Stable Clifford Theory for Divisorially Graded Rings

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Introduction

Dade [$D1$, Theorem 7.4] obtained an important result on the equivalence of categories, extending the classical stable Clifford theory. He used the theory of strongly graded rings. Recently, this work has been generalized to arbitrary graded rings, see E. Dade [$D2$], [$D3$], J.L. Gómez Pardo and C. Năstăsescu [GN], C. Năstăsescu and F. Van Oystaeyen [$NVO2$]. In the classical case the stable Clifford theory relates isomorphism classes of simple modules on a strongly graded ring R which are direct sums of a fixed simple R_e-module, where R_e is the component of degree e, with the isomorphism classes of simple modules on a crossed product. The aim of this paper is extend the foregoing result to $\mathcal{C}$-cocritical modules, where $\mathcal{C}$ is a localizing subcategory, on divisorially graded rings. We start with a relative version of Clifford theory using the simple objects of the quotient category. We investigated the situation of the so-called divisorially graded rings introduced by F. Van Oystaeyen in the commutative case and then generalized by many other author to more general situations (see the monograph [$LVVO$] and the references quoted there). We will work in the categories of R-Mod and $R - gr$, thus we prefer use the a general Grothendieck category and the concept of static objects in this kind of category to establish our basic results.

The paper is organized as follows. After a Section of preliminaries, we introduce the notion of static objects in quotient categories in the next section. If we have adjoints functors between two Grothendieck categories $\mathcal{A}$ and $\mathcal{B}$ and a localizing subcategory $\mathcal{C}$ of $\mathcal{A}$, then we show that under certain conditions it is possible to obtain an equivalence between the category of static objects in $\mathcal{A}/\mathcal{C}$ and the category of co-static objects of some quotient category of $\mathcal{B}$. In the last Section, we apply

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this result to study the stable Clifford theory on divisorially graded rings, extending (see Theorem 3.14) the main result of Dade (cf. [$D1$, Theorem 8.2]).

1 Preliminaries

All the rings considered in this paper are associative with identity element. Let R be a ring, R-Mod will denote the category of the unital left R-modules.

Let G be a multiplicative group with identity element e. A G-graded ring R is a ring with identity 1, together with a direct decomposition $R = \oplus_{g\in G} R_g$ (as additive subgroups) such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in$ G. Thus R_e is a subring of $R, 1 \in R_e$ and for every $g \in G, R_g$ is an R_e -bimodule. A G-graded left R-module is a left R-module M endowed with an internal direct sum decomposition $M = \oplus_{g\in G} M_g$ where each M_g is a subgroup of the additive group of M such that $R_g M_h \subseteq M_{gh}$ for all $g, h \in$ G. Let M and N be graded left modules over the graded ring R. For every $g \in G$ we set

$$\mathrm{HOM}_R(M,N)_g = \{f : M \to N \mid f \text{ is } R-\text{linear and } f(M_h) \subseteq M_{hg}\}$$

$\mathrm{HOM}_R(M,N)_g$ is an additive subgroup of the group $\mathrm{Hom}_R(M,N)$ of all R-linear maps from M to N. Observe that

$$\mathrm{HOM}_R(M,N) = \oplus_{g\in G}\, \mathrm{HOM}_R(M,N)_g$$

is a subgroup of $\mathrm{Hom}_R(M,N)$ and it is a graded abelian group of type G. Clearly $\mathrm{HOM}_R(M,N)_e$ is just $\mathrm{Hom}_{R-gr}(M,N)$, i.e. the group of all morphisms from M to N in the category $R - gr$ of all graded left R-modules. Define for $g \in G$ the g-suspension $M(g)$ of a graded left R-module M as follows: $M(g)$ is the left R-module M graded by G by putting $M(g)_h = M_{hg}$ for all $h \in$ G. Observe that

$$\mathrm{HOM}_R(M,N)_g = \mathrm{Hom}_{R-gr}(M, N(g)) = \mathrm{Hom}_{R-gr}(M(g^{-1}), N)$$

It is well known that $R - gr$ is a Grothendieck category (See [*NVO1*]).

We recall some ideas from torsion theories on Grothendieck categories. Let $\mathcal{A}$ be a Grothendieck category. A non empty subclass $\mathcal{T}$ of $\mathcal{A}$ is a *torsion class* if $\mathcal{T}$ is closed under quotient objects, coproducts and extensions. In this case for any $M \in \mathcal{A}$ one can consider the greatest suboject $t_{\mathcal{T}}(M)$ of M belonging to $\mathcal{T}$. A torsion class is said to be *hereditary* if it is closed under subobjects.

Let us now recall the concept of quotient category. A *Serre class* (or Serre subcategory) of an abelian category $\mathcal{A}$ is a non-empty class $\mathcal{S}$ which is closed under subobjects, quotient objects and extensions. The quotient category $\mathcal{A}/\mathcal{S}$ of $\mathcal{A}$ by $\mathcal{S}$ is the category defined as follows: the objects of $\mathcal{A}/\mathcal{S}$ are those of $\mathcal{A}$ and the morphism are defined by

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(A,B) = \lim_{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}(A', B/B')$$

where A' runs over the subojects of A such that $A/A' \in \mathcal{S}$ and B' runs over the subobjects of B such that $B' \in \mathcal{S}$. $\mathcal{A}/\mathcal{S}$ is an abelian category and the canonical functor $\mathbf{T}_{\mathcal{S}} : \mathcal{A} \to \mathcal{A}/\mathcal{S}$, which is the identity on objects and maps morphisms in $\mathrm{Hom}_{\mathcal{A}}(A, B)$ onto their canonical image in the direct limit $\mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(A, B)$, is an exact functor. The Serre class $\mathcal{S}$ is called a *localizing subcategory* of $\mathcal{A}$ if the canonical functor $\mathbf{T}$ has a right adjoint. If $\mathcal{A}$ is a Grothendieck category then the concept of localizing subcategory coincides with that of hereditary torsion class. If $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$, then for any $X \in \mathcal{A}$ we consider the greatest subobject $t_{\mathcal{C}}(X)$ of X belonging to $\mathcal{C}$. If $t_{\mathcal{C}}(X) = 0$ then X is called a $\mathcal{C}$-torsionfree object, if $t_{\mathcal{C}}(X) = X$, then M is said to be a $\mathcal{C}$-torsion object. Following Gabriel $[G]$, if $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$, we can define the quotient category $\mathcal{A}/\mathcal{C}$ which is also a Grothendieck category. We denote by $\mathbf{T}_{\mathcal{C}} : \mathcal{A} \to \mathcal{A}/\mathcal{C}, \mathbf{S}_{\mathcal{C}} : \mathcal{A}/\mathcal{C} \to \mathcal{A}$, the canonical functors. It is well known $[G]$ that $\mathbf{T}_{\mathcal{C}}$ is an exact functor, and $\mathbf{S}_{\mathcal{C}}$ is right adjoint of $\mathbf{T}_{\mathcal{C}}$. Moreover, $\mathbf{S}_{\mathcal{C}}$ is a left exact functor.

2 Static objects in quotient categories.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Grothendieck categories, and consider two additive functors F and G, such that F is a left adjoint of G.

$$\begin{array}{c}\mathcal{B}\\ F\Big\downarrow\Big\uparrow G\\ \mathcal{A}\end{array}$$

For an $R-S$-bimodule M over associative rings R and S, the adjunction $M \otimes_S - \dashv \mathrm{Hom}_R(M, -)$ was used in $[Na]$ to define full subcategories of R-Mod and S-Mod in order to have that the restrictions of the functors $M \otimes_S -$ and $\mathrm{Hom}_R(M, -)$ to such subcategories establish an equivalence of categories between them. The aim of this section is to carry this construction to a quotient category of $\mathcal{A}$.

If $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$ we can induce a subcategory $\mathcal{D}$ of $\mathcal{B}$ setting

$$\mathcal{D} = \{Y \in \mathcal{B} \mid F(Y) \text{ is } \mathcal{C} - \text{torsion}\}.$$

It is evident that $\mathcal{D}$ is stable under homomorphic images and direct sums. To check that $\mathcal{D}$ is closed by extensions, consider

$$0 \to X \to Y \to Z \to 0$$

an exact sequence in $\mathcal{B}$ with $X, Z \in \mathcal{D}$. Applying F, we obtain an exact sequence in $\mathcal{A}$,

$$F(X) \xrightarrow{f} F(Y) \xrightarrow{g} F(Z) \to 0$$

with $F(X)$ and $F(Z)$ $\mathcal{C}$-torsion. Construct the exact sequence

$$0 \to \text{ Ker } g \to F(Y) \to F(Z) \to 0.$$

Since Kerg = Imf and $F(X)$ is $\mathcal{C}$-torsion it follows that Kerg is $\mathcal{C}$-torsion and therefore $F(Y)$ is $\mathcal{C}$-torsion. This gives $Y \in \mathcal{D}$. In order to have that $\mathcal{D}$ is a localizing subcategory of $\mathcal{B}$, we require that F satisfies certain property of exactness, as reflects the following result.

Proposition 2.1. *Let $\mathcal{A}, \mathcal{B}$ be two Grothendieck categories. Consider the following situation of adjoints functors.*

$$\begin{array}{c} \mathcal{B} \\ F \Big\downarrow \Big\uparrow G \\ \mathcal{A} \end{array}$$

Let $\mathcal{C}$ be a localizing subcategory of $\mathcal{A}$ and $\mathcal{D} = \{Y \in \mathcal{B} \mid F(Y)$ is $\mathcal{C}$-torsion$\}$ the induced subcategory of $\mathcal{B}$. The class $\mathcal{D}$ is a localizing subcategory of $\mathcal{B}$ whenever for every monomorphism $f : X \to Y$ in $\mathcal{B}$, $Ker(F(f))$ is $\mathcal{C}$-torsion.

Proof. For every monomorphism in $\mathcal{B}$, $f : X \to Y$ we have the exact sequence

$$0 \to Ker(F(f)) \to F(X) \to Im(F(f)) \to 0.$$

If Y is $\mathcal{C}$-torsion then $Im(F(f))$ is $\mathcal{C}$-torsion. Thus $F(X)$ is $\mathcal{C}$-torsion if and only if Ker$(F(f))$ is $\mathcal{C}$-torsion. The proposition follows from this fact.

Definition 2.2. A functor F satisfying Proposition 2.1 is said to be $\mathcal{C}$-*exact*.

Throughout this section the functor F will be assumed to be $\mathcal{C}$-exact. Following Gabriel [G] we can define the quotient categories $\mathcal{A}/\mathcal{C}$ and $\mathcal{B}/\mathcal{D}$. We will denote by $\mathbf{T}_{\mathcal{C}} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ and $\mathbf{S}_{\mathcal{C}} : \mathcal{A}/\mathcal{C} \to \mathcal{A}$ (resp. $\mathbf{T}_{\mathcal{D}} : \mathcal{B} \to \mathcal{B}/\mathcal{D}$ and $\mathbf{S}_{\mathcal{D}} : \mathcal{B}/\mathcal{D} \to \mathcal{B}$) the canonical functors (see [G, ch. III]). We have also natural transformations $\Phi_{\mathcal{C}} : \mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}} \to \text{id}$ and $\Psi_{\mathcal{C}}:\text{id}\to \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}$ such that $\Phi_{\mathcal{C}}$ is a natural isomorphism and for each object X in $\mathcal{A}$ the morphism $(\Psi_{\mathcal{C}})_X : X \to \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}(X)$ has kernel and cokernel $\mathcal{C}$-torsion. Analogous notations will be used for $\mathcal{D}$. The adjunction

$$\begin{array}{c} \mathcal{B} \\ F \Big\downarrow \Big\uparrow G \\ \mathcal{A} \end{array}$$

induces functors

$$\begin{array}{ccc} \mathcal{A} & \underset{F}{\overset{G}{\leftrightarrows}} & \mathcal{B} \\ \mathbf{T}_{\mathcal{C}} \downarrow \uparrow \mathbf{S}_{\mathcal{C}} & & \mathbf{T}_{\mathcal{D}} \downarrow \uparrow \mathbf{S}_{\mathcal{D}} \\ \mathcal{A}/\mathcal{C} & \underset{H}{\overset{K}{\leftrightarrows}} & \mathcal{B}/\mathcal{D} \end{array}$$

defined as $K = \mathbf{T}_{\mathcal{D}} G \mathbf{S}_{\mathcal{C}}$ and $H = \mathbf{T}_{\mathcal{C}} F \mathbf{S}_{\mathcal{D}}$. There exist natural transformations $\chi : HK \to \mathrm{id}$ and $\upsilon : \mathrm{id} \to KH$ described as follows: For each object Y in $\mathcal{B}/\mathcal{D}$ there is a natural $\mathcal{B}$-morphism

$$\mathbf{S}_{\mathcal{D}} Y \to GF\mathbf{S}_{\mathcal{D}} Y$$

Applying $\mathbf{T}_{\mathcal{D}}$ we obtain a natural morphism in $\mathcal{B}/\mathcal{D}$,

$$Y \cong \mathbf{T}_{\mathcal{D}} \mathbf{S}_{\mathcal{D}} Y \to \mathbf{T}_{\mathcal{D}} GF\mathbf{S}_{\mathcal{D}} Y.$$

The natural $\mathcal{B}$-morphism

$$F\mathbf{S}_{\mathcal{D}} Y \to \mathbf{S}_{\mathcal{C}} \mathbf{T}_{\mathcal{C}} F \mathbf{S}_{\mathcal{D}} Y$$

induces a natural $\mathcal{A}$-morphism

$$GF\mathbf{S}_{\mathcal{D}} Y \to G\mathbf{S}_{\mathcal{C}} \mathbf{T}_{\mathcal{C}} F \mathbf{S}_{\mathcal{D}} Y.$$

Applying again $\mathbf{T}_{\mathcal{D}}$ and composing we obtain the natural homomorphism in $\mathcal{B}/\mathcal{D}$

$$\upsilon_Y : Y \cong \mathbf{T}_{\mathcal{D}} \mathbf{S}_{\mathcal{D}} Y \to \mathbf{T}_{\mathcal{D}} GF\mathbf{S}_{\mathcal{D}} Y \to \mathbf{T}_{\mathcal{D}} G\mathbf{S}_{\mathcal{C}} \mathbf{T}_{\mathcal{C}} F \mathbf{S}_{\mathcal{D}} Y = KHY$$

On the other hand, given X in $\mathcal{A}/\mathcal{C}$, we can use the natural morphism in $\mathcal{B}$,

$$G\mathbf{S}_{\mathcal{C}} X \to \mathbf{S}_{\mathcal{D}} \mathbf{T}_{\mathcal{D}} G \mathbf{S}_{\mathcal{C}} X$$

with kernel and cokernel $\mathcal{D}$-torsion to define the canonical morphism

$$FG\mathbf{S}_{\mathcal{C}} X \to F\mathbf{S}_{\mathcal{D}} \mathbf{T}_{\mathcal{D}} G \mathbf{S}_{\mathcal{C}} X.$$

By a standard argument and the definition of $\mathcal{D}$ it is not difficult to check that this morphism permits us to obtain a natural isomorphism in $\mathcal{A}/\mathcal{C}$

$$\mathbf{T}_{\mathcal{C}} FG\mathbf{S}_{\mathcal{C}} X \cong \mathbf{T}_{\mathcal{C}} F\mathbf{S}_{\mathcal{D}} \mathbf{T}_{\mathcal{D}} G \mathbf{S}_{\mathcal{C}} X.$$

Now, the canonical morphism $FG\mathbf{S}_{\mathcal{C}} X \to \mathbf{S}_{\mathcal{C}} X$ gives a morphism in $\mathcal{A}/\mathcal{C}$

$$\mathbf{T}_{\mathcal{C}} FG\mathbf{S}_{\mathcal{C}} X \to \mathbf{T}_{\mathcal{C}} \mathbf{S}_{\mathcal{C}} X.$$

Therefore we achieve the definition of a natural homomorphism in $\mathcal{A}/\mathcal{C}$

$$\chi_X : HKX = \mathbf{T}_{\mathcal{C}} F\mathbf{S}_{\mathcal{D}} \mathbf{T}_{\mathcal{D}} G \mathbf{S}_{\mathcal{C}} X \cong \mathbf{T}_{\mathcal{C}} FG\mathbf{S}_{\mathcal{C}} X \to \mathbf{T}_{\mathcal{C}} \mathbf{S}_{\mathcal{C}} X \to X$$

We are now ready to define full subcategories of $\mathcal{A}/\mathcal{C}$ and $\mathcal{B}/\mathcal{D}$ equivalent by restriction of K and H.

Definition 2.3. An object X of $\mathcal{A}/\mathcal{C}$ is said to be F-*static* whenever $\chi_X : HKX \to X$ is an isomorphism. The category of all F-static objects in $\mathcal{A}/\mathcal{C}$ will be denoted by $\mathcal{A}/\mathcal{C}_F$ which is a full additive subcategory of $\mathcal{A}/\mathcal{C}$. An object Y of $\mathcal{B}/\mathcal{D}$ is said to be F-*co-static* if $\upsilon_Y : Y \to KHY$ is an isomorphism. The category of all F-co-static objects will be denoted by $\mathcal{B}/\mathcal{D}^F$ which is a full additive subcategory of $\mathcal{B}/\mathcal{D}$. When the original adjunction is $M \otimes_S - \dashv \mathrm{Hom}_R(M, -)$ for some left R-bimodule with $S = \mathrm{End}_R(M)$, we will speak of M-*static* and M-*co-static* objects of R-Mod$/\mathcal{C}$ and S-Mod$/\mathcal{D}$.

We have immediately the following theorem, that extends [Na, Theorem 2.5].

Theorem 2.4. *The restrictions of the additive functors*

$$K = \mathbf{T}_{\mathcal{D}} G \mathbf{S}_{\mathcal{C}} : \mathcal{A}/\mathcal{C}_F \to \mathcal{B}/\mathcal{D}^F$$

and

$$H = \mathbf{T}_{\mathcal{C}} F \mathbf{S}_{\mathcal{D}} : \mathcal{B}/\mathcal{D}^F \to \mathcal{A}/\mathcal{C}_F$$

form an equivalence between the categories $\mathcal{A}/\mathcal{C}_F$ and $\mathcal{B}/\mathcal{D}^F$.

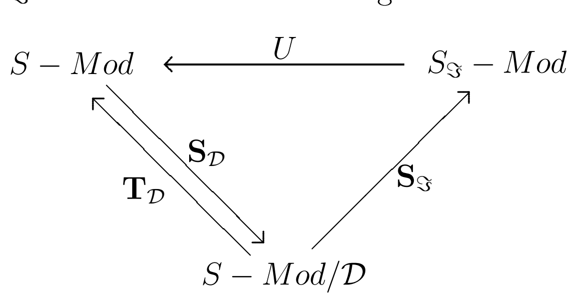
As usual, we will say that an object X in $\mathcal{A}$ *divides* an object U if there is an object X' in $\mathcal{A}$ and an isomorphism $U \cong X \oplus X'$. When X divides a finite direct sum of copies of U, we say that X *weakly divides* U. We will say that two objects of $\mathcal{A}$ are *weakly isomorphic* if each weakly divides the other. It is clear that both functors K as H preserve finite direct sums. Therefore, the following result has an easy straightforward proof.

Proposition 2.5. *The subcategories $\mathcal{A}/\mathcal{C}_F$ and $\mathcal{B}/\mathcal{D}^F$ are closed under finite direct sums and direct summands.*

Let M be a $\mathcal{C}$-*closed* object in $\mathcal{A}$, i.e., $M \cong \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}M$ naturally, and $S = \mathrm{End}_{\mathcal{A}}(M)$. We can take $G = \mathrm{Hom}_{\mathcal{A}}(M, -)$ and we know by [P, Corollary 7.3] that there exists a left adjoint F of G, satisfying $F(S) = M$. Assume that F is $\mathcal{C}$-exact. As in a foregoing argument, we have that

$$\begin{aligned} HK(\mathbf{T}_{\mathcal{C}}M) = \mathbf{T}_{\mathcal{C}}(F\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}G\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}M) &\cong \mathbf{T}_{\mathcal{C}}(FG\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}M) \\ &\cong \mathbf{T}_{\mathcal{C}}(FS) \cong \mathbf{T}_{\mathcal{C}}M. \end{aligned}$$

This gives immediately that M is F-static. Therefore $\mathbf{T}_{\mathcal{D}}S$ is G-co-static. If $\Im$ is the filter of left ideals of S associated with the torsion theory $(\mathcal{D}, \mathcal{G})$ and Y is a left S-module, then $Y_{\Im} = lim_{\to \alpha \in \Im} \mathrm{Hom}_S(\alpha, Y/t_{\mathcal{D}}(Y))$ denotes the localized S-module. Indeed, $S_{\Im}$ is an S-algebra via the canonical map $S \to S_{\Im}$ and $Y_{\Im}$ is a left $S_{\Im}$-module. It is well known [S, Ch. X] that there exists a full and faithful functor $\mathbf{S}_{\Im} : S - \mathrm{Mod}/\mathcal{D} \to S_{\Im} - \mathrm{Mod}$ such that the diagram of functors



commutes, where U is the restriction of scalars. Moreover, $U\mathbf{S}_{\mathfrak{I}}$ is isomorphic to $\mathbf{S}_{\mathcal{D}}$. It follows from this that the restriction of $\mathbf{S}_{\mathfrak{I}}$ to the full subcategory of S-Mod$/\mathcal{D}$ of the objects that weakly divides $\mathbf{T}_{\mathcal{D}}S$ gives an isomorphism between this category and the category of the projective $S_{\mathfrak{I}}$-modules of finite type. This last category will be denoted by $(S_{\mathfrak{I}} \mid \ weak\, S_{\mathfrak{I}})$. In what follows, these categories will be identified. Analogously, the subcategory of $\mathcal{A}/\mathcal{C}_M$ consisting of the objects that weakly divides $\mathbf{T}_{\mathcal{C}}M$ will be denoted by $(\mathcal{A}/\mathcal{C} \mid weak\ M)$. These are equivalent categories, as shows the following result, that generalizes [Na, Theorem 3.7].

Corollary 2.6. *The restrictions of the additive functors*

$$H = \mathbf{T}_{\mathcal{D}}G\mathbf{S}_{\mathcal{C}} : (\mathcal{A}/\mathcal{C} \mid weak\ M) \to (S_{\mathfrak{I}} \mid \ weak\ S_{\mathfrak{I}})$$

and

$$K = \mathbf{T}_{\mathcal{C}}F\mathbf{S}_{\mathcal{D}} : (S_{\mathfrak{I}} \mid \ weak\ S_{\mathfrak{I}}) \to (\mathcal{A}/\mathcal{C} \mid weak\ M)$$

form an equivalence between the categories $(\mathcal{A}/\mathcal{C} \mid weak\ M)$ *and* $(S_{\mathfrak{I}} \mid weak\ S_{\mathfrak{I}})$.

We specialize to the case of modules. For a left R-module M we have the adjoint functors

$$\begin{array}{c} S-Mod \\ M\otimes_S - \Big\downarrow \Big\uparrow Hom_R(M,-) \\ R-Mod \end{array}$$

where $S = \mathrm{End}_R(M)$. Let $\mathcal{C}$ denote a localizing subcategory of R-Mod. In order to assure that the subcategory of S-Mod

$$\mathcal{D} = \{X \in S-\mathrm{Mod} : M \otimes_S X \in \mathcal{C}\}$$

is a localizing subcategory of S-Mod we will assume that the functor $M \otimes_S -$ is $\mathcal{C}$-exact (see Proposition 2.1 and Definition 2.2). In this case we will say that M is *weakly $\mathcal{C}$-flat*. In the absolute case [Na, Theorem 3.7], under the condition that M is finitely generated, it is proved that there is an equivalence between the category of left R-modules that divide M and the category of all projective left S-modules. In our quotient situation we will need some finiteness condition on the torsion theory $(\mathcal{C},\mathcal{F})$ defined by the localizing subcategory $\mathcal{C}$ of $\mathcal{A} = R$-Mod. Actually, $(\mathcal{C},\mathcal{F})$ is said to be a torsion theory of *finite type* if the Gabriel filter of left ideals $\mathfrak{R}$ associated with $(\mathcal{C},\mathcal{F})$ has a cofinal subset of finitely generated left ideals. On the other hand, the assumption on M can be weakened, and we will prove a generalization of [Na, Theorem 3.7] for a $\mathcal{C}$-closed left R-module M which is *$\mathcal{C}$-finitely generated*, i.e., there is a finitely generated submodule N of M such that M/N is $\mathcal{C}$-torsion. When $(\mathcal{C},\mathcal{F})$ is of finite type, the functor $\mathbf{T}_{\mathcal{C}}$ preserves arbitrary direct sums. In such case, we will denote by $(R\text{-Mod}/\mathcal{C} \mid M)$ the full subcategory of R-Mod$/\mathcal{C}$ consisting of the objects that divide some direct sum of copies of $\mathbf{T}_{\mathcal{C}}M$ and by $(S_{\Im} \mid S_{\Im})$ the category of all projective $S_{\Im}$-modules.

Theorem 2.7. *Assume that the torsion theory $(\mathcal{T}, \mathcal{F})$ on R-Mod is of finite type and let M be a $\mathcal{C}$-finitely generated, $\mathcal{C}$-closed and $\mathcal{C}$-flat left R-module. The restrictions of the additive functors*

$$H = \mathbf{T}_{\mathcal{D}}(\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}}(-)) : (R - \mathrm{Mod}/\mathcal{C} \mid M) \to (S_{\Im} \mid S_{\Im})$$

and

$$K = \mathbf{T}_{\mathcal{C}}(M \otimes_S \mathbf{S}_{\mathcal{D}}(-)) : (S_{\Im} \mid S_{\Im}) \to (R - \mathrm{Mod}/\mathcal{C} \mid M)$$

form an equivalence between the categories $(R\text{-Mod}/\mathcal{C} \mid M)$ and $(S_{\Im} \mid S_{\Im})$.

Proof. In view of Corollary 2.6 and the discussions made throughout this section, we need only prove that K and H preserve arbitrary direct sums and that they are well defined. First, we claim that $\mathcal{D}$ is of finite type. If this claim is assumed, then $\mathbf{T}_{\mathcal{D}}$ preserves direct sums by [*S,* Proposition XIII.2.1]. Since M is $\mathcal{C}$-finitely generated and $\mathcal{C}$ is of finite type, $\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}}(-))$ preserves direct sums and this shows that K preserves direct sums. Moreover, the fact that $\mathcal{D}$ is of finite type forces that every projective left $S_{\Im}$-module is $\mathcal{D}$-closed as left S-module. Since $\mathbf{S}_{\Im}$ is full and faithful, $(S_{\Im} \mid S_{\Im})$ can be identified with the full subcategory of $\mathcal{B}/\mathcal{D}$ consisting of the objects that divide some direct sum of copies of $\mathbf{T}_{\mathcal{D}}$S. This shows that H is well defined. Analogous arguments give that K preserves direct sums and it is well defined. In this way, the proof will be complete if we prove the claim. For, observe that the filter of left ideals of S associated with $\mathcal{D}$ is $\Im = \{I \leq_S S : M/MI$ is $\mathcal{C}$-torsion$\}$. Take I a left ideal in $\Im$ and let $m_1, \ldots, m_n$ in M such that $N = Rm_1 + \ldots + Rm_n$ is $\mathcal{C}$-dense in M. For each $i = 1, \ldots, n$ there is an finitely generated left ideal α_i of R in the filter $\Re$ such that $\alpha_i m_i \subseteq MI$. It is straightforward to check that $\alpha_1 m_1 + \ldots + \alpha_n m_n$ is $\mathcal{C}$-dense in N. Since N is $\mathcal{C}$-dense in M this implies that $\alpha_1 m_1 + \ldots + \alpha_n m_n$ is $\mathcal{C}$-dense in M. Now, for each $i = 1, \ldots, n$ the left ideal α_i can be expressed as $\alpha_i = \sum_{j \in A_i} Ra_{ij}$, for some a_{ij} in R and A_i a finite index set. Thus $\alpha_1 m_1 + \ldots + \alpha_n m_n = \sum_{i=1}^{n} \sum_{j \in A_i} Ra_{ij} m_i \subseteq MI$. For every $i = 1, \ldots, n$ and for every $j \in A_i$, there is $f_{ij} \in I$ such that $a_{ij} m_i \in \mathrm{Imf}_{ij}$. This implies that $\sum_{i=1}^{n} \sum_{j \in A_i} \mathrm{Imf}_{ij} = M\{f_{ij}\}$ is $\mathcal{C}$-dense in M. Hence, the left S-ideal I_0 generated by the f_{ij} verifies that MI_0 is $\mathcal{C}$-dense in M. Hence, I_0 is in $\Im$ and this proves that $\mathcal{D}$ is of finite type.

3 Divisorial Clifford theory

In this section the notation will be slightly modified. So, $R = \oplus_{g \in G} R_g$ will denote a G-graded ring for G an arbitrary group with neutral element e and $\mathcal{C}$ will be a localizing subcategory of R_e-Mod that induces an hereditary torsion theory $(\mathcal{C}, \mathcal{F})$ with associated filter of left ideals of R_e denoted by $\Re$. We will denote by $R - gr$ the Grothendieck category whose objects are the G-graded left R-modules. The morphisms in $R - gr$ are the R-linear graded maps of degree e. Every graded left R-module decomposes, when it is considered as left R_e-module, as a direct sum $M = \oplus_{g \in G} M_g$ of R_e-modules and every morphism $f : M \to N$ in $R - gr$ is, after forgetting the R-linear structure, a morphism of left R_e-modules $f : M \to N$ that maps the g-th component M_g of M to the g-th component N_g of N. This construction defines an exact functor

$$(\_)_e : R - gr \to R_e\text{−Mod}$$

This permits us to induce a localizing subcategory $\mathcal{C}^g$ from $\mathcal{C}$ by putting

$$\mathcal{C}^g = \{X \in R - gr \mid X \in \mathcal{C}\}$$

This localizing subcategory of $R - gr$ define a rigid $[NVO1]$ torsion theory $(\mathcal{C}^g, \mathcal{F}^g)$ in $R - gr$ with associated filter of graded left ideals $\Re^g$.

Now we can consider the following diagram of functors

$$\begin{array}{ccc} R - gr & \underset{(\_)_e}{\overset{R \otimes_{R_e} -}{\leftrightarrows}} & R_e - Mod \\ \mathbf{T}_{\mathcal{C}^g} \Big\downarrow \Big\uparrow \mathbf{S}_{\mathcal{C}^g} & & \mathbf{T}_{\mathcal{C}} \Big\downarrow \Big\uparrow \mathbf{S}_{\mathcal{C}} \\ R - gr/\mathcal{C}^g & \underset{B}{\overset{A}{\leftrightarrows}} & R - Mod/\mathcal{C} \end{array}$$

where $R - gr/\mathcal{C}^g$ (resp. R_e-Mod$/\mathcal{C}$) is the quotient category of $R - gr$ (resp. R_e-Mod) under the localizing subcategory $\mathcal{C}$ (resp. $\mathcal{C}^g$). B is defined as $\mathbf{T}_{\mathcal{C}} \circ (R \otimes_{R_e} -) \circ \mathbf{S}_{\mathcal{C}^g}$ and analogously A is defined as $\mathbf{T}_{\mathcal{C}^g} \circ ()_e \circ \mathbf{S}_{\mathcal{C}}$.

We remember $[D1]$ that R is strongly graded by G (i.e. $R_g R_h = R_{gh}$ for all $g, h \in G$) if and only if $(\_)_e$ and $R \otimes_{R_e} -$ establish an equivalence between the categories $R - gr$ and R_e-Mod. In $[AGT$, Theorem 1.1$]$ we analyzed when B and A give an equivalence between the categories $R - gr/\mathcal{C}^g$ and R_e-Mod$/\mathcal{C}$. The results there obtained were expressed in the equivalent language of torsion theories. Now, we will restate some of these facts in the formalism of quotient categories.

Definition 3.1. Let R be a G-graded ring and $\mathcal{C}$ a localizing subcategory of R_e-Mod. Following $[NR]$ we will say that $\mathcal{C}$ is *G-stable* if $R \otimes_{R_e} A$ is $\mathcal{C}$-torsion for every $\mathcal{C}$-torsion left R_e-module A.

The ring morphism $R_e \to R$ permits to define canonically the localizing subcategory $\mathcal{C}^*$ of R-Mod consisting of the left R-modules that are $\mathcal{C}$-torsion considered as left R_e-modules. The following Lemma relates $\mathcal{C}^*$ and $\mathcal{C}^g$ under the hypothesis of G-stability for $\mathcal{C}$.

Lemma 3.2. *Let R be a G-graded ring and $\mathcal{C}$ a G -stable localizing subcategory of* R_e-Mod. *Let us denote by $\Re$ the filter of left ideals of* R_e-Mod *associated with $\mathcal{C}$ and by $\Re^*$ the filter associated with $\mathcal{C}^*$. The localizing subcategory $\mathcal{C}^*$ is the smallest localizing subcategory of* R-Mod *that contains the underlying R-modules of the objects in $\mathcal{C}^g$. Moreover, $(\mathcal{C}^*, \mathcal{F}^*)$ is a graded torsion theory on* R-Mod *and*

$$\Re^* = \{I \leq_R R \mid \text{ there is } H \in \Re^g \text{ with } H \subseteq I\} = \{I \leq_R R \mid I \cap R_e \in \Re\}$$

Proof: If we prove that $\mathcal{C}^*$ is the smallest localizing subcategory of R-Mod containing the underlying R-modules of the objects in $\mathcal{C}^g$, then, by $[NR$, Proposition 1.1$]$, the rest of the assertions follows. Let $\mathcal{E}$ be any localizing subcategory of R-Mod containing $\mathcal{C}^g$ and take A any $\mathcal{C}^*$-torsion left R-module. Then ${}_{R_e}A$ is $\mathcal{C}$-torsion. By G-stability, $R \otimes_{R_e} A$ is $\mathcal{C}$-torsion. But $R \otimes_{R_e} A$ is a graded left R-module and, so, $R \otimes_{R_e} A$ is $\mathcal{C}^g$-torsion. This implies that $R \otimes_{R_e} A$ is $\mathcal{E}$-torsion. Observe that there is a canonical epimorphism of left R-modules from $R \otimes_{R_e} A$ onto A. This shows that A is $\mathcal{E}$-torsion. Therefore $\mathcal{C}^* \subseteq \mathcal{E}$ and the Lemma is proved.

Definition 3.3. A G-graded ring R is said to be $\mathcal{C}$-*divisorially* graded (see [$LVVO$]) whenever for every $g, h \in G$, $R_g R_h$ is $\mathcal{C}$-dense in R_e.

In [AGT, Theorem 1.1] it is essentially proved that the categories $R-gr/\mathcal{C}^g$ and R_e-Mod$/\mathcal{C}$ are canonically equivalent if and only if R is $\mathcal{C}$-divisorially graded and $\mathcal{C}$ is G-stable. We recall this result now.

Theorem 3.4. *Let $R = \oplus_{g\in G} R_g$ be a graded ring and consider $\mathcal{C}$ a localizing subcategory of R_e-*Mod*. Then the following assertions are equivalent:*

*(i) B and A establish an equivalence between the categories $R - gr/\mathcal{C}^g$ and R_e-*Mod$/\mathcal{C}$.

(ii) A graded left R-module X is $\mathcal{C}^g$-torsion if and only if X_e is $\mathcal{C}$-torsion.

(iii) R is $\mathcal{C}$-divisorially graded and $\mathcal{C}$ is G-stable.

A graded ring R satisfying Theorem 3.4 is said to be $\mathcal{C}$-*strongly graded.* Note that it is possible that a strongly graded ring fails to be $\mathcal{C}$-strongly graded. This happens if $\mathcal{C}$ is not G-stable. This last condition was studied on strongly graded rings in [NR].

In the part (ii) $\Leftrightarrow$ (i) of the proof of the Theorem 3.4 [AGT, Theorem 1.1] was proved a property of relative flatness on the ring extension $R_e \to R$ that we record in the following proposition.

Proposition 3.5. *Let $R = \oplus_{g\in G} R_g$ be a $\mathcal{C}$-strongly graded ring. For each exact sequence $0 \to K \to L \to N \to 0$ in R_e-*Mod *with K $\mathcal{C}$-torsion, the kernel of the canonical morphism $R \otimes_{R_e} L \to R \otimes_{R_e} N$ is $\mathcal{C}$-torsion.*

For a graded left R-module M, consider the ring $S = \mathrm{END}_R(M)$ consisting of the graded endomorphism of M. S is canonically G-graded [D, Sections 3 and 4] by putting

$$S_g = \{f \in \mathrm{END}_R(M) : f(M_h) \subseteq M_{gh} \text{ for all } h \in G\}.$$

Therefore $S_e = \mathrm{End}_{R-gr}(M, M)$. Before to prove the main results on equivalence of certain categories constructed from the module M and the localizing subcategory $\mathcal{C}$, we need some technical results. These facts will be stated in the following lemmas.

Lemma 3.6. *Let R be a $\mathcal{C}$-strongly graded ring and M a $\mathcal{C}^g$-torsionfree graded left R-module. The map ρ:$End_{R-gr}(M) \to End_{R_e}(M_e)$, given by $\rho(f) = f_e$ for every $f \in End_{R-gr}(M)$ is a ring isomorphism.*

Proof: It is clear that ρ is a ring homomorphism. We will prove that it has trivial kernel and it is surjective. Note that

$$\mathrm{Ker}\rho = \{f \in \ \mathrm{End}_{R-gr}(M) : f_e = 0\}$$

Observe that for $f \in \mathrm{End}_{R-gr}(M)$, $f_e = 0$ if and only if $(\mathrm{Im}\, f)_e = 0$. By Theorem 3.4, $\mathrm{Im}\, f$ is a $\mathcal{C}^g$-torsion graded left R-module. Since $\mathrm{Imf} \subseteq M$ and M is $\mathcal{C}^g$-torsionfree, $\mathrm{Im}\, f$ must be trivial, i.e., $f = 0$. This proves that $\mathrm{Ker}\rho = 0$.

To prove that ρ is surjective, take $f \in \mathrm{End}_{R_e}(M_e)$. We construct the morphism of graded left R-modules

$$R \otimes_{R_e} f : R \otimes_{R_e} M_e \to R \otimes_{R_e} M_e$$

Consider the canonical morphism of graded left R-modules

$$\zeta : R \otimes_{R_e} M_e \to M$$

Because ζ_e is an isomorphism, it follows from Theorem 3.4 that ζ has kernel and cokernel $\mathcal{C}^g$-torsion. This implies that

$$\zeta \circ (R \otimes_{R_e} f) : R \otimes_{R_e} M_e \to M$$

annihilates $\mathrm{Ker}\zeta$, since M is $\mathcal{C}^g$-torsionfree. But this implies that there exists a morphism of graded left R-modules $\bar{f} : M \to M$ such that

$$\bar{f} \circ \zeta = \zeta \circ (R \otimes_{R_e} f)$$

It is immediate to check that $\bar{f}_e = f$. Therefore, ρ is surjective and the Lemma is proved.

Assume that M is a $\mathcal{C}^g$-torsionfree graded left R-module over a $\mathcal{C}$-strongly graded ring R. The localizing subcategory $\mathcal{C}$ of R_e-Mod induces (see Section 1) a subcategory $\mathcal{D}$ of the category of left modules on $\mathrm{End}_{R_e}(M_e)$. From Lemma 3.6 it is possible to identify $\mathrm{End}_{R_e}(M_e)$ with S_e and, therefore, M_e can be considered as an $R_e - S_e$-bimodule. Therefore, up to this identification, we rewrite

$$\mathcal{D} = \{B \in S_e - \mathrm{Mod} : M_e \otimes_{S_e} B \text{ is } \mathcal{C} - \text{torsion}\}.$$

Proposition 2.1. assures that if the functor $M_e \otimes_{S_e} B : S_e\text{-Mod} \to R_e\text{-Mod}$ is $\mathcal{C}$-exact then $\mathcal{D}$ is a localizing subcategory of S_e-Mod. We will change for this functor the nomenclature and we will say that M_e is *weakly $\mathcal{C}$-flat*, i.e., for each monomorphism $0 \to A \xrightarrow{f} B$ in S_e-Mod the canonical morphism in R_e-Mod, $M_e \otimes_{S_e} f : M_e \otimes_{S_e} A \to M_e \otimes_{S_e} B$, has $\mathcal{C}$-torsion kernel.

At this point we can induce a localizing subcategory of S-Mod canonically from $\mathcal{C}$ in two ways. The first idea is to use the restriction of scalars $S_e \to S$ to define a localizing subcategory $\mathcal{D}^*$ of S-Mod. The $\mathcal{D}^*$-torsion left S-modules are the left S-modules that are $\mathcal{D}$-torsion considered as left S_e-modules. The second possibility is to define a (at this moment, possibly not localizing), subcategory $\mathcal{P}$ of S-Mod by using the tensor product $M \otimes_S -$. A left S-module Y is in $\mathcal{P}$ if and only the left R-module $M \otimes_S Y$ is $\mathcal{C}^*$-torsion.

Dade, in [D, Theorem 4.6], characterized in terms of the graded module M when is $S = \mathrm{END}_R(M)$ strongly graded. Concretely, he found that S is strongly graded if and only if M is *weakly G-invariant*, i.e., M is weakly isomorphic in $R - gr$ to all its suspensions $M(g)$. It is not hard to prove that a weakly G-invariant graded left R-module M with $\mathcal{C}$-torsionfree M_e must be $\mathcal{C}^g$-torsionfree.

Lemma 3.7. *Let M be a weakly G-invariant graded left R-module such that M_e is weakly $\mathcal{C}$-flat and $\mathcal{C}$-torsionfree as left R_e-module. Assume that R is $\mathcal{C}$-strongly graded. The classes of left S-modules*

$$\mathcal{D}^* = \{Y \in S - \text{Mod} :_{S_e} Y \textit{ is } \mathcal{D} - \textit{torsion}\}$$

and

$$\mathcal{P} = \{Y \in S - \text{Mod} : M \otimes_S Y \textit{ is } \mathcal{C}^* - \textit{torsion}\}$$

coincide.

Proof: We make the following computation: Given Y a left S-module, Y is $\mathcal{D}^*$-torsion if and only if ${}_{S_e}Y$ is $\mathcal{D}$-torsion if and only if $M_e \otimes_{S_e} Y$ is $\mathcal{C}$-torsion. But, since S is strongly graded, $M_e \otimes_{S_e} Y \cong M_e \otimes_{S_e} S \otimes_S Y \cong M \otimes_S$Y. Thus, Y is $\mathcal{D}^*$-torsion if and only if $M \otimes_S Y$ is $\mathcal{C}$-torsion if and only if $M \otimes_S Y$ is $\mathcal{C}^*$-torsion.

Lemma 3.8. *Let M be a weakly G-invariant graded left R-module such that M_e is weakly $\mathcal{C}$-flat and $\mathcal{C}$-torsionfree as left R_e-module. If R is $\mathcal{C}$-strongly graded, then the strongly graded ring S is $\mathcal{D}$-strongly graded. Moreover, the torsion theory on S-*Mod *determined by $\mathcal{D}^*$ is a graded torsion theory.*

Proof: Given B in S_e-Mod, $S \otimes_{S_e} B$ is $\mathcal{D}$-torsion if and only if $M_e \otimes_{S_e} S \otimes_{S_e} B$ is $\mathcal{C}$-torsion if and only if $M \otimes_{S_e} B$ is $\mathcal{C}$-torsion. But $M \otimes_{S_e} B$ is a graded left R-module by putting $(M \otimes_{S_e} B)_g = M_g \otimes_{S_e} B$ for each g in the group G. So, $M \otimes_{S_e} B$ is $\mathcal{C}$-torsion if and only if $M \otimes_{S_e} B$ is $\mathcal{C}^g$-torsion and, by Theorem 3.4, this occurs if and only if $(M \otimes_{S_e} B)_e = M_e \otimes_{S_e} B$ is $\mathcal{C}$-torsion if and only if B is $\mathcal{D}$-torsion. Therefore, B is $\mathcal{D}$-torsion if and only if $S \otimes_{S_e} B$ is $\mathcal{D}$-torsion and we can use Theorem 3.4 to obtain that S is $\mathcal{D}$-strongly graded.

As in Section 1, we have functors

$$\begin{array}{c} R - Mod/\mathcal{C}^* \\ F^* \Big\downarrow \Big\uparrow G^* \\ S - Mod/\mathcal{D}^* \end{array}$$

defined as $F^* = \mathbf{T}_{\mathcal{D}^*}\text{Hom}_R(M, \mathbf{S}_{\mathcal{D}^*}(-))$ and $G^* = \mathbf{T}_{\mathcal{C}^*}(M \otimes_S \mathbf{S}_{\mathcal{C}^*}(-))$, where R-Mod$/\mathcal{C}^*$ is the quotient category of R-Mod constructed from $\mathcal{C}^*, S$-Mod$/\mathcal{D}^*$ is the quotient category of S-Mod defined by $\mathcal{D}^*$ and $\mathbf{T}_{\mathcal{D}^*}, \mathbf{S}_{\mathcal{D}^*}, \mathbf{T}_{\mathcal{C}^*}, \mathbf{S}_{\mathcal{C}^*}$ denote the canonical functors. For every object X in R-Mod$/\mathcal{C}^*$ we can consider $\mathbf{S}_{\mathcal{C}^*}X$ as a left R_e-module and it is natural ask if ${}_{R_e}\mathbf{S}_{\mathcal{C}^*}X$ is $\mathcal{C}$-closed, i.e., if $\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X$ is isomorphic to $\mathbf{S}_{\mathcal{C}^*}X$ as left R_e-modules. But this is true if R is a $\mathcal{C}$-strongly graded ring, and the proof of this fact can be constructed analogously to that of [NR, Proposition 2.1]. In this case, S is also $\mathcal{D}$-strongly graded by Lemma 3.8 and so we have that every object in S-Mod$/\mathcal{D}^*$ can be regarded in S_e-Mod$/\mathcal{D}$ via the functor $\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}$. We are now ready to establish, by restriction of the functors F^* and G^*, an equivalence of categories between certain subcategories of R-Mod$/\mathcal{C}^*$ and S-Mod$/\mathcal{D}^*$.

Theorem 3.9. *Consider R a $\mathcal{C}$-strongly graded ring and M a weakly G-invariant graded left R-module such that M_e is a $\mathcal{C}$-torsionfree $\mathcal{C}$-flat $R_e - S_e$-bimodule. Let*

$$R-\mathrm{Mod}/\mathcal{C}^*_{rest.M_e} = \{X \in R-\mathrm{Mod}/\mathcal{C}^* \mid \mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X \ \textit{is}\ M_e-\textit{static}\}$$

and

$$S-\mathrm{Mod}/\mathcal{D}^{*rest.M_e} = \{Y \in S-\mathrm{Mod}/\mathcal{D}^* \mid \mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}Y \ \textit{is}\ M_e-\textit{co}-\textit{static}\}$$

The restriction of the functors

$$F^* : R-\mathrm{Mod}/\mathcal{C}^*_{rest.M_e} \to S-\mathrm{Mod}/\mathcal{D}^{*rest.M_e}$$

and

$$G^* : S-\mathrm{Mod}/\mathcal{D}^{*rest.M_e} \to R-\mathrm{Mod}/\mathcal{C}^*_{rest.M_e}$$

establish an equivalence of categories between the full subcategories

$$R\text{–}\mathrm{Mod}/\mathcal{C}^*_{rest.M_e}$$

of R-Mod$/\mathcal{C}^*$ and

$$S\text{–}\mathrm{Mod}/\mathcal{D}^{*rest.M_e}$$

of S-Mod$/\mathcal{D}^*$.

Proof: First, we need to prove that the restriction of

$$F^* : R\text{–}\mathrm{Mod}/\mathcal{C}^*_{rest.M_e} \to S\text{–}\mathrm{Mod}/\mathcal{D}^{*rest.M_e}$$

and

$$G^* : S-\mathrm{Mod}/\mathcal{D}^{*rest.M_e} \to R-\mathrm{Mod}/\mathcal{C}^*_{rest.M_e}$$

are well defined. For, take $X \in R$-Mod$/\mathcal{C}^*_{rest.M_e}$ and observe that

$$\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}X) \cong \mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(\mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}M, \mathbf{S}_{\mathcal{C}^*}X) \cong$$

$$\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(\mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(R \otimes_{R_e} M_e), \mathbf{S}_{\mathcal{C}^*}X) \cong \mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(R \otimes_{R_e} M_e, \mathbf{S}_{\mathcal{C}^*}X),$$

where the second isomorphism is given by Theorem 3.4. We have an exact sequence in S-Mod

$$0 \to T \to \mathrm{Hom}_R(R \otimes_{R_e} M_e, \mathbf{S}_{\mathcal{C}^*}X)$$

$$\to \mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(R \otimes_{R_e} M_e, \mathbf{S}_{\mathcal{C}^*}X) \to C \to 0$$

where T and C are $\mathcal{D}^*$-torsion left S-modules. If we consider this sequence in S_e-Mod, then T and C are $\mathcal{D}$-torsion then

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_R(R \otimes_{R_e} M_e, \mathbf{S}_{\mathcal{C}^*}X) \cong \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(R \otimes_{R_e} M_e, \mathbf{S}_{\mathcal{C}^*}X),$$

since $\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(R\otimes_{R_e} M_e, \mathbf{S}_{\mathcal{C}^*}X)$ is $\mathcal{D}$-closed. Therefore,

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}X) \cong \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_R(R\otimes_{R_e} M_e, \mathbf{S}_{\mathcal{C}^*}X)$$

$$\cong \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X).$$

Since

$$X \in R\text{−Mod}/\mathcal{C}^*_{rest.M_e},$$

$\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X$ is M_e-static by definition. Hence, $\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X)$ is M_e-co-static by Theorem 2.4. But, by the foregoing computations,

$$\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X) \cong \mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}X) = \mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}F^*X,$$

and we have that $\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}F^*X$ is M_e-co-static, that is, $F^*X \in R\text{-Mod}/\mathcal{C}^*_{rest.M_e}$. Analogously, for each $Y \in S\text{-Mod}/\mathcal{D}^{*rest.M_e}$, $G^*Y \in R\text{-Mod}/\mathcal{C}^*_{rest.M_e}$. Concretely, we have an exact sequence in R-Mod

$$0 \to T \to M\otimes_S \mathbf{S}_{\mathcal{D}^*}Y \to \mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(M\otimes_S \mathbf{S}_{\mathcal{D}^*}Y) \to C \to 0$$

with both T and C, $\mathcal{C}^*$-torsion. As in the foregoing argument, we can deduce that

$$\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}(M\otimes_S \mathbf{S}_{\mathcal{D}^*}Y) \cong \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(M\otimes_S \mathbf{S}_{\mathcal{D}^*}Y)$$

as R_e-modules. The claim follows as in the foregoing argument after observing that $M \cong M_e \otimes_{S_e} S$ since S is strongly graded. Now, we will check that F^* and G^* give the equivalence. For $X \in R\text{-Mod}/\mathcal{C}^*_{rest.M_e}$,

$$\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}G^*F^*X = \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(M\otimes_S \mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}X)) \cong$$

$$\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}(M_e\otimes_{S_e} S\otimes_S \mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}X)) \cong$$

$$\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}(M_e\otimes_{S_e} \mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}X)) \cong$$

$$\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}(M_e\otimes_{S_e} \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_{R_e}(M, \mathbf{S}_{\mathcal{C}^*}X)) \cong$$

$$\mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}(M_e\otimes_{S_e} \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X)) \cong \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X.$$

As $\mathbf{S}_{\mathcal{C}^*}$ maps objects from R-Mod$/\mathcal{C}^*$ to $\mathcal{C}$-closed R_e-modules, we can deduce that $\mathbf{S}_{\mathcal{C}^*}G^*F^*X \cong \mathbf{S}_{\mathcal{C}^*}X$. Since $\mathbf{S}_{\mathcal{C}^*}$ is full and faithful, it follows that $G^*F^*X \cong X$.

Given $Y \in S - \mathrm{Mod}/\mathcal{D}^{*rest.M_e}$,

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}F^*G^*Y = \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(M\otimes_S \mathbf{S}_{\mathcal{D}^*}Y)) \cong$$

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(\mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(R\otimes_{R_e} M_e), \mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(M\otimes_S \mathbf{S}_{\mathcal{D}^*}Y)) \cong$$

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\mathrm{Hom}_R(R\otimes_{R_e} M_e, \mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(M\otimes_S \mathbf{S}_{\mathcal{D}^*}Y)) \cong$$

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}\mathbf{T}_{\mathcal{C}^*}(M\otimes_S \mathbf{S}_{\mathcal{D}^*}Y)) \cong$$

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}(M_e\otimes_{S_e} S\otimes_S \mathbf{S}_{\mathcal{D}^*}Y)) \cong$$

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathrm{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}(M_e\otimes_{S_e} \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}Y)) \cong \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}Y.$$

We can again deduce from this that $F^*G^*Y \cong Y$.

In order to extend [$D1$, Theorem 7.4], we will denote by (R-Mod/$\mathcal{C}^*$ |*weak* M_e) the full subcategory of R-Mod/$\mathcal{C}^*$ consisting of the objects X such that $\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X$ weakly divides $\mathbf{T}_{\mathcal{C}}M_e$. If the hereditary torsion theory $(\mathcal{C}, \mathcal{F})$ determined by the localizing subcategory $\mathcal{C}$ is of finite type, then the functor $\mathbf{T}_{\mathcal{C}}$ preserves direct sums. So, it makes sense to consider (R-Mod/$\mathcal{C}^*$ | M_e), the category of all the objects X in R-Mod/$\mathcal{C}^*$ such that $\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X$ divides some direct sum of copies of $\mathbf{T}_{\mathcal{C}}M_e$. Analogously, ($S$-Mod/$\mathcal{D}^*$ |*weak* S_e) denotes the full subcategory of S-Mod/$\mathcal{D}^*$ whose objects Y satisfy that $\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}Y$ weakly divides $\mathbf{T}_{\mathcal{D}}S_e$. From the proof of Theorem 2.7, if $(\mathcal{C}, \mathcal{F})$ is of finite type and M_e is $\mathcal{C}$-finitely generated, then $(\mathcal{D}, \mathcal{G})$ is also of finite type and we can define the category (S-Mod/$\mathcal{D}^*$ | S_e) whose objects Y verify that $\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}Y$ divides some direct sum of copies of $\mathbf{T}_{\mathcal{D}}S_e$. Recall from Section 1 that we denote by $\mathbf{S}_{\Im^*} : S{-}\mathrm{Mod}/\mathcal{D}^* \to S_{\Im^*}$-Mod the canonical functor that associates uniquely a left $S_{\Im^*}$-module to each object in S-Mod/$\mathcal{D}^*$. Analogously, we have the full and faithful functor $\mathbf{S}_{\Im}$:S_e-Mod/$\mathcal{D} \to (S_e)_{\Im}$-Mod. First, we remark the following easy result.

Proposition 3.10. *The subcategories*

$$R{-}\mathrm{Mod}/\mathcal{C}^*_{rest.M_e}$$

and

$$S{-}\mathrm{Mod}/\mathcal{D}^{*rest.M_e}$$

are closed under finite direct sums and direct summands.

Lemma 3.11. *If R is $\mathcal{C}$-strongly graded, $(\mathcal{C}, F)$ is of finite type and M is a graded left R-module such that M_e is $\mathcal{C}$-finitely generated, then $(\mathcal{C}^*, \mathcal{F}^*)$ is of finite type and M is $\mathcal{C}^*$-finitely generated.*

Proof. Let I be a left ideal in the Gabriel topology $\Re^*$ associated with $\mathcal{C}^*$. By Lemma 3.2, $R_e \cap I$ is a left ideal of R_e that it is in $\Re$. As $(\mathcal{C}, \mathcal{F})$ is of finite type, $R_e \cap I$ contains a finitely generated member J of $\Re$. But RJ is finitely generated as left ideal of R and thus we have proved that $(\mathcal{C}^*, \mathcal{F}^*)$ is of finite type, since RJ is in $\Re^*$. Now assume that M is a graded left R-module such that M_e is $\mathcal{C}$-finitely generated. Thus, M_e contains a $\mathcal{C}$-dense finitely generated R_e-submodule A. Consider the exact sequence of left R-modules

$$0 \to T \to R \otimes_{R_e} A \to R \otimes_{R_e} M_e \to R \otimes_{R_e} (M_e/A) \to 0$$

Since R is $\mathcal{C}$-strongly graded, R is $\mathcal{C}$-flat by Proposition 3.5 and $\mathcal{C}$ is G-stable. Thus, the exact sequence

$$0 \to T \to R \otimes_{R_e} A \to R \otimes_{R_e} M_e \to R \otimes_{R_e} (M_e/A) \to 0$$

has starting and final points $\mathcal{C}$-torsion or, equivalently, $\mathcal{C}^*$-torsion. This means that the R-finitely generated image on $R \otimes_{R_e} A$ in $R \otimes_{R_e} M_e$ is $\mathcal{C}^*$-dense. An analogous argument shows that the canonical image of $R \otimes_{R_e} M_e$ in M is $\mathcal{C}^*$-dense. By composing these two canonical morphisms we obtain that the image of $R \otimes_{R_e} A$ in M is $\mathcal{C}^*$-dense and finitely generated.

Theorem 3.12. *Consider a localizing subcategory $\mathcal{C}$ of R_e-Mod for which R is $\mathcal{C}$-strongly graded. Let M be a weakly G-invariant graded left module over a G-graded ring R such that the R_e-S_e-bimodule M_e is weakly $\mathcal{C}$-flat and $\mathcal{C}$-closed. The following statements hold:*

(I) The restriction of the functors

$$F^* : (R - \text{Mod}/\mathcal{C}^* \mid \textit{weak } M_e) \to (S - \text{Mod}/\mathcal{D}^* \mid \textit{weak } S_e)$$

and

$$G^* : (S - \text{Mod}/\mathcal{D}^* \mid \textit{weak } S_e) \to (R - \text{Mod}/\mathcal{C}^* \mid \textit{weak } M_e)$$

establish an equivalence of categories between the full subcategories

$$(R\text{–Mod}/\mathcal{C}^* \mid \textit{weak} M_e)$$

of R-Mod$/\mathcal{C}^*$ and

$$(S\text{–Mod}/\mathcal{D}^* \mid \textit{weak} \mathbf{T}_{\mathcal{D}} S_e)$$

of S-Mod$/\mathcal{D}^*$.

(II) If $(\mathcal{C}, \mathcal{F})$ is of finite type and M_e is $\mathcal{C}$-finitely generated, then the restriction of the functors

$$F^* : (R - \text{Mod}/\mathcal{C}^* \mid M_e) \to (S - \text{Mod}/\mathcal{D}^* \mid S_e)$$

and

$$G^* : (S - \text{Mod}/\mathcal{D}^* \mid S_e) \to (R - \text{Mod}/\mathcal{C}^* \mid M_e)$$

establish an equivalence of categories between the full subcategories

$$(R\text{–Mod}/\mathcal{C}^* \mid M_e)$$

of R-Mod$/\mathcal{C}^*$ and

$$(S\text{–Mod}/\mathcal{D}^* \mid S_e)$$

of S-Mod$/\mathcal{D}^*$.

Proof: (I) We need to check that if X is an object in $(R\text{-Mod}/\mathcal{C}^* \mid \textit{weak } M_e)$ then F^*X is an object in $(S\text{-Mod}/\mathcal{D}^* \mid \text{weak } S_e)$ and that if Y is in $(S\text{-Mod}/\mathcal{D}^* \mid \text{weak } S_e)$ then G^*Y is in $(R\text{-Mod}/\mathcal{C}^* \mid \textit{weak } M_e)$. For, take X in $(R\text{-Mod}/\mathcal{C}^* \mid \textit{weak } M_e)$. There is a splitting monomorphism in R-Mod$/\mathcal{C}$, $\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X \to (\mathbf{T}_{\mathcal{C}} M_e)^n$. Now compute

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}F^*X = \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}} * \text{Hom}_R(M, \mathbf{S}_{\mathcal{C}} * X) \cong$$

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}\text{Hom}_R(M_e \otimes_{S_e} S, \mathbf{S}_{\mathcal{C}^*}X) \cong \mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\text{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}^*}X) \cong$$

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\text{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X)$$

and observe that if we apply the functor $\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\text{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}(-))$ to the morphism $\mathbf{T}_{\mathcal{C}}\mathbf{S}_{\mathcal{C}^*}X \to (\mathbf{T}_{\mathcal{C}}M_e)^n$ then we obtain a splitting monomorphism of left S_e-modules. But

$$\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\text{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}(\mathbf{T}_{\mathcal{C}}M_e)^n) \cong (\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\text{Hom}_{R_e}(M_e, \mathbf{S}_{\mathcal{C}}\mathbf{T}_{\mathcal{C}}M_e))^n \cong$$

$$(\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}\text{Hom}_{R_e}(M_e, M_e))^n \cong (\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}S_e)^n$$

Since $\mathbf{S}_{\mathcal{D}}$ reflects splitting monomorphisms, we conclude that $\mathbf{T}_{\mathcal{D}}\mathbf{S}_{\mathcal{D}^*}F^*X$ weakly divides $\mathbf{T}_{\mathcal{D}}S_e$. Following a similar argument it is possible to prove that $G^*Y \in (R\text{-Mod}/\mathcal{C}^* \mid \textit{weak } M_e)$ whenever $Y \in (S\text{-Mod}/\mathcal{D}^* \mid \text{weak } S_e)$. Part (I) follows now from Proposition 3.10 and Theorem 3.9.
(II) By Lemma 3.11, if $(\mathcal{C}, \mathcal{F})$ is of finite type and M_e is $\mathcal{C}$-finitely generate, then $(\mathcal{C}^*, \mathcal{F}^*)$ is of finite type and M is $\mathcal{C}^*$-finitely generated. Therefore the categories $R\text{-Mod}/\mathcal{C}^*_{rest.M_e}$ and $S\text{-Mod}/\mathcal{D}^{*rest.M_e}$ have arbitrary direct sums and the functors $F^*, G^*, \mathbf{T}_{\mathcal{C}}, \mathbf{T}_{\mathcal{D}}, \mathbf{T}_{\mathcal{C}^*}$ and $\mathbf{T}_{\mathcal{D}^*}$ preserve direct sums. By using these facts, one can prove part (II) in a similar way that part (I).

Recall that if $\Im^*$ and $\Im$ denotes respectively the Gabriel topologies of left ideals of S and of S_e associated with $\mathcal{D}^*$ and $\mathcal{D}$, we have a canonical ring morphism $(S_e)_{\Im} \to S_{\Im^*}$ since $(\mathbf{S}_{\mathcal{D}^*}\mathbf{T}_{\mathcal{D}^*}S)_e$ is isomorphic to $\mathbf{S}_{\mathcal{D}}\mathbf{T}_{\mathcal{D}}S_e$ whenever S is $\mathcal{D}$-strongly graded. By $\text{Mod}(S_{\Im^*} \mid \text{weak } (S_e)_{\Im})$ we denote the category of all the left $S_{\Im^*}$-modules that are projective of finite type considered as left $(S_e)_{\Im}$-modules. By $\text{Mod}(S_{\Im^*} \mid (S_e)_{\Im})$ we denote the category of all the left $S_{\Im^*}$-modules that are projective as left $(S_e)_{\Im}$-modules.

Let Y be a left $S_{\Im^*}$-module such that there exists an isomorphisms of left $(S_e)_{\Im}$-modules $f : Y \oplus C \to Z$, where Z is a $\mathcal{D}$-closed left S_e-module. Assume that G is finite. Following [$NRVO$, Theorem 3.1], the forgetful functor $(\_) : S - gr \to S$-Mod has a left and right adjoint $\_[G] : S\text{-Mod} \to S - gr$ that construct for the S-module Y the G-graded left S-module $Y[G] = \oplus_{g\in G}{}^gY$, where ${}^gY$ denotes a copy of the abelian group Y. If ${}^gy$ denotes the natural image of an element y of Y in the subgroup ${}^gY$ of $Y[G]$ then the structure of left graded S-module in $Y[G]$ is given by setting $(s_h^g y) ={}^{hg}(s_h y)$ for $s_h \in S_h$, and $g, h \in G$. We have [$NRVO$, Remark 3.2.1] a canonical S-homomorphism $\alpha : Y \to Y[G]$ that is actually an injective map. It is evident that $Y[G]_e \cong Y$ as left S_e-modules and we have that Y is $\mathcal{D}$-closed since it is a direct summand of the $\mathcal{D}$-closed left S_e-module Z. By [NR, Proposition 2.1] this implies that $Y[G]$ is $\mathcal{D}^*$-closed and, therefore, Y is $\mathcal{D}^*$-torsionfree. Now we have a canonical monomorphism of left S (or $S_{\Im^*}$)-modules $Y \to Y_{\Im^*}$ that we extend trivially to a monomorphism of left S_e-modules $g : Y \oplus C \to Y_{\Im^*} \oplus$C. Since the cokernel of this monomorphism is $\mathcal{D}$-torsion, the isomorphism f extends uniquely to a isomorphism $\bar{f} : Y_{\Im^*} \oplus C \to Z$. This forces that g is an isomorphism. Hence, $Y \cong Y_{\Im^*}$ and we have obtained that Y is $\mathcal{D}^*$-closed. Taking Z a free left $(S_e)_{\Im}$-module (of finite rank in the case that $\mathcal{D}$ is not of finite type), it is possible to deduce the following result.

Theorem 3.13. *Let R be a $\mathcal{C}$-strongly graded ring by a finite group G, where $\mathcal{C}$ is a localizing subcategory of R_e-Mod and let M be a G-invariant graded left module over a G-graded ring R such that the $R_e - S_e$-bimodule M_e is weakly $\mathcal{C}$-flat and $\mathcal{C}$-closed. The following statements hold:*

(I) The restriction of the functors

$$F^* : (R - \text{Mod}/\mathcal{C}^* \mid \textit{weak } M_e) \to (S_{\Im^*} \mid \ \textit{weak } (S_e)_{\Im})$$

and

$$G^* : (S_{\Im^*} \mid \textit{weak}(S_e)_{\Im}) \to (R\text{–Mod}/\mathcal{C}^* \mid \textit{weak} M_e)$$

establish an equivalence of categories between the full subcategories

$$(R\text{–Mod}/\mathcal{C}^* \mid weak M_e)$$

of R-Mod$/\mathcal{C}^*$ *and*

$$(S_{\Im^*} \mid weak(S_e)_{\Im})$$

of $S_{\Im^*}$ -Mod.

(II) If $(\mathcal{C}, \mathcal{F})$ *is of finite type and* M_e *is* $\mathcal{C}$*-finitely generated, then the restriction of the functors*

$$F^* : (R - \text{Mod}/\mathcal{C}^* \mid M_e) \to (S_{\Im^*} \mid (S_e)_{\Im})$$

and

$$G^* : (S_{\Im^*} \mid (S_e)_{\Im}) \to (R - \text{Mod}/\mathcal{C}^* \mid M_e)$$

establish an equivalence of categories between the full subcategories

$$(R\text{–Mod}/\mathcal{C}^* \mid M_e)$$

of R-Mod$/\mathcal{C}^*$ *and*

$$(S_{\Im^*} \mid (S_e)_{\Im})$$

of $S_{\Im^*}$-Mod.

Let M be a weakly G-invariant graded left R-module with M_e $\mathcal{C}$-closed. We will assume that M_e is $\mathcal{C}$*-cocritical* as left R_e-module, that is, $\mathbf{T}_{\mathcal{C}} M_e$ is a simple object in R_e-Mod$/\mathcal{C}$. It is not hard to see that $S_e = \text{End}_{R_e}(M_e)$ is a division ring and S is a crossed product. Hence M_e is flat as right S_e-module. Since S_e is a division ring, the localizing subcategory of S_e-Mod, $\mathcal{D} = \{A \in S_e\text{-Mod: } M_e \otimes_{R_e} A = 0\} = \{0\}$. Therefore we have that the localizing subcategory $\mathcal{D}^*$ of S-Mod is trivial too. Moreover, every left S_e-module is free and, therefore, $(S \mid S_e) = S$-Mod and $(S \mid weak\, S_e)$ is the category of such left S-modules that are finitely generated as left S_e-modules. In the last case, for G a finite group, it is possible to prove that a left S-module is finitely generated as S-module if and only if it is finitely generated as left S_e-module. This happens because S is strongly graded and, so, S is a projective of finite type left S_e-module. As a corollary of Theorem 3.12 and the foregoing observations, we have

Theorem 3.14. *Let* R *be a* $\mathcal{C}$*-strongly* G*-graded ring for* $\mathcal{C}$ *a localizing subcategory of* R_e-Mod. *Let* M *be a weakly* G*-invariant graded left* R*-module such that* M_e *is a* $\mathcal{C}$*-closed and* $\mathcal{C}$*-cocritical left* R_e*-module. The following assertions hold:*

(I) If $\mathcal{C}$ *is of finite type then the restriction of the functors*

$$\text{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}(-)) : (R - \text{Mod}/\mathcal{C}^* \mid M_e) \to S - \text{Mod}$$

and

$$\mathbf{T}_{\mathcal{C}^*}(M \otimes_S -) : S - \text{Mod} \ \to (R - \text{Mod}/\mathcal{C}^* \mid M_e)$$

establish an equivalence of categories between the full subcategory

$$(R\text{–Mod}/\mathcal{C}^* \mid M_e)$$

of $R\text{-Mod}/\mathcal{C}^*$ *and the category* S-Mod.

(II) If the group G *is finite then the restriction of the functors*

$$\text{Hom}_R(M, \mathbf{S}_{\mathcal{C}^*}(-)) : (R - \text{Mod}/\mathcal{C}^* \mid \ \textit{weak } M_e) \to S - \text{mod}$$

and

$$\mathbf{T}_{\mathcal{C}^*}(M \otimes_S -) : S - \text{mod } \to (R - \text{Mod}/\mathcal{C}^* \mid \ \textit{weak } M_e)$$

establish an equivalence of categories between the full subcategory

$$(R\text{–Mod}/\mathcal{C}^* \mid M_e)$$

of $R\text{-Mod}/\mathcal{C}^*$ *and the category* S-mod *of the finitely generated left* S*-modules.*

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