Collineations of the Subiaco generalized quadrangles

S. E. Payne

Dedicated to J. A. Thas on his fiftieth birthday

Abstract

Each generalized quadrangle (GQ) of order (q^2, q) derived in the standard way from a conical flock via a q-clan with $q = 2^e$ has subquadrangles of order q associated with a family of $q+1$ (not necessarily projectively equivalent) ovals in $\mathrm{PG}(2, q)$. A new family of these GQ is announced in [1] and named the Subiaco GQ. We begin a study of their collineation groups. When e is odd, $e \geq 5$, the group is determined. In the standard notation for the GQ, the collineation group is transitive on the lines through the point (∞). As a corollary we have that up to the usual notions of equivalence, just one conical flock, one oval in $\mathrm{PG}(2, q)$, and one subquadrangle of order (q, q) arise.

1 Introduction

The objects studied in this paper are introduced in [1], and we thank its authors for making their work available to us as it was being developed. Moreover, Tim Penttila and Gordon Royle helped us eliminate a serious error in an early version of this work.

Let $F = \mathrm{GF}(q)$, $q = 2^e$. For each $t \in F$, let $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$ be a 2×2 matrix over F. Put $\mathcal{C} = \{A_t : t \in F\}$. Then $\mathcal{C}$ is a q-clan provided $A_t - A_s$ is anisotropic (i.e., $\alpha(A_t - A_s)\alpha^T = 0$ if and only if $\alpha = (0, 0)$) whenever $t, s \in F$, $t \neq s$. This holds if and only if $(x_t - x_s)(z_t - z_s)(y_t - y_s)^{-2}$ has trace 1 whenever $s \neq t$. From

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now on we assume that $\mathcal{C}$ is a q-clan, so the three maps $t \mapsto x_t$, $t \mapsto y_t$, $t \mapsto z_t$ are all permutations. Put $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. And for $A_t \in \mathcal{C}$, put $K_t = A_t + A_t^T = y_t P$.

There is a standard construction of a generalized quadrangle (GQ) $\mathcal{S} = \mathcal{S}(\mathcal{C})$ as a coset geometry starting with the group

$$\overline{G} = \{(\alpha, c, \beta) : \alpha, \beta \in F^2,\ c \in F\}$$

whose binary operation is given by

$$(\alpha, c, \beta) * (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta'), \tag{1}$$

and a certain 4-gonal family of subgroups. Specifically, put $\overline{A}(\infty) = \{(\vec{0}, 0, \beta) \in \overline{G} : \beta \in F^2\}$, and for $t \in F$, $\overline{A}(t) = \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) \in \overline{G} : \alpha \in F^2\}$. Put $\overline{\mathcal{F}} = \{\overline{A}(t) : t \in F \cup \{\infty\}\}$, and $C = \{(\vec{0}, 0, \vec{0}) \in \overline{G} : c \in F\}$. For $A \in \overline{\mathcal{F}}$, put $A^* = AC$. Then $\overline{\mathcal{F}}$ is a 4-gonal family for $\overline{G}$ with associated groups (tangent spaces) $\overline{\mathcal{F}}^* = \{A^* : A \in \overline{\mathcal{F}}\}$. We assume the reader is familiar with W. M. Kantor's construction of a GQ $\mathcal{S}(\overline{G}, \overline{\mathcal{F}})$ (cf. [3], [4], [8]). In [8], pp. 213–214, it is shown that for a fixed $t_0 \in F$, a new q-clan may be constructed so that each $A_t \in \mathcal{C}$ is replaced with $A_t - A_{t_0}$, and the "new" GQ is isomorphic to the original. Then the new matrices may be reindexed so that $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

For two matrices $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, $B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ over F, let $A \equiv B$ mean that $x = r$, $w = u$, and $y + z = s + t$. So $A \equiv B$ if and only if $\alpha A \alpha^T = \alpha B \alpha^T$ for all $\alpha \in F^2$.

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, F)$. For $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} \in \mathcal{C}$, put

$$\overline{A}_t = B\, A_t B^T \equiv \begin{pmatrix} a^2 x_t + ab y_t + b^2 z_t & (ad + bc) y_t \\ 0 & c^2 x_t + cd y_t + d^2 z_t \end{pmatrix}.$$

Then

$$(\alpha, c, \beta) \mapsto (\alpha B^{-1}, c, \beta B^T) \tag{2}$$

is an automorphism of $\overline{G}$ that replaces $\overline{\mathcal{F}}$ with a 4-gonal family derived from the q-clan $\overline{\mathcal{C}} = \{\overline{A}_t : t \in F\}$ and that produces a GQ isomorphic to the original.

First, by reindexing the members of $\mathcal{C}$ we may assume $x_t = t$ for all $t \in F$. So $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 1 & y_1 \\ 0 & z_1 \end{pmatrix}$. Then using $B = \begin{pmatrix} 1 & 0 \\ 0 & y_1^{-1} \end{pmatrix}$ in equation (2), we may assume $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix}$, where $\delta \in F$ is some element with trace 1. We again reindex the members of $\mathcal{C}$ to obtain $y_t = t$ (probably destroying $x_t = t$) for all $t \in F$. So without loss of generality we may assume that the q-clan $\mathcal{C}$ has been normalized to satisfy the following:

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix} (\text{with } \mathrm{tr}(\delta) = 1); \quad A_t = \begin{pmatrix} x_t & t \\ 0 & z_t \end{pmatrix}, \ t \in F. \quad (3)$$

From [5] recall the following notation: For $\alpha \in F^2$, $[\alpha]_\infty = (\overrightarrow{0}, 0, \alpha) \in \overline{G}$; for $t \in F$, $[\alpha]_t = (\alpha, \alpha A_t \alpha^T, \alpha K_t)$; for $c \in F$, $[c] = (\overrightarrow{0}, c, \overrightarrow{0}) \in \overline{G}$. For $t, u \in F \cup \{\infty\}$, $t \neq u$, put $([\alpha]_t, [c], [\beta]_u) := [\alpha]_t * [c] * [\beta]_u$. A simple computation shows that

$$([\alpha]_\infty, [c], [\beta]_0) * ([\alpha']_\infty, [c'], [\beta']_0) = ([\alpha + \alpha']_\infty, [c + c' + \beta(\alpha')^T], [\beta + \beta']_0). \quad (4)$$

And with $\gamma = \alpha K_t$,

$$\begin{aligned} [\alpha]_t &= (\alpha, \alpha A_t \alpha^T, \alpha K_t) = ([\alpha K_t]_\infty, [\alpha A_t \alpha^T], [\alpha]_0) \\ &= ([\gamma]_\infty, [\gamma K_t^{-1} A_t K_t^{-1} \gamma^T], [\gamma K_t^{-1}]_0). \end{aligned} \quad (5)$$

Since in the original description of $\overline{G}$, $(\alpha, c, \beta) = ([\alpha]_0, [c], [\beta]_\infty)$, it follows that by interchanging the roles of ∞ and 0 in the description of $\overline{G}$ the matrix $A_t \in \mathcal{C}$ is replaced with $K_t^{-1} A_t K_t^{-1} \equiv y_t^{-2} \begin{pmatrix} z_t & y_t \\ 0 & x_t \end{pmatrix}$. Now by using $B = P$ in equation (2), we have that $\hat{\mathcal{C}} = \left\{ A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \{ y_t^{-2} A_t : 0 \neq t \in F \}$ is a q-clan associated with a GQ isomorphic to that derived from $\mathcal{C}$. By combining this operation with the shift $\overline{A}_t = A_t - A_{t_0}$ mentioned earlier, we obtain

$$\hat{\mathcal{C}} = \{ (y_t - y_{t_0})^{-2} (A_t - A_{t_0}) : t_0 \neq t \in F \} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \quad (6)$$

is a q-clan associated with a GQ isomorphic to (essentially the same as) that derived from $\mathcal{C}$, but recoordinatized so that the line $[A(t_0)]$ of $\mathcal{S}(\mathcal{C})$ through the point (∞) corresponds to the line $[A(\infty)]$ through (∞) in $\mathcal{S}(\hat{\mathcal{C}})$. (Also see [7], [9].)

A truly satisfactory geometric interpretation of the construction of a GQ from a q-clan (equivalent to a conical flock by J. A. Thas [10]) must somehow explain the existence of these $q+1$ distinct (and not always equivalent) q-clans. For the purpose of distinguishing flocks, it is important to note that there is a collineation of $\mathcal{S}(\mathcal{C})$ (fixing (∞) and $(\overrightarrow{0}, 0, \overrightarrow{0})$) moving the line $[A(t_0)]$ to the line $[A(\infty)]$ if and only if the flock associated with $\mathcal{C}$ is equivalent to that associated with the q-clan $\hat{\mathcal{C}}$ obtained in equation (6).

We now recall the slightly modified description of $\overline{G}$ introduced in [6] (cf. also [4]). In characteristic 2, this revised version seems to us to be more natural and useful.

Let $E = F(\zeta) = \mathrm{GF}(q^2)$, $\zeta^2 + \zeta + \delta = 0$ (for some $\delta \in F$ with $\mathrm{tr}(\delta) = 1$). Then $x \mapsto \overline{x} = x^q$ is the unique involutionary automorphism of E with fixed field F. Here $\zeta + \overline{\zeta} = 1$ and $\zeta\overline{\zeta} = \delta$. The element $\alpha = a + b\zeta \in E$ $(a, b \in F)$ is often (without notice) identified with the pair $(a, b) \in F^2$. For example, the inner product

$$\alpha \circ \beta = \alpha\overline{\beta} + \overline{\alpha}\beta \quad (7)$$

on E as a vector space over F may also appear as $\alpha \circ \beta = \alpha P \beta^T$. Note that $\alpha \circ \beta = 0$ if and only if $\{\alpha, \beta\}$ is F-dependent.

Now put $G = \{(\alpha, c, \beta) : \alpha, \beta \in E = F^2,\ c \in F\}$ with binary operation

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha',\ c + c' + \sqrt{\beta \circ \alpha'},\ \beta + \beta'). \tag{8}$$

It is straightforward to check that $\overline{G} \to G : (\alpha, c, \beta) \mapsto (\alpha, \sqrt{c}, \beta P)$ is an isomorphism mapping the 4-gonal family $\overline{\mathcal{F}}$ for $\overline{G}$ to a 4-gonal family $\mathcal{F}$ of G defined as follows:

$$\begin{aligned} A(\infty) &= \{(\overrightarrow{0}, 0, \beta) \in G : \beta \in E\}; \\ A^*(\infty) &= \{(\overrightarrow{0}, c, \beta) \in G : c \in F,\ \beta \in F^2\}. \end{aligned} \tag{9}$$

And for $t \in F$,

$$\begin{aligned} A(t) &= \{(\alpha, \sqrt{\alpha A_t \alpha^T}, y_t \alpha) \in G : \alpha \in F\}, \\ A^*(t) &= \{(\alpha, c, y_t \alpha) \in G : \alpha \in E,\ c \in F\}. \end{aligned} \tag{10}$$

Clearly $\mathcal{F} = \{A(t) : t \in F \cup \{\infty\}\}$ yields a GQ $\mathcal{S}(G, \mathcal{F})$ isomorphic to $\mathcal{S}(\overline{G}, \overline{\mathcal{F}})$. The revised description $\mathcal{S}(G, \mathcal{F})$ makes it easy to recognize subquadrangles.

2 Subquadrangles and ovals

Let $\mathcal{C}$ be a q-clan (normalized as in equation (3)) with corresponding 4-gonal family $\mathcal{F}$ for G (in the revised form just given), etc., and let $\mathcal{S} = \mathcal{S}(G, \mathcal{F})$ be the associated GQ. For $\vec{0} \neq \alpha \in E$, put

$$G_\alpha = \{(a\alpha, c, b\alpha) \in G : a, b, c \in F\}. \tag{11}$$

Since $\alpha \circ \beta = 0$ if $\beta = c\alpha$, $c \in F$, in G_α we have

$$(a\alpha, c, b\alpha) \cdot (a'\alpha, c', b'\alpha) = ((a + a')\alpha, c + c', (b + b')\alpha). \tag{12}$$

So G_α is an elementary abelian group with order q^3. Define the following subgroups of G_α:

$$\begin{aligned} A_\alpha(\infty) &= A(\infty) \cap G_\alpha = \{(\vec{0}, 0, b\alpha) \in G : b \in F\}; \\ A^*_\alpha(\infty) &= A^*(\infty) \cap G_\alpha = \{(\vec{0}, c, b\alpha) \in G : c, b \in F\}. \end{aligned} \tag{13}$$

And for $t \in F$,

$$\begin{aligned} A_\alpha(t) &= A(t) \cap G_\alpha = \{(a\alpha, a\sqrt{\alpha A_t \alpha^T},\ at\alpha) \in G : a \in F\}; \\ A^*_\alpha(t) &= A^*(t) \cap G_\alpha = \{(a\alpha, c, at\alpha) \in G : a, c \in F\}. \end{aligned} \tag{14}$$

Here $\mathcal{F}_\alpha = \{A_\alpha(t) : t \in F \cup \{\infty\}\}$ is immediately seen to be a 4-gonal family for G_α. Moreover, by [6] we may view $\mathcal{S}_\alpha = \mathcal{S}(G_\alpha, \mathcal{F}_\alpha)$ as a subquadrangle (of order

q) of $\mathcal{S} = \mathcal{S}(G, \mathcal{F})$. Clearly G_α is isomorphic to F^3 under componentwise addition. Moreover, we can define a scalar multiplication on G_α by

$$d(a\alpha, c, b\alpha) = (da\alpha, dc, db\alpha), a, b, c, d \in F. \tag{15}$$

Then it is also clear that $\mathcal{F}_\alpha$ is an oval in the projective plane naturally associated with the 3-dimensional F-linear space G_α.

Consider the three projective points $p_1 = (\alpha, 0, \vec{0})$, $p_2 = (\vec{0}, 1, \vec{0})$, $p_3 = (\vec{0}, 0, \alpha)$. The scalar triple $(1, \sqrt{\alpha A_t \alpha^T}, t)$ on the points p_1, p_2 and p_3 results in $(\alpha, 0, \vec{0}) \cdot (\vec{0}, \sqrt{\alpha A_t \alpha^T}, \vec{0}) \cdot (\vec{0}, 0, t\alpha) = (\alpha, \sqrt{\alpha A_t \alpha^T}, t\alpha) \leftrightarrow A_\alpha(t)$ considered as a projective point. Hence the oval $\mathcal{F}_\alpha$ is the set of $q+1$ points represented by the following set of triples of coordinates:

$$\mathcal{O}_\alpha = \{(0,0,1)\} \cup \{(1, \sqrt{\alpha A_t \alpha^T}, t) : t \in F\}, \text{with nucleus } (0,1,0). \tag{16}$$

We say that the map $t \mapsto \sqrt{\alpha A_t \alpha^T}$ is an $\mathcal{O}$-permutation. Specifically, a permutation $\gamma : F \to F$ is an *$\mathcal{O}$-permutation* provided

$$\begin{array}{ll} \text{(i)} & \gamma : 0 \mapsto 0, \text{and} \\ \text{(ii)} & \dfrac{s^\gamma - t^\gamma}{s - t} \neq \dfrac{s^\gamma - u^\gamma}{s - u} \end{array} \tag{17}$$

whenever s, t and u are distinct members of F.

Note. This language is suggested by the term $\mathcal{O}$-polynomial in [2], except that we find it more convenient NOT to require that γ be normalized so that $\gamma : 1 \mapsto 1$. (We could also avoid $\gamma : 0 \mapsto 0$, but this holds in all the specific examples we consider.)

For $\alpha = (a_1, a_2) \neq (0,0)$, $A_t = \begin{pmatrix} x_t & t \\ 0 & z_t \end{pmatrix}$, the above becomes

$$t \mapsto a_1\sqrt{x_t} + \sqrt{a_1 a_2}\sqrt{t} + a_2\sqrt{z_t} \text{ is an } \mathcal{O}\text{-permutation.} \tag{18}$$

In the next section it will be convenient to represent q-clans in the form $\mathcal{C} = \left\{A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix} : t \in F\right\}$. If $\sigma : x \mapsto x^2$ is the Frobenius automorphism of F, then clearly $\mathcal{C}^\sigma = \left\{A_t^\sigma = \begin{pmatrix} f(t)^2 & t \\ 0 & g(t)^2 \end{pmatrix} : t \in F\right\}$ is also a q-clan, so

$$t \mapsto a^2 f(t) + abt^{1/2} + b^2 g(t) \tag{19}$$

is an $\mathcal{O}$-permutation whenever $(a,b) \neq (0,0)$.

Note. T. Penttila (private communication) has given a direct proof of (19) without mentioning GQ.

Using standard notation for the GQ $\mathcal{S} = \mathcal{S}(G, \mathcal{F})$, let Γ be the dual grid spanned by the points (∞) and $(\vec{0}, 0, \vec{0})$. All the subquadrangles $\mathcal{S}_\alpha$ constructed above contain Γ. We know that two subquadrangles of order q in $\mathcal{S}$ that have a dual grid (with $2(q+1)$ points) in common must have in common just the points and lines of that dual grid. Hence $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are identical if $\{\alpha, \beta\}$ is F-dependent, and they meet in Γ otherwise. It follows that we have constructed a family of $q+1$ subquadrangles $\mathcal{S}_\alpha$ of order q pairwise intersecting in Γ.

3 Collineations

Suppose without loss of generality that the matrices of the q-clan $\mathcal{C}$ are given in the form $A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix}$, $t \in F$, with $f(0) = g(0) = 0$ and $f(1) = 1$. Then $K_t^{-1} A_t K_t^{-1} \equiv \begin{pmatrix} t^{-1}g(t) & t^{-1/2} \\ 0 & t^{-1}f(t) \end{pmatrix}$. Starting with the paragraph following equation (5), it is straightforward to prove the following.

Theorem 3.1 *If $f(t^{-1}) = t^{-1}g(t)$ (equivalently, $g(t^{-1}) = t^{-1}f(t)$), then the automorphism $\theta : G \to G : (\alpha, c, \beta) \mapsto (\beta P, c + \sqrt{\alpha P \beta^T}, \alpha P)$ induces a collineation of $\mathcal{S}(G, F)$ that interchanges $A(\infty)$ and $A(0)$, and interchanges $A(t)$ and $A(t^{-1})$ for $0 \neq t \in F$.*

We now recall a result from [7], but modified to fit our group G introduced at the end of section 1.

Theorem 3.2 *Let $\mathcal{C} = \{A_t : t \in F\}$ be a q-clan with $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let θ be a collineation of the* GQ *$\mathcal{S} = \mathcal{S}(G, \mathcal{F})$ derived from $\mathcal{C}$ which fixes the point (∞), the line $[A(\infty)]$ and the point $(\vec{0}, 0, \vec{0})$. Then the following must exist:*

(i) *A permutation $t \mapsto t'$ of the elements of F;*

(ii) *$\lambda \in F$, $\lambda \neq 0$;*

(iii *$\sigma \in \mathrm{Aut}(F)$;*

(iv) *$D \in \mathrm{GL}(2, q)$ for which $A_{t'} \equiv \lambda D^T A_t^\sigma D - A_{0'}$ for all $t \in F$.*

Conversely, given σ, D, λ and a permutation $t \mapsto t'$ satisfying the above conditions, the following automorphism θ of G induces a collineation of $\mathcal{S}(G, \mathcal{F})$ fixing (∞), $[A(\infty)]$ and $(\vec{0}, 0, \vec{0})$:

$$\begin{aligned} \theta &= \theta(\sigma, D, \lambda) : G \to G : \\ (\alpha, c, \beta) &\mapsto (\lambda^{-1}\alpha^\sigma D^{-T}, \lambda^{-1/2}c^\sigma + \lambda^{-1}\sqrt{\alpha^\sigma D^{-T} A_{0'} D^{-1} (\alpha^\sigma)^T}, \\ &\qquad \beta^\sigma PDP + \lambda^{-1} y_{0'} \alpha^\sigma D^{-T}). \end{aligned} \tag{20}$$

Theorem 3.3 *For $A_t = \begin{pmatrix} f(t) & t^{1/2} \\ 0 & g(t) \end{pmatrix}$, the conditions in theorem 3.2 are equivalent to having a permutation $t \mapsto t'$, $0 \neq \lambda \in F$, $D = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}(2, F)$, $\sigma \in \mathrm{Aut}(F)$, for which*

$$\begin{array}{rl} \text{(i)} & t' = \lambda^2 (ad + bc)^2 t^\sigma + 0', \text{for all } t \in F. \\ \text{(ii)} & f(t') = \lambda[a^2 f(t)^\sigma + ab t^{\sigma/2} + b^2 g(t)^\sigma] + f(0'), \text{for all } t \in F. \\ \text{(iii)} & g(t') = \lambda[c^2 f(t)^\sigma + cd t^{\sigma/2} + d^2 g(t)^\sigma] + g(0'), \text{for all } t \in F. \end{array} \tag{21}$$

For completeness, we note that right multiplication by elements of G induces a group of q^5 collineations of $\mathcal{S}(G,\mathcal{F})$ acting regularly on the set of points not collinear with (∞) and fixing each line through (∞).

The Subiaco GQ introduced in [1] all have q-clans of the form used in theorem 3.3 with the following additional specializations:

$$\begin{array}{rl} \text{(i)} & f(t) = \frac{f'(t)}{k(t)} + Ht^{1/2},\ t \in F; \\ \text{(ii)} & g(t) = \frac{g'(t)}{k(t)} + Kt^{1/2},\ t \in F; \text{where} \\ \text{(iii)} & k(t) \text{ is the square of an irreducible quadratic polynomial} \\ & (\text{say } k(t) = t^4 + c_2t^2 + c_0); \\ \text{(iv)} & f'(t) \text{ and } g'(t) \text{ are polynomials over } F \text{ of degree at most 4} \\ & \text{with } f'(0) = g'(0) = 0 \text{ (and } f(1) = 1\text{); and} \\ \text{(v)} & H \text{and } K \text{are nonzero elements of } F. \end{array} \tag{22}$$

Then the conditions of theorem 3.3 can be rewritten.

Theorem 3.4 *Suppose* $A_t = \begin{pmatrix} \frac{f'(t)}{k(t)} + Ht^{1/2} & t^{1/2} \\ 0 & \frac{g'(t)}{k(t)} + Kt^{1/2} \end{pmatrix}$, $t \in F$, *with the conditions of* (22) *satisfied. Then the conditions in equation* (21) *of theorem* 3.3 *take on the following form:*

$$\begin{array}{rl} \text{(i)} & t' = \lambda^2(ad+bc)^2t^\sigma + 0'. \\ \text{(ii)} & f'(t')k(t)^\sigma k(0') + \lambda(a^2f'(t)^\sigma + b^2g'(t)^\sigma)k(t')k(0') + k(t')k(t)^\sigma f'(0') \\ & +\lambda k(t')k(t)^\sigma k(0')[a^2H^\sigma + ab + b^2K^\sigma + H(ad+bc)]t^{\sigma/2} = 0 \\ \text{(iii)} & g'(t')k(t)^\sigma k(0') + \lambda(c^2f'(t)^\sigma + d^2g'(t)^\sigma)k(t')k(0') + k(t')k(t)^\sigma g'(0') \\ & +\lambda k(t')k(t)^\sigma k(0')[c^2H^\sigma + cd + d^2K^\sigma + K(ad+bc)]t^{\sigma/2}. \end{array} \tag{23}$$

In equations (23)(ii) and (iii) replace t' with $\lambda^2(ad+bc)^2t^\sigma + 0'$ and write the resulting expressions as polynomials in t^σ. Now square both sides. The terms touched by $t^{\sigma/2}$ (before squaring) have odd positive integer exponents ≤ 17. The other terms have even exponents ≤ 16. Since $e \geq 5$, the coefficients on $t^{\sigma/2}$ in equations (23)(ii) and (iii) must be zero. Hence equation (23) can be replaced with

$$\begin{array}{rl} \text{(i)} & (a^2H^\sigma + ab + b^2K^\sigma)/H = ad + bc = (c^2H^\sigma + cd + d^2K^\sigma)/K \neq 0. \\ \text{(ii)} & f'(t')k(t)^\sigma k(0') + \lambda[a^2f'(t)^\sigma + b^2g'(t)^\sigma]k(t')k(0') = k(t')k(t)^\sigma f'(0'). \\ \text{(iii)} & g'(t')k(t)^\sigma k(0') + \lambda[c^2f'(t)^\sigma + d^2g'(t)^\sigma]k(t')k(0') = k(t')k(t)^\sigma g'(0'). \end{array} \tag{24}$$

4 Subiaco GQ with $q = 2^e$, e odd

From now on we assume $F = \mathrm{GF}(q)$, $q = 2^e$, e odd and $e \geq 5$. So $1 + t^2 + t^4 \neq 0$ for all $t \in F$. $\mathcal{C} = \left\{ A_t = \begin{pmatrix} h(t) + t^{1/2} & t^{1/2} \\ 0 & t^2h(t) + t^{1/2} \end{pmatrix} : t \in F \right\}$, where $h(t) = (t+t^2)/(1+t^2+t^4)$. Let $\mathcal{F}$ denote the corresponding 4-gonal family for G, and let $\mathcal{S} = \mathcal{S}(\mathcal{C}) = \mathcal{S}(G,\mathcal{F})$ be the associated GQ. This example arises as a specialization of the general construction of Subiaco GQ in [1].

Theorem 4.1 *Each* $\sigma \in \mathrm{Aut}\,(F)$ *induces a collineation of* $\mathcal{S}$ *fixing* $[A(\infty)]$ *and mapping* $[A(t)]$ *to* $[A(t^\sigma)]$ *for* $t \in F$.

Proof. Clearly $f(t)^\sigma = f(t^\sigma)$ and $g(t)^\sigma = g(t^\sigma)$. In equation (21) put $\lambda = a = d = 1$, $b = c = 0' = 0$. Then the conditions are all satisfied with $t' = t^\sigma$. □

Theorem 4.2 *There is a collineation of* $\mathcal{S}$ *interchanging* $[A(\infty)]$ *and* $[A(0)]$ *and interchanging* $[A(t)]$ *and* $[A(t^{-1})]$ *for* $0 \neq t \in F$.

Proof. Check that $f(t^{-1}) = t^{-1}g(t)$, with $f(t) = h(t)+t^{1/2}$ and $g(t) = t^2h(t)+t^{1/2}$, and use theorem 3.1. □

From the form of f and g we know that $\mathcal{S}$ is not classical. (Alternatively, we will show that the group of collineations fixing the point (∞) and the line $[A(\infty)]$ is not transitive on the other lines through (∞).) Hence the point (∞) is fixed by the full collineation group of $\mathcal{S}$ (cf. [9]). And because of theorem 4.1, to find all collineations fixing $[A(\infty)]$, it suffices to find all solutions of equation (24) (since the q-clan of this section has the form given in theorem 3.4 with $\sigma = id$. And we use $g'(t) = t^2f'(t)$, $f'(t) = t + t^2$, $t' = \lambda^2(ad + bc)^2t + 0'$. Now compute $(t')^2$ times equation (24)(ii) added to equation (24)(iii), and divide by $k(t')$ to obtain:

$$\begin{aligned} &\lambda(t + t^2)k(0')[a^2(\lambda^4(ad + bc)^4t^2 + (0')^2 \\ &\qquad +b^2t^2(\lambda^4(ad + bc)^4t^2 + (0')^2) + c^2 + d^2t^2] \\ &= \lambda^4(ad + bc)^4t^2(1 + t^2 + t^4)f'(0'). \end{aligned} \tag{25}$$

The coefficient on t in equation (25) is $\lambda k(0')[a^2(0')^2 + c^2]$, implying

$$c = a0'. \tag{26}$$

The coefficient on t^2 is then $\lambda k(0')\lambda^4(ad + bc)^4f'(0')$, implying $f'(0') = 0$. Hence

$$0' \in \{0, 1\}. \tag{27}$$

The coefficient on t^5 is $\lambda k(0')b^2\lambda^4(ad + bc)$, implying

$$b = 0 \text{ and } ad \neq 0. \tag{28}$$

Then from equation (24)(i), $a^2 = ad$, so

$$a = d. \tag{29}$$

Now equation (25) appears as $\lambda(t + t^2)k(0')[a^2\lambda^4a^8t^2 + a^2t^2] = 0$, from which we conclude

$$\lambda a^2 = 1. \tag{30}$$

We now have established the following:

$$\begin{array}{rl} \text{(i)} & 0' \in \{0, 1\} \\ \text{(ii)} & c = a0' \\ \text{(iii)} & b = 0 \neq a = d \\ \text{(iv)} & \lambda a^2 = 1. \end{array} \tag{31}$$

A straightforward check shows that if the conditions of equation (31) all hold, then the conditions of equation (24) all hold with $t' = t+0'$. This establishes the following:

Theorem 4.3 *The group of collineations of $\mathcal{S}$ fixing $[A(\infty)]$ and $(\vec{0},0,\vec{0})$ (and of course (∞)) has order $2e(q-1)$ and has $\{[A(0)],[A(1)]\}$ as an orbit.*

At this point, including collineations induced by right multiplication by elements of G, we have a group of collineations of $\mathcal{S} = \mathcal{S}(G,\mathcal{F})$ with order $6e(q-1)q^5$, and which as a permutation group acting on the set of indices of the lines through the point (∞) includes the following:

$$\begin{array}{rl} \text{(i)} & \text{For } \sigma \in \text{Aut }(F),\ \sigma : \infty \mapsto \infty \text{ and } \sigma : t \mapsto t^\sigma \text{ for } t \in F. \\ \text{(ii)} & \theta : \infty \leftrightarrow 0, \text{ and } \theta : t \leftrightarrow t^{-1} \text{ for } 0 \neq t \in F. \\ \text{(iii)} & \phi : \infty \leftrightarrow \infty, \text{ and } \phi : t \mapsto t+1 \text{ for } t \in F. \end{array} \tag{32}$$

The set $\{\infty, 0, 1\}$ is invariant under the permutations exhibited in equation (32). $\mathcal{S}_3 \cong \langle\theta,\phi\rangle$ has $\{\infty,0,1\}$ as one orbit, and for $t \in F\backslash\{0,1\}$ has $\Omega_t = \{t, t+1, (t+1)^{-1}, t/(t+1), (t+1)/t, t^{-1}\}$ as an orbit. Note that $|\Omega_t| = 6$ since $q = 2^e$ with e odd. But the automorphisms $\sigma \in \text{Aut }(F)$ act on the Ω_t differently for different t and different e.

For example, when $e = 5$, there are five disjoint Ω_t on which Aut(F) acts transitively: $\Omega_t \cap \Omega_{t^\sigma} \neq \phi$ if and only if $\sigma = id$. So in this case (i.e., $q = 32$), the collineation group of $\mathcal{S}$ either has two orbits on the lines through the point (∞) (one of which is $\{[A(\infty)],[A(0)],[A(1)]\}$), or it has just one orbit. (We will show in section 5 that the full collineation group is transitive on the set of $q+1$ lines through (∞) for all odd $e \geq 5$.)

Note. If $3|e$, then $x^3+x^2+1 = 0$ has a root $t_0 \in F$. In this case Ω_{t_0} is broken into two orbits under Aut (F), since for each $t \in \Omega_{t_0}$, $t^2 = (t+1)^{-1}$.

5 Recoordinatizing the GQ

Start with the *standard form* of the q-clan given at the beginning of section 4. Fix $w \in F$. The idea is to let w play the role of ∞, i.e., w plays the role of t_0 in equation (6).

$$\mathcal{C}^w = \left\{ A_t^w = \begin{pmatrix} \frac{h(t)+h(w)}{t+w} + (t+w)^{-1/2} & (t+w)^{-1/2} \\ 0 & \frac{t^2h(t)+w^2h(w)}{t+w} + (t+w)^{-1/2} \end{pmatrix} : w \neq t \in F \right\}$$
$$\cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{33}$$

Put $x = (t+w)^{-1}$, so $t = w + x^{-1}$, and substitute into equation (33) to obtain

$$\mathcal{C}^w = \left\{ A_x^w = \begin{pmatrix} f^w(x) + x^{1/2} & x^{1/2} \\ 0 & g^w(x) + x^{1/2} \end{pmatrix} : x \in F \right\}, \tag{34}$$

where

$$\begin{array}{rl} \text{(i)} & f^w(x) = f'(x)/k(x),\ g^w(x) = g'(x)/k(x), \text{ and} \\ \text{(ii)} & f'(t) = t^4(1+w^2+w^4) + t^3(1+w+w^4) + t(w+w^2), \\ \text{(iii)} & g'(t) = t^4(w^2+w^4+w^6) + t^3(w+w^4+w^5) \\ & \quad +t^2(1+w^2+w^4) + t(1+w^2+w^3), \\ \text{(iv)} & k(t) = t^4(1+w^2+w^4)^2 + (t^2+1)(1+w^2+w^4). \end{array} \tag{35}$$

If there is a collineation of the GQ $\mathcal{S}$ (with q-clan in standard form) mapping $[A(\infty)]$ to $[A(w)]$, then there must be a collineation $\theta = \theta(\sigma, D, \lambda)$ (in the *new* coordinatization) which is an involution fixing $[A(\infty)]$ and interchanging all other lines through (∞) in pairs. So there is an involution θ of the form

$$\theta : (\alpha, c, \beta) \mapsto \qquad (36)$$
$$(\lambda^{-1}\alpha^\sigma D^{-T},\ \lambda^{-1/2}c^\sigma + \lambda^{-1}\sqrt{\alpha^\sigma D^{-T} A_{0'} D^{-1} (\alpha^\sigma)^T}\beta^\sigma PDP + \lambda^{-1} y_{0'}\alpha^\sigma D^{-T}),$$

with $D = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GF}(2, q)$.

Since theorem 3.4 applies (with the new coordinates), equation (24)(i) holds with $H = K = 1$.

$$ad + bc = a^2 + ab + b^2 = c^2 + cd + d^2. \qquad (37)$$

Compute the effect of $\theta^2 = id$ on $(\vec{0}, c, \vec{0}) \in G : (\vec{0}, c, \vec{0}) = (\vec{0}, \lambda^{-(\sigma+1)/2}c^{\sigma^2}, \vec{0})$ for all $c \in F$. This implies $\sigma^2 = id$, so $\sigma = id$ since e is odd, and also $\lambda = 1$. Similarly, $id = \theta^2 : (\vec{0}, 0, \beta) \mapsto (\vec{0}, 0, \beta(PDP)^2)$ for all $\beta \in F^2$, implying $D^2 = I$. This implies $a = d$ and $1 = \det(D) = a^2 + bc = a^2 + ab + b^2 = a^2 + ac + c^2$, forcing $1 - a^2 = bc = ab + b^2 = ac + c^2$. Hence $b(a + b + c) = c(a + b + c) = 0$. Suppose first that $a + b + c \neq 0$. Then $D = I$, and from the fact that the coefficient on $t^\sigma = t$ in equation (24)(ii) must be 0 it follows that $0' = 0$. This means that θ is not the involution we seek, hence $a + b + c = 0$.

The coefficients on $(t^\sigma)^7 = t^7$ in equation (24)(ii) and (iii) must both be 0. This leads to a system of two linear equations in a^2 and b^2 which is easily solved. And then $c^2 = a^2 + b^2$ leads to the following:

Put $v = (1 + w + w^2)^{1/2}$. Then

$$\begin{aligned} &\text{(i)} && a = d = (1 + w^5)/v^5, \\ &\text{(ii)} && b = (1 + w + w^4)/v^5, \\ &\text{(iii)} && c = (w + w^4 + w^5)/v^5. \end{aligned} \qquad (38)$$

Finally, considering the coefficient of $t^\sigma = t$ in equation (24)(ii), we compute

$$0' = 1/v^2. \qquad (39)$$

It is now a tedious but uninspired task to show that the unique possible involution θ determined by equations (38) and (39) does satisfy the conditions of equation (24). Tracing back through the recoordinatization process, we find that the involution θ induces the following permutation on the indices of the lines through (∞) in the *original coordinatization* of section 4:

$$t \mapsto (t(1 + w^2) + w^2)/(t + 1 + w^2). \qquad (40)$$

In particular, $\infty \leftrightarrow 1 + w^2$ and $w \leftrightarrow w$. Here w can be any element of F, and $w = 1$ gives the original involution found in theorem 4.2. This proves the following:

Theorem 5.1 *The full collineation group $\mathcal{G}$ of the* GQ *given in section* 4 *(with $q \geq 32$) has order $2e(q^2 - 1)q^5$ and acts transitively on the lines through the point (∞).*

6 The Action of $\mathcal{G}$ on the subquadrangles $\mathcal{S}_\alpha$

The action of the full collineation group $\mathcal{G}$ on the subquadrangles $\mathcal{S}_\alpha$ is determined by its action on the subgroups G_α. Clearly the stabilizer $\mathcal{G}_0$ of the point $(\vec{0}, 0, \vec{0})$ must leave the dual grid Γ invariant, so it must permute the $\mathcal{S}_\alpha$ among themselves.

Let $w \in F$. Then the map ϕ_w defined by

$$\phi_w : G \to G : (\alpha, c, \beta) \mapsto ((y_w\alpha + \beta)P,\ c + \sqrt{\alpha A_w \alpha^T + \alpha P \beta^T},\ \alpha P) \tag{41}$$

is an automorphism of G that corresponds to the recoordinatization of section 5. It is convenient to have its inverse

$$\phi_w^{-1} : G \to G : (\gamma, d, \delta) \mapsto (\delta P,\ d + \sqrt{\delta P A_w P \delta^T + \delta P \gamma^T},\ (y_w\delta + \gamma)P). \tag{42}$$

Now consider an involution of the recoordinatized GQ of the type $\theta_w = \theta_w(id, D, 1)$ with D as in equation (38). For such a D, $PD^{-T}P = D$ and $PDP = D^T = D^{-T}$.

$$\begin{aligned} \theta_w : G &\to G : \\ (\alpha, c, \beta) &\mapsto (\alpha D^{-T}, c + \sqrt{\alpha D^{-T} A_{0'} D^{-1} \alpha^T}, (\beta + 0'\alpha)D^{-T}), \end{aligned} \tag{43}$$

with $0' = (1 + w + w^2)^{-1}$.

Then $\overline{\theta}_w = \phi_w \circ \theta_w \circ \phi_w^{-1}$ (doing ϕ_w first) as an automorphism of G is an involution of the GQ $\mathcal{S}$ expressed in the original coordinates. To consider its effect on the G_α we do not need to compute the middle coordinate.

$$\overline{\theta}_w : (\alpha, c, \beta) \mapsto ((\alpha + 0'(y_w\alpha + \beta))D, -, 0'y_w^2\alpha D + (y_w 0' + 1)\beta D). \tag{44}$$

From equation (44) it is clear that $G_\alpha \to G_{\alpha D}$.

From here on we use just the group $\mathcal{G}_0$. The stabilizer $\mathcal{G}_{0,\infty}$ of $[A(\infty)]$ (see theorem 4.3) has order $2e(q-1)$, and for $0 \neq a \in F$, $\sigma \in \mathrm{Aut}(F)$, $0' \in \{0, 1\}$, consists of the following maps:

$$(\alpha, c, \beta) \mapsto (a\alpha^\sigma K, ac^\sigma + a\sqrt{\alpha^\sigma K A_{0'} K^T (\alpha^\sigma)^T}, a(\beta^\sigma + 0'\alpha^\sigma)K), \tag{45}$$

with

$$K = \begin{pmatrix} 1 & 0 \\ 0' & 1 \end{pmatrix}.$$

Put $\alpha = (0, 1)$ and determine the stabilizer of *this* α. Since $\alpha K = (0', 1)$, the stabilizer of α in $\mathcal{G}_{0,\infty}$ has order $e(q-1)$, since $0'$ must be 0. Write $D(w)$ for the D given by equation (38). The collineations $\theta(id, D(w), 1)$ are coset representatives for those cosets of $\mathcal{G}_{0,\infty}$ in $\mathcal{G}_0$ different from $\mathcal{G}_{0,\infty}$. So to find the stabilizer of α in $\mathcal{G}_{0,\infty} \cdot D(w)$ we consider $\alpha K D(w) = (0', 1) \cdot D(w) = (0'(1 + w^5) + 1 + w + w^4, 0'(w + w^4 + w^5) + 1 + w^5)/(1 + w + w^2)^{5/2}$. The first coordinate is 0 if and only if $0' = 1$ and $w = 0$. So there are $e(q-1)$ such collineations, implying that the stabilizer of G_α in $\mathcal{G}_0$ has order $2e(q-1)$. As $\mathcal{G}_0$ has order $2e(q^2-1)$, $\mathcal{G}_0$ is transitive on the set of $q+1\,G_\alpha$, and hence on the $q+1\,\mathcal{S}_\alpha$ as well, implying also that only one oval arises.

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S. E. Payne
Department of Mathematics
CU-Denver Camp Box 170
PO Box 173364
Denver CO 80217-3364
USA.