. Math. (Basel)*, 42, pp. 89–96, 1984.

[28] **N. Nizette**. Variétés linéaires et automorphismses des espaces affins sur un presque-corp non planaire. *J. Geometry*, 2, pp. 175–183, 1972.

[29] **A. Pasini**. Gluing two copies of an affine space. In preparation.

[30] **A. Pasini**. An Introduction to Diagram Geometry. Book manuscript, to appear at Oxford University Press.

[31] **A. Pasini**. On a problem on chamber systems. Submitted.

[32] **A. Pasini**. Quotients of affine polar spaces. *Bull. Soc. Math. Belg.*, 42, pp. 643–658, 1990.

[33] **A. Pasini and S. Yoshiara**. Generalized towers of flag-transitive circular extensions of a non-classical C_3-geometry. To appear in *J. Combin. Theory, Ser. A.*

[34] **S. E. Payne**. Hyperovals and generalized quadrangles. In C. A. Baker and L. A. Batten, editors, *Finite Geometries*, volume 103 of *Lecture Notes in Pure and Applied Mathematics*, pages 251–271. M. Dekker, 1985.

[35] **S. E. Payne and J. A. Thas**. *Finite Generalized Quadrangles*, volume 104 of *Research Notes in Mathematics*. Pitman, 1984.

[36] **S. Rees**. Finite C_3-geometries in which all lines are thin. *Math. Zeit.*, 189, pp. 263–271, 1985.

[37] **M. A. Ronan**. Coverings and automorphisms of chamber systems. *European J. Combinatorics*, 1, pp. 259–269, 1980.

[38] **S. Schmidt**. Grundlegungen zu einer Allgemeinen Affinen Geometrie, 1992. Habilitationsschrift, Mainz.

[39] **J. Tits**. *Buildings of Spherical Type and Finite BN-pairs*, volume 386 of *Lect. Notes in Math.* Springer Verlag, 1974.

[40] **J. Tits**. A local approach to buildings. In *The Geometric Vein (Coxeter Festschrift)*, pages 519–547. Springer Verlag, 1981.

[41] **O. Veblen and J. Young**. *Projective Geometry.* Ginn, 1917.

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