The reduction of a double covering of a Mumford curve

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The technique of analytic reductions is one of the main tools in the study of curves over a non-archimedean valued field. In [2] a theorem is given which describes such a reduction for an unramified abelian covering of a Mumford curve.

In this paper we prove a similar theorem for two-sheeted coverings of a Mumford curve which are possibly ramified. We aply the theorem to determine the reduction in the case that the underlying curve has genus two.

I thank M. Van Der Put for his suggestions.

Notations The fiels k is supposed to be algebraically closed and complete with respect to a non-archimedean absolute value. The residue field $\bar{k}$ has characteristic different from 2.

1 The reduction of a double covering

We study a double covering $\phi: \ X \to Y$ where X and Y are non-singular projective curves defined over k. The morphism ϕ is ramified in the points of $S = \{p_1, \cdots, p_n\} \subset Y$.

We assume Y to be a Mumford curve. This means that Y has a finite admissible covering $\mathcal{U} = (U_i)_{i \in I}$ such that each affinoid set U_i is isomorphic to an affinoid subset of $\mathbf{P}^1(k)$.
The covering can be chosen such that the corresponding analytic reuction $r: \ Y \to \bar{Y}$ has the following properties, (cf. [1]) :

a) each irreducible component of $\bar{Y}$ is a non-singular projective curve over the residue field $\bar{k}$ with genus zero;

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b) the only singularities of $\bar{Y}$ are the intersection points of the irreducible components and these are ordinary double points;

c) the images $r(p_1), \cdots, r(p_n)$ are all regular points.

The inverse images $V_i = \phi^{-1}(U_i)$ are affinoid subsets of X and the covering $\mathcal{V} = (V_i)_{i \in I}$ is an admissible covering of X. Let $R: X \to \bar{X}$ be the coreponding analytic reduction. The map $\phi: X \to Y$ induces an algebraic morphism $\bar{\phi}: \bar{X} \to \bar{Y}$ such that $r \circ \phi = \bar{\phi} \circ R$. We will prove the following theorem.

Theorem 1.1.
1) The irreducible components of $\bar{X}$ are non-singular curves.
2) The only singularities of $\bar{X}$ are the intersections of the irreducible components and these points are ordinary double points.
3) The map $\bar{\phi}$ is a double covering, ramified in the points of $r(S)$ and possibly ramified in the double points of $\bar{Y}$.

Proof a) *The affinoids of $\mathcal{U}$ and $\mathcal{V}$.*

An affinoid set $U \in \mathcal{U}$ can be decribed in the folowing way :

$$U = \{x \in \mathbf{P}^1(k) \mid |q| \leq |x| \leq 1, |x - a_k| \geq 1, |x - b_l| \geq 1;\ k = 1, \cdots, s \text{ and } l = 1, \cdots t\}$$

$$\text{with} \quad \begin{cases} q \in k^* \text{ and } |q| < 1 \\ |a_k| = 1, \quad |a_k - a_i| = 1 \qquad (k \neq i = 1, \cdots, s) \\ |b_l| = |q|, \quad |b_l - b_j| = |q| \qquad (l \neq j = 1, \cdots, t) \end{cases}$$

The ring of holomorphic functions on U is then

$$A = k < x, y, z_1, z_2 > /(xy - q,\ \prod_{i=1}^{s}(x - a_i)z_1 - 1,\ \prod_{j=1}^{t}(y - \frac{q}{b_j})z_2 - 1).$$

Let $V = \phi^{-1}(U)$. The ring of holomorphic functions on V has the form $B = A[z]/(z^2 - t)$ with $t \in A$.
According to [3] t has a decomposition as:

$$t = \lambda(1 + t_0)x^{\epsilon_0} . \prod_{i=1}^{s}(x - a_i)^{\epsilon_i} . \prod_{j=1}^{t}(y - \frac{q}{b_j})^{\nu_j} . \prod_{p_i \in U}(x - p_i)$$

with $t_0 \in A$ and $||t_0|| < 1,\ \lambda \in k^*$.
Since $r(p_1), \cdots, r(p_n)$ are regular points of $\bar{Y}$ the set $\{p_i \mid p_i \in U\}$ can be split in two parts $(p_i)_{i \in K}$ and $(p_i)_{i \in L}$

$$\text{with} \quad \begin{cases} |p_k| = |p_k - a_i| = 1,\ k \in K \text{ and } 1 = 1, \cdots, t \\ |p_l| = |p_l - b_j| = |q|,\ l \in L \text{ and } j = 1, \cdots, t. \end{cases}$$

Furthermore z is only defined up to multiplication with an element of A^*.
So we may assume that $B = A[z]/(z^2 - t)$ with $t = x^{\epsilon_0} m. \prod_{k \in K}(x - p_k). \prod_{l \in L}(y - \frac{q}{bj})$

$$\text{and} \quad m = \prod_{i=1}^{s}(x - a_i)^{\epsilon_i}. \prod_{j=1}^{b}(y - \frac{q}{b_j})^{\nu^j}\ ;\ \epsilon_i, \nu_j = 0 \text{ or } 1.$$

b) *The reduction of A and B.*

The reduction $\bar{A}$ of A is the localization of $\bar{k}[x,y]/(xy)$ at the element $\prod_{i=1}^{s}(x-\bar{a}_i).\prod_{j=1}^{t}(y-\frac{\bar{q}}{b_j})$.

The reduction $\bar{U}$ of U has two components l_x and l_y corresponding to $x=0$ and $y=0$ respectively. Each of them is an open affine subset of $\mathbf{P}^1(k)$.

We calculate the reduction of B and V in the following cases.

1) $K=L=\emptyset$

We have the same situation as is [2],pages 42-43.
We find that $\bar{V}$ has irreducible components which are open affine subsets of non-singular curves over $\bar{k}$.
The intersections of the components are ordinary double points. The reduced map $\bar{\phi}:\ \bar{V}\to\bar{U}$ has degree two and can only be ramified in the double points of $\bar{U}$.

2) $K\neq\emptyset, \epsilon_0\neq 0$

Let $k_0\in K$ and let $w=\frac{zy}{\sqrt{q}}$. We have

$$w^2=m.(q-p_{k_0}y)\cdot\prod_{k_0\neq k\in K}(x-p_k)\cdot\prod_{l\in L}(y-\frac{q}{p_l})$$

So $B=A[z,w]$, divided by the ideal generated by

$$\begin{cases} zw-\sqrt{q}.m.\prod_{k\in K}(x-p_k).\prod_{l\in L}(y-\frac{q}{p_l}) \\ z^2-x.m.\prod_{k\in K}(x-p_k).\prod_{l\in L}(y-\frac{q}{p_l}) \\ w^2-m(q-p_{k_0}y)\cdot\prod_{k_0\neq k\in K}(x-p_k).\prod_{l\in L}(y-\frac{q}{p_l}). \end{cases}$$

The ring that one obtaines by taking residues of the coefficients has no nilpotents and hence is isomorphic to the reduction of B. So $\bar{B}=A[z,w]$, divided by the ideal generated by

$$\begin{cases} zw \\ z^2-x.\bar{m}.\prod_{k\in K}(x-\bar{p_k}).\prod_{l\in L}(y-\frac{\bar{q}}{p_l}) \\ w^2-\bar{m}.\bar{p_{k_0}}.y.\prod_{k_0\neq k\in K}\cdot\prod_{l\in L}(y-\frac{\bar{q}}{p_l}). \end{cases}$$

It follows that $\bar{V}$ has two components L_z and L_w corresponding to $z=0$ and $w=0$ respectively. Both are open affine subsets of a non-singular curve.
The components intersect in the point corresponding to $z=x=0$. The map $\bar{\phi}:\ \bar{V}\to\bar{U}$ has degree two an maps L_z onto l_x and L_w onto l_y. Furthermore $\bar{\phi}$ is ramified in the intersection point of l_x and l_y and in the reductions of the points $(p_k)_{k\in K}$ and $(p_l)_{l\in L}$.

3) $K \neq \emptyset, \epsilon_0 = 0$

In a similar way as in (b) we find that

$$\bar{B} = \bar{A}[z]/(z^2 - \bar{m}. \prod_{k \in K}(x - \bar{p_k}). \prod_{l \in L}(y - \frac{\bar{q}}{p_l})).$$

It follows that $\bar{V}$ has two components L_x and L_y lying above l_x and l_y respectively. The map $\bar{\phi}$: $\bar{V} \to \bar{U}$ is ramified in the reduction of the points $(p_k)_{k \in K}$ and $(p_l)_{l \in L}$.

4) $K = \emptyset, L \neq \emptyset$

This case gives a similar result as in (b) and (c). ∎

2 Unramified double coverings of a Mumford curve

Let $\phi :\ X \to Y$ and $\bar{\phi} :\ \bar{X} \to \bar{Y}$ be as in section 1.
The reductions $\bar{Y}$ and $\bar{X}$ are taken with respect to the coverings $\mathcal{U}$ and $\mathcal{V} = \phi^{-1}(\mathcal{U})$ respectively.
We assume now that ϕ is unramified. It is clear that ϕ is an analytic covering in the sense of [2] if and only if for each $U \in \mathcal{U}$ the restriction $\phi :\ V = \phi^{-1}(U) \to U$ is an analytic covering. Such coverings are easy to describe.

Lemma 2.1. *Let*

$$U = \{x \in k \mid |q| \leq |x| \leq 1,\ |x - a_i| \geq 1,\ |x - b_j| \geq |q|;\ i = 1, \cdots, s \textit{ and } j = 1, \cdots, t\}$$

$$\textit{with} \quad \begin{cases} q \in k^* \textit{ and } |q| < 1 \\ |a_i| = |a_i - a_k| = 1; \quad 1 \neq k = 1, \cdots, s \\ |b_j| = |b_j - b_l| = |q|; \quad j \neq l = 1, \cdots, t \end{cases}$$

An affinoid map $u :\ V \to U$ of degree two is an analytic covering if and only if V has two connected components V_1 and V_2 such that the restriction of u to V_i is an isomorphism, $(i =, 1, 2)$.

Proof Let A and B be the rings of holomorphic functions of U and V respectively.
We have $B = A[z]/(z^2 - x^{\epsilon_0}. \prod_{i=1}^{s}(x - a_i)^{\epsilon_i}. \prod_{j=1}^{t}(\frac{q}{x} - \frac{q}{b_j})^{\nu_j})$ with $\epsilon_1, \cdots, \epsilon_s, \nu_1, \cdots, \nu_t = 0$ or 1. Let $U_1 = \{x \in U \mid |x| = 1\}$ and $U_2 = \{x \in U \mid |x| = |q|\}$. The restrictions $u :\ u^{-1}(U_i) \to U_i$ are also analytic coverings. There is an admissible covering $\mathcal{W}_1$ of U_1 such that for each $W \in \mathcal{W}_1$ the inverse image $u^{-1}(W)$ has two connected components isomorphic to W.
At least one $W \in \mathcal{W}_1$ has the form $W = U_1 - \{$ open discs with radius 1 $\}$. Let D and D' be the holomorphic function rings of W and $u^{-1}(W)$ respectively.
It follows that $D' = D[w]/(w^2 - t)$ where $t \in D^*$ is a squarefree element divisible by $x^{\epsilon_0}. \prod_{i=1}^{s}(x - a_i)^{\epsilon_i}$. But since $u^{-1}(W)$ has two components t has to be constant and hence $\epsilon_0, \cdots, \epsilon_s = 0$. In a similar way we find that $\nu_1, \cdots, \nu_t = 0$. It follows that V has two connected components as required. ∎

Proposition 2.2. *The map $\phi : X \to Y$ is an analitic covering if and only if for each irreducible component l of $\bar{Y}$ the invers image $u^{-1}(l)$ consists of two disjoint irreducible components of X, each one isomorphic to l.*

Proof Each element U of $\mathcal{U}$ satifies the conditions of the lemma. ∎

The proof of the lemma also shows that for general ϕ the inverse image of an irreducible component l of $\bar{Y}$ can have the following forms :

a) $\bar{\phi}^{-1}(l)$ is an irreducible component L of $\bar{X}$ and $\bar{\phi} : L \to l$ is ramified ;

b) $\bar{\phi}^{-1}(l)$ consists of two irreducible components L_1 and L_2 of $\bar{X}$ and $\bar{\phi} : L_i \to l$ is an isomorphism.

3 Example : Mumford curves with genus 2

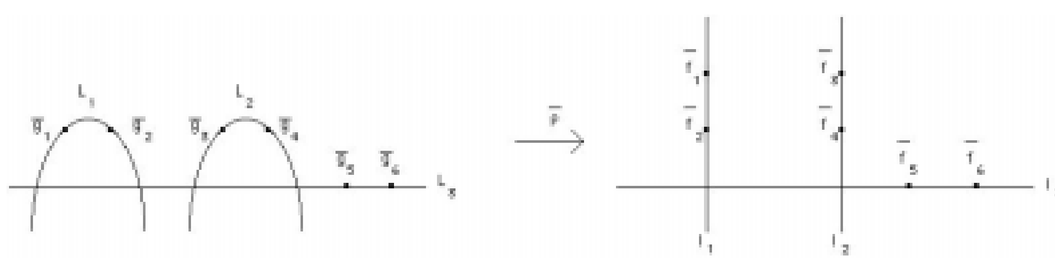
Let Y be a curve with genus 2. Hence Y is hyperelliptic; i.e. there exists a two-sheeted covering $\rho : Y \to \mathbf{P}^1$ which is ramified in six points $f_1, \cdots, f_6$.

The finite set $S = \{f_1, \cdots, f_6\}$ determines an analityc reduction $\overline{(\mathbf{P}^1, S)}$ of $\mathbf{P}^1(k)$, see [1]. This reduction satifies the conditions of Theorem 1.1. So we have a reduction $\bar{Y}$ of Y and an induced map $\bar{\phi} : \bar{Y} \to \overline{(\mathbf{P}^1, S)}$.

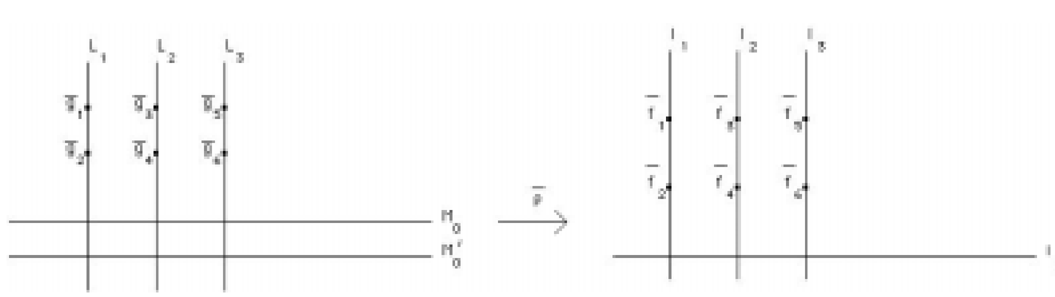
Considering the different possibilities for $\overline{(\mathbf{P}^1, S)}$ we find all the possibilities for $\bar{Y}$, see [1], page 168.

We find that Y is a Mumford curve in the following three cases :

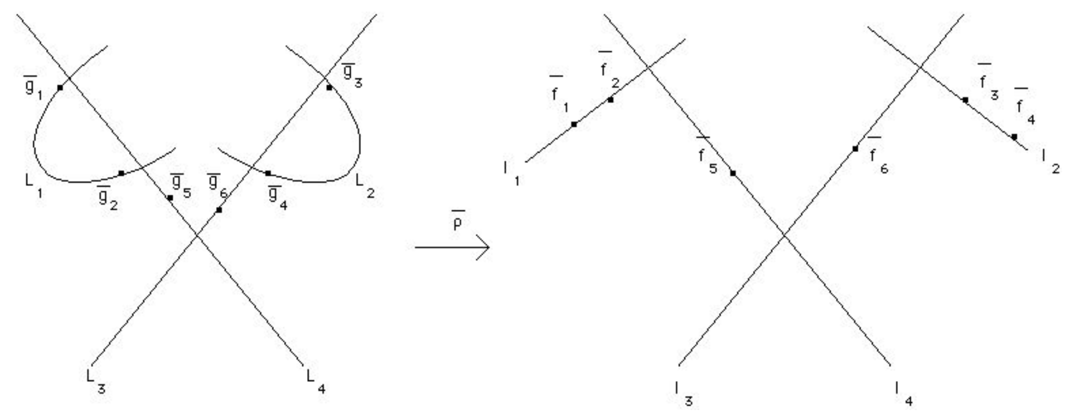
a)



b)



c)



In these pictures each line is a rational curve. The points $\bar{f}_i$ are the reductions of f_i and $\bar{\rho}(\bar{g}_i) = \bar{f}_i$.

Each restriction $\bar{\rho}: \ L_i \to l_i$ is a two- sheeted covering, ramified in the points $\bar{f}_j$ on l_i.

In case (c) the intersection of l_3 and l_4 is also a ramification point for $\bar{\rho}: \ L_4 \to l_4$.

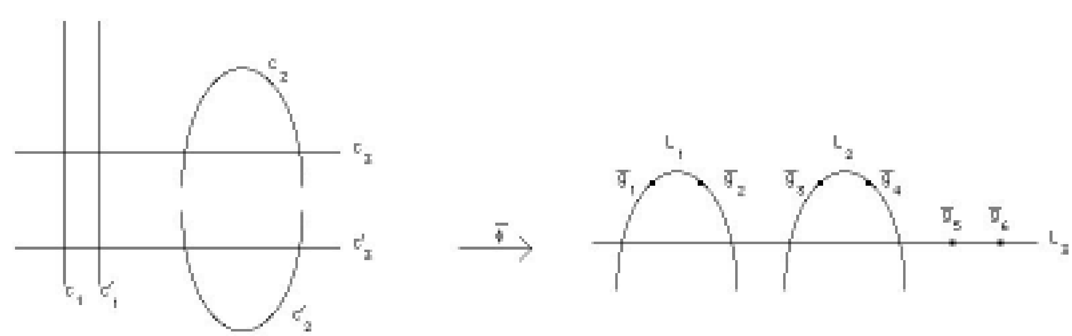
In case (b) The curves M_0 and M_0' are mapped onto l_0.

Let $\phi: \ X \to Y$ be a double unramified covering with X a non-singular projective curve with genus 3. The kernel of the dual morphism $\phi^*: \ \mathrm{Jac}(Y) \to \mathrm{Jac}(X)$ is a group of order two. The non-trivial element in this group is a divisor class of the form $D_{ij} = cl(g_i - g_j)$ with $i < j$. The divisor class D_{ij} detemines X up to an isomorphism. So we have 15 possibilities for $\phi: \ X \to Y$. We proved in [4] that three of these coverings are analytic and in each of these cases X is also hyperelliptic.

Since the reduction $\bar{Y}$ also satisfies the conditions of section 1, we have a reduction $\bar{X}$ and an induced map $\bar{\phi}: \ \bar{X} \to \bar{Y}$ such as in Theorem 1.1.

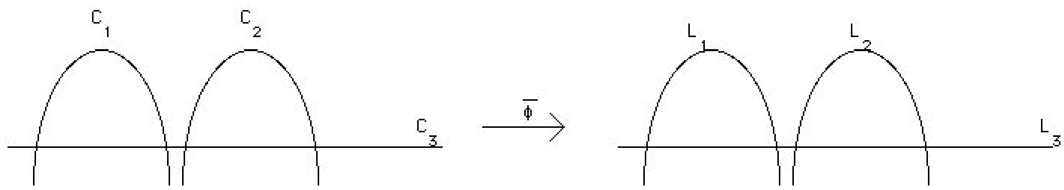
It is not difficult to calculate this reduction. We give only two examples for case (a).

If $D_{12} \in \mathrm{Ker}\phi^*$ then we have



Each line is a rational curve and the restrictions $\bar{\phi}: \ C_i \to L_i$ and $\bar{\phi}: \ C_i' \to L_i$ are isomorphisms. So X is a Mumford curve and ϕ is an analytic covering. The covering ϕ is also analytic when $D_{56} \in \ker\phi^*$ or $D_{34} \in \mathrm{Ker}\phi^*$.

If D_{15}, D_{25}, D_{16} or $D_{26} \in \ker\phi^*$ then we have



The curves C_1 and C_2 are rational and the restrictions $\bar{\phi}:\ C_i \to L_i,\ i = 1, 2$ are double coverings, ramified in the intersection points of L_i with L_3.
The curve C_3 is hyperelliptic and the restriction $\bar{\phi}:\ C_3 \to L_3$ is a double covering ramified in the intersection points of L_3 with L_1 and L_2. Since C_3 is not rational, X is not a Mumford curve. In fact these are the only cases where X is not a Mumford curve.

References

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