

# On the Expansion of a Function in Terms of Spherical Harmonics in Arbitrary Dimensions

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*Dedicated to Johann Walter*

## 1 A historical survey

It is well known that a function which is continuously differentiable on the unit circle  $S^1$  can be expressed as a uniformly convergent Fourier series. It is less known that a function which is continuously differentiable on the unit sphere  $S^2$  in  $\mathbb{R}^3$  can be expanded in terms of a uniformly convergent series of spherical harmonics, a so-called Laplace series<sup>1</sup> (Kellogg [20,p.259]). Both results can be traced back at least to Dirichlet (1829 and 1837, respectively), although the notion of uniform convergence was brought out a little later through the work of Gudermann (the teacher of Weierstrass), Seidel (a student of Dirichlet), Stokes and Weierstrass. The importance of such an expansion is due to the fact that the solution of the Dirichlet problem for the Laplace equation on the unit disc or unit ball can then be given in terms of a uniformly convergent series of elementary functions. The first edition of Heine's handbook of spherical harmonics which appeared in 1861 and reproduced Dirichlet's 1837 proof does not contain the notion of uniform convergence. The second edition of 1878 does have this notion [14,p.478f.]; at the same time it takes a critical attitude towards Dirichlet's proof [14,p.434]. We shall comment on this proof in Remark 1 at the end of §2.

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<sup>1</sup>The classical treatise of Courant–Hilbert [7,p.513] obtains this result as a special case of an expansion of a function in terms of eigenfunctions of a second-order elliptic operator [7,p.369] and is forced to assume that the function be in  $C^2(S^2)$ . - The three editions of MacRobert's book [24,p.131] contain the assertion that every function in  $C^0(S^2)$  has a convergent Laplace series. Experience from Fourier series renders this claim at once highly improbable and it is in fact false (see, for example, [2,p.211]).

After Cayley had introduced in 1848 spherical harmonics in arbitrary dimensions [14,p.463], Mehler (a student of Dirichlet), in 1866, was the first to attempt to give a general expansion theorem for a function on the unit sphere  $S^{D-1}$  in  $\mathbb{R}^D$  in terms of such functions<sup>2</sup>[27]. We shall refer to such an expansion also as a Laplace series. Mehler obtained the correct form of this expansion and noticed that the case of Fourier or ordinary Laplace series was typical of even or odd dimensions, respectively, but failed as he himself remarked to relate the properties of the function which was to be expanded to those required of a certain spherical means of this function (see eq.(15) below). (In principle this problem occurs already in three dimensions where it can, however, easily be solved. Nevertheless it is frequently disregarded, for example in the books of Hobson [16,p.344f.] and Jordan[18,p.296], or insufficiently treated [47,p.319].)

Ironically, a formula found by Reinhold Hoppe in 1845 [17] for an arbitrary derivative of the composition of two functions provides the information that Mehler lacked. We should like to show in our paper that, given Hoppe's formula, Mehler's arguments establish that every function in  $C^{[\frac{D-1}{2}]}(S^{D-1})$  can be expanded in a uniformly convergent Laplace series (this regularity assumption cannot be weakened without adding other conditions). What will be needed, more precisely, is a multi-dimensional extension of Hoppe's formula, but with the multi-index notation this is a straightforward generalisation (see eq.(34)of §4). Without this notation the extension is rather painful (see [44,p.88] for early such attempts; cf. also our Remark 6).

It was not until 1972 that the above-mentioned expansion theorem was proved by Ragozin ([36], together with [34] or [35]). He worked in the more general context of polynomial approximation on compact smooth manifolds. Although Ragozin's paper [36] is very elegant, a need for a more elementary and more explicit proof in the case of a sphere was felt, and such a proof was given by Roetman [39] in 1976. At an important stage, however, he has to invoke the Whitney extension theorem. We shall replace this with Hoppe's formula, thus giving for the first time a proof that, we believe, Mehler only slightly missed. An important insight one gains (by this and by Roetman's proof) is that the Riemann localisation principle which is familiar from Fourier series holds for Laplace series in even dimensions, but

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<sup>2</sup>In the second half of his paper Mehler performs with remarkable insight the limit  $D \rightarrow \infty$  in the uniform measure on  $S^{D-1}$  and in the Gegenbauer polynomials. (A century later, with better abstract background, this was taken up again in [42], apparently unaware of Mehler's achievements.) In this context he obtained what is known as Mehler's formula for generating functions of Hermite polynomials (cf.,e.g.,[8,p.181], [25,p.252]). In [46] Watson gives a proof of this formula which was communicated to him by Hardy and he ends his paper with the remark "Prof. Hardy tells me that he has not seen his proof in print, though the inevitability of the successive steps makes him think that it is unlikely to be new". Hardy's argument is in fact exactly that given by Mehler himself [27,p.173f.].

fails in odd dimensions (Remark 4). (In connexion with the result Roetman has on pointwise convergence, cf. [33].)

To continue with the history of our subject, the only paper in the last century after Mehler's to consider a Laplace series in arbitrary dimensions appears to be one by M.J.M. Hill in 1883 [15]. Unfortunately, he proceeded in an entirely formal way and apparently without any knowledge of previous work on the subject. The paper contains the following disarming confession [15,p.291]. "The proof here set forth is similar to one of those given of Laplace's expansion of a function of two variables. It is subject to similar criticisms."<sup>3</sup>

It is a strange coincidence, but really only a coincidence, that the case  $D = 4$  was treated independently by Caccioppoli [4] and Koschmieder [22] at a time when four-dimensional spherical harmonics were first used in physics. In 1935 it was observed by Fock [11] that the "accidental" degeneracy of the eigenvalues of the quantum mechanical Kepler problem in three dimensions is due to the invariance of the corresponding Schrödinger equation under  $O(4)$ , the group of orthogonal four-by-four matrices. As representations of this group spherical harmonics on the unit sphere  $S^3$  in  $\mathbb{R}^4$  therefore occur naturally in this problem. More generally, the quantum mechanical Kepler problem in  $D - 1$  dimensions leads to the group  $O(D)$  and thus to spherical harmonics on the unit sphere  $S^{D-1}$  in  $\mathbb{R}^D$  [1].

In [23] Koschmieder considered arbitrary dimensions, but set aside the central problem of connecting the smoothness properties of the function to be expanded with those of its generalised circle of latitude means.

In all the results that we mentioned so far the fact that the spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator does not play any role. This is different in a theorem of Vekua's of 1943 [43]. Using sufficiently high powers of the Laplace-Beltrami operator to compensate for the growth of the spherical harmonics with increasing degree, he shows that every function  $f \in C^k(S^{D-1})$  with  $k := 2\lfloor \frac{D+4}{4} \rfloor$  has a uniformly absolutely convergent Laplace series. (In view of other purposes he assumes analyticity of  $f$ , but from his proof it is obvious that this is all he needs.) Independently and slightly more effectively, the same idea was pursued by Rellich [38] who arrived at the same result with  $d := 2\lfloor \frac{D+3}{4} \rfloor$  replacing  $k$ . If one is willing to employ fractional powers of the Laplace-Beltrami operator, the numbers  $k$  and  $d$  can immediately be improved to  $\lfloor \frac{D+2}{2} \rfloor$  or  $\lfloor \frac{D+1}{2} \rfloor$ , respectively. A substantially revised version of Rellich's lecture notes appeared in [19], but his chapter on spherical harmonics was left out in this edition. As a consequence, his quick and transparent proof remained largely unknown. In the hope of a wider dissemination we reproduce it in §4. (Rellich's result and proof were recently rediscovered in [32].)

While it is sad, but not surprising that Vekua's result was overlooked in the west until the English edition of his book appeared in 1967, it is somewhat peculiar that it is not mentioned in [29,30]. In these books Mikhlin uses an inequality of A.A. Markov to estimate the growth of the spherical harmonics, but the resulting bound is less precise than the one the addition theorem immediately supplies (compare

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<sup>3</sup>As a curiosity we mention that M.J.M. Hill has an entry in [28] because he is, in an essential point, confused with the great American astronomer George William Hill. The mistake is faithfully copied in [12].

Mikhlin's inequality (6) in [30,p.114] with our inequality (33) in §4). Thus he has to put up with the hypothesis  $f \in C^{D-1}(S^{D-1})$  without gaining in simplicity of proof. ([29,p.272] has a smoothness assumption that is even stronger. When the Sobolev embedding theorem is called in to control the spherical harmonics, as in [41,p.431], the smoothness requirement increases further.)

S.N. Bernstein's well-known result of 1914 that a periodic function which is Hölder continuous with exponent larger than  $\frac{1}{2}$  has a uniformly absolutely convergent Fourier series [51,p.240f.] was generalised to Laplace series in any dimension by V.L. Shapiro in 1961 [40]. Ragozin [37] treats this problem in the case where a compact connected Lie group replaces the sphere  $S^{D-1}$ .

It is clear that each of the finer questions in the theory of Fourier series has its counterpart in Laplace series. We restrict ourselves to mentioning the following items. Uniqueness of the expansion can be proved under very weak conditions [26]. The possibility of summing a Laplace series according to the Cesàro method was extensively studied and [2] is a good source of references (Abel–Poisson summability is easier [31,p.42f.]). Also the famous Wilbraham–Gibbs phenomenon ([21,Ch.17], [51,p.61f.]) for the Fourier series of a function with a jump discontinuity persists for Laplace series, but the ensuing analysis is naturally much more intricate than in the case  $D = 2$  [48]. A great many further references can be found in the survey article [50].

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## 2 Prerequisites

In this section we collect and describe without proof those properties of the spherical harmonics that will be needed in §3, and partly in §4. They were derived by Claus Müller [31] in a particularly simple and elegant way. His approach is based on two fundamental observations, firstly, that it is frequently advantageous to avoid representing points of  $S^{D-1}$  by  $D - 1$  angular variables, and secondly, that one should delay representing spherical harmonics explicitly as long as possible.

$D$  will always be a natural number greater than one. For  $D \geq 3$  the integration of a function  $g \in C^0(S^{D-1})$  can be greatly facilitated by representing a point  $\vartheta \in S^{D-1}$  as

$$\vartheta = \zeta(t, \mu) := t \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sqrt{1-t^2} \begin{pmatrix} 0 \\ \mu \end{pmatrix} \quad (1)$$

where  $t \in [-1, 1]$  and  $\mu \in S^{D-2}$ . An elementary but troublesome calculation yields the fundamental formula

$$\int_{|\vartheta|=1} g(\vartheta) d\sigma_D(\vartheta) = \int_{-1}^1 (1-t^2)^{\frac{D-3}{2}} \left( \int_{|\mu|=1} g(\zeta(t, \mu)) d\sigma_{D-1}(\mu) \right) dt. \tag{2}$$

In particular,

$$\sigma_D := \int_{|\vartheta|=1} d\sigma_D(\vartheta) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \tag{3}$$

is the area of the unit sphere.

Let  $D \geq 2$  and  $l \in \mathbb{N}_0$ .

- a) A spherical harmonic of degree  $l$  in  $D$  dimensions is by definition the restriction to the unit sphere  $S^{D-1}$  of a polynomial in  $\mathbb{R}^D$  which is homogeneous of degree  $l$  and satisfies the Laplace equation  $\Delta u = 0$ . The spherical harmonics of a given degree  $l$  in  $D$  dimensions form a finite-dimensional vector space. Its dimension  $N(D, l)$  is given by

$$N(D, l) = \frac{(2l + D - 2)(l + D - 3)!}{(D - 2)!!} \tag{4}$$

if  $l \geq 1$  and 1 if  $l = 0$  ([14,p.462], [31,p.4]). Thus we have

$$N(2, l) = 2 \quad (l \in \mathbb{N}); N(3, l) = 2l + 1, N(4, l) = (l + 1)^2 \quad (l \in \mathbb{N}_0). \tag{5}$$

- b) Subject to a normalisation, there is exactly one harmonic polynomial  $L_l(D, \cdot)$ , homogeneous of degree  $l$ , which is invariant under rotations around a fixed point in  $S^{D-1}$  [31,p.8]. Using the parametrization (1),  $L_l(D, \zeta(t, \mu))$  is independent of  $\mu$  and a polynomial of degree  $l$  in  $t \in [-1, 1]$ . It is called the Legendre polynomial of degree  $l$  in dimension  $D$  and denoted by  $P_l(D, t)$ , the normalisation being such that

$$P_l(D, 1) = 1. \tag{6}$$

By homogeneity it follows that

$$P_l(D, -1) = (-1)^l. \tag{7}$$

For  $D = 2$  a simple argument reveals [31,p.11]

$$P_l(2, t) = \cos(l \arccos t) \quad (t \in [-1, 1]) \tag{8}$$

where arccos is the inverse of the restriction of the cosine to  $[0, \pi]$ . (8) is the well-known Chebyshev polynomial.  $P_l(3, t)$  is the ordinary Legendre polynomial, usually denoted by  $P_l(t)$ . For  $D \geq 4$ ,  $P_l(D, t)$  is, up to a factor, the Gegenbauer polynomial.

- c) Let  $(S_{lj}(D, \cdot))(j \in \{1, \dots, N(D, l)\})$  be an orthonormal basis of spherical harmonics of degree  $l$  in  $D$  dimensions. Then the following addition theorem holds,

$$\sum_{j=1}^{N(D,l)} S_{lj}(D, \xi) \overline{S_{lj}(D, \eta)} = \frac{N(D, l)}{\sigma_D} P_l(D, \xi \eta) \quad (\xi, \eta \in S^{D-1}) \quad (9)$$

[31,p.10]. Applying the Cauchy-Schwarz inequality to (9) and using (6) we see

$$|P_l(D, t)| \leq 1 \quad (t \in [-1, 1]) \quad (10)$$

[31,p.15]. As a second consequence of (9) we mention that the Legendre polynomials are orthogonal polynomials in the sense

$$\frac{\sigma_{D-1}}{\sigma_D} \int_{-1}^1 P_l(D, t) P_m(D, t) (1-t^2)^{\frac{D-3}{2}} dt = \frac{1}{N(D, l)} \delta_{lm} \quad (m \in \mathbb{N}_0) \quad (11)$$

([14,p.458f.],[31,p.15]).

- d) Let  $f \in C^0(S^{D-1})$ ,  $l \in \mathbb{N}_0$  and  $n \in \mathbb{N}$ . Again, if  $(S_{lj}(D, \cdot))$  is an orthonormal basis of spherical harmonics of degree  $l$  in  $D$  dimensions and

$$c_{lj}(f) := \langle f, S_{lj}(D, \cdot) \rangle := \int_{|\eta|=1} f(\eta) \overline{S_{lj}(D, \eta)} d\omega_D(\eta) \quad (12)$$

( $j \in \{1, \dots, N(D, l)\}$ ),

we wish to show that the  $n$ -th partial sum

$$s_n := \sum_{l=0}^n \sum_{j=1}^{N(D,l)} c_{lj}(f) S_{lj}(D, \xi) \quad (13)$$

tends to  $f(\xi)$  as  $n \rightarrow \infty$ , uniformly in  $\xi \in S^{D-1}$ . By virtue of the addition theorem (9), (13) can be represented as

$$s_n = \sum_{l=0}^n \frac{N(D, l)}{\sigma_D} \int_{|\eta|=1} P_l(D, A^T \xi \cdot A^T \eta) f(\eta) d\omega_D(\eta) \quad (14)$$

for all  $A \in O(D)$ , the group of  $D$ -dimensional orthogonal matrices ( $T$  denoting the transpose).

Let  $D \geq 3$  and  $\xi \in S^{D-1}$ . Then there is an  $A \in O(D)$  such that  $A^T \xi = (1, 0, \dots, 0)^T$ . Taking advantage of the parametrization (1) and employing (2), we can write the integral in (14) as

$$\begin{aligned}
 & |\det A| \int_{|\vartheta|=1} P_l(D, A^T \xi \cdot \vartheta) f(A\vartheta) d\omega_D(\vartheta) = \\
 & = \int_{-1}^1 P_l(D, t) (1 - t^2)^{\frac{D-3}{2}} \left( \int_{|\mu|=1} f(A\zeta(t, \mu)) d\sigma_{D-1}(\mu) \right) dt.
 \end{aligned}$$

It is therefore convenient to introduce the function

$$F_A(t) := \frac{1}{\sigma_{D-1}} \int_{|\mu|=1} f(A\zeta(t, \mu)) d\sigma_{D-1}(\mu) \quad (t \in [-1, 1]). \tag{15}$$

Note that

$$F_A(1) = f(\xi). \tag{16}$$

Taking the point  $\xi$  as the north pole of the sphere,  $F_A(t)$  is, for  $D = 3$ , the average of  $f$  over the parallel circle whose latitude is determined by  $t$ .

Thus we can bring the  $n$ -th partial sum (13) into the final form

$$s_n = \frac{\sigma_{D-1}}{\sigma_D} \int_{-1}^1 K_n(D, t) F_A(t) (1 - t^2)^{\frac{D-3}{2}} dt \tag{17}$$

where

$$K_n(D, t) := \sum_{l=0}^n N(D, l) P_l(D, t) \quad (t \in [-1, 1]). \tag{18}$$

With a suitable interpretation of  $F_A$ , relationship (17) remains valid if  $D = 2$ . In view of (5) and (8), (18) can immediately be evaluated in this case, and the result is, as is to be expected, the Dirichlet kernel,

$$K_n(2, \cos \Theta) = \begin{cases} \frac{\sin[(2n+1)\Theta/2]}{\sin(\Theta/2)} & \text{if } \Theta \in (0, \pi) \\ 2n + 1 & \text{if } \Theta \in \{0, \pi\}. \end{cases} \tag{19}$$

For  $D \geq 3$  a calculation that is a great deal more involved gives

$$K_n(D, t) = \frac{1}{(D-2)!} \left[ \frac{(n+D-3)!}{n!} P'_n(D, t) + \frac{(n+D-2)!}{(n+1)!} P'_{n+1}(D, t) \right] \tag{20}$$

( $t \in [-1, 1]$ )

where the dash indicates differentiation with respect to  $t$  [31,p.36f.]. It is readily verified that (20) continues to hold for  $D = 2$ . For  $D = 3$  relationship (20) is sufficiently simple to commence the convergence proof of (13) at once. This was first done in 1874 independently by Darboux and Dini ([14,p.435], [45,p.717]).

For general  $D$  we observe that the formula

$$P_m(D, t) = \frac{\Gamma(\frac{D}{2} - j)}{2^j \Gamma(\frac{D}{2})} \cdot \frac{N(D - 2j, m + j)}{N(D, m)} \left(\frac{d}{dt}\right)^j P_{m+j}(D - 2j, t) \quad (21)$$

( $t \in [-1, 1]$ )

( $m, j \in \mathbb{N}; 0 \leq j < \frac{D}{2}$ ) ([14,p.252; Vol.2,p.380], [31,p.25]) enables us to express the Legendre polynomials in even (odd) dimensions in terms of the more familiar two (three)-dimensional Legendre polynomials (albeit of higher degree). The main ingredient in the proof of (21) is the orthogonality relation (11).

**Remark 1.** The idea of Dirichlet's original convergence proof was to reduce the case  $D = 3$  to  $D = 2$ . Using his integral representation for the Legendre polynomials that was later put into a more symmetric form by Mehler, he obtained a formula for the  $n$ -th partial sum the crucial term of which is, in our notation,

$$\int_0^\pi g_A(\Theta) \frac{d}{d\Theta} K_n(2, \cos \Theta) d\Theta$$

where

$$g_A(\Theta) := \cos \frac{\Theta}{2} \int_0^\Theta \frac{F_A(\cos \vartheta) \sin \vartheta}{\sqrt{2(\cos \vartheta - \cos \Theta)}} d\vartheta \\ - \sin \frac{\Theta}{2} \int_\Theta^\pi \frac{F_A(\cos \vartheta) \sin \vartheta}{\sqrt{2(\cos \Theta - \cos \vartheta)}} d\vartheta \quad (\Theta \in [0, \pi])$$

[9,§3]. Before integrating by parts one has of course to ascertain that  $g'_A$  exists and is integrable. Dirichlet confines himself to showing that  $g'_A(0)$  exists. More generally, one could note that the first integral, for example, equals

$$\sqrt{2(1 - \cos \Theta)} F_A(1) - \int_0^\Theta \sqrt{2(\cos \vartheta - \cos \Theta)} F'_A(\cos \vartheta) \sin \vartheta d\vartheta.$$

To make sure that the Leibniz rule is applicable, write

$$\int_0^\Theta \left( \int_0^\varphi \frac{F'_A(\cos \vartheta) \sin \vartheta \sin \varphi}{\sqrt{2(\cos \vartheta - \cos \varphi)}} d\vartheta \right) d\varphi \\ = \int_0^\Theta \left( \int_\vartheta^\Theta \frac{F'_A(\cos \vartheta) \sin \vartheta \sin \varphi}{\sqrt{2(\cos \vartheta - \cos \varphi)}} d\varphi \right) d\vartheta \\ = \int_0^\Theta \sqrt{2(\cos \vartheta - \cos \Theta)} F'_A(\cos \vartheta) \sin \vartheta d\vartheta,$$

using Dirichlet's integral formula. In this way it would be possible to salvage Dirichlet's argument if  $f \in C^1(S^2)$ . The critical remarks in [14,p.434], [45,p.716] are therefore not quite justified.



### 3 Expansion in a uniformly convergent Laplace series

Our object here is to prove the following result that was slightly missed by Mehler [27] and that was first established by Ragozin [36].

**Theorem 1.** Let  $D \geq 3$ ,  $f \in C^{[\frac{D-1}{2}]}(S^{D-1})$  and  $c_{l_j}(f)$  the Fourier coefficients of  $f$  with respect to an orthonormal basis  $(S_{l_j}(D, \cdot))$  of spherical harmonics of degree  $l$  in dimension  $D$  ( $j \in \{1, \dots, N(D, l)\}, l \in \mathbb{N}_0$ ). Then

$$\sum_{l=0}^{\infty} \sum_{j=1}^{N(D,l)} c_{l_j}(f) S_{l_j}(D, \xi) = f(\xi) \tag{22}$$

uniformly with respect to  $\xi \in S^{D-1}$ .

*Proof.* We recall that  $s_n$ , the  $n$ -th partial sum of (22), can be brought into the form

$$s_n = \frac{\sigma_{D-1}}{\sigma_D} \int_{-1}^1 K_n(D, t) G_A(t) dt$$

where

$$G_A(t) := (1 - t^2)^{\frac{D-3}{2}} F_A(t)$$

and  $F_A(t)$  is the generalised circle of latitude means of  $f$ , introduced in (15). For the Dirichlet kernel  $K_n(D, t)$  we have the convenient expression (20). Moreover,

$$\frac{\sigma_{D-1}}{\sigma_D} = \frac{\Gamma\left(\frac{D}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{D-1}{2}\right)} \tag{23}$$

in view of (3). To proceed further we have to distinguish two cases.

- a) **D odd.** We write  $D = 2k + 3$  for some  $k \in \mathbb{N}_0$ . Formula (21), together with (4), then yields

$$\frac{(m + D - 3)!}{(D - 2)!m!} P'_m(D, t) = \frac{\sqrt{\pi}}{(2k + 1)2^k \Gamma(k + \frac{1}{2})} P_{m+k}^{(k+1)}(3, t).$$

Since

$$\frac{\sigma_{D-1}}{\sigma_D} = \frac{2k + 1}{2\sqrt{\pi}k!} \Gamma\left(k + \frac{1}{2}\right)$$

by (23), we find

$$s_n = \frac{1}{2^{k+1}k!} \int_{-1}^1 \left[ P_{n+k}^{(k+1)}(3, t) + P_{n+1+k}^{(k+1)}(3, t) \right] G_A(t) dt. \tag{24}$$

Under our assumptions on  $f$ ,  $G_A$  is certainly  $(k + 1)$ -times continuously differentiable on the open interval  $(-1, 1)$ . Let  $t \in (-1, 1)$  and  $l \in \{0, \dots, k + 1\}$ . By

virtue of Lemma 1a) and Lemma 2 of the Appendix we have the existence of polynomials  $P_{j,k}$  of degree  $\leq j$  and functions  $C_{j,A}$  ( $j \in \{0, \dots, l\}$ ), continuous on  $[-1, 1]$ , such that

$$\begin{aligned} G_A^{(l)}(t) &= \sum_{j=0}^l \binom{l}{j} F_A^{(l-j)}(t) \left(\frac{d}{dt}\right)^j (1-t^2)^k \\ &= \sum_{j=0}^l \binom{l}{j} (1-t^2)^{k-l} (1-t^2)^{l-j} F_A^{(l-j)}(t) P_{j,k}(t) \\ &= \sum_{j=0}^l \binom{l}{j} (1-t^2)^{k-l+\frac{1}{2}} C_{l-j,A}(t) P_{j,k}(t). \end{aligned}$$

For  $l \leq k$   $G_A^{(l)}$  therefore admits of a continuous extension to the whole interval  $[-1, 1]$ . In addition,  $G_A^{(l)}(\pm 1) = 0$  for  $l \leq k-1$  and, by Lemma 1a,b),

$$G_A^{(k)}(1) = (-1)^k 2^k k! F_A(1). \quad (25)$$

Furthermore,

$$\left| G_A^{(k+1)}(t) \right| \leq K(1-t^2)^{-\frac{1}{2}} \quad (t \in (-1, 1)) \quad (26)$$

with a number  $K > 0$  which is independent of  $A$  (see Lemma 2). So we can integrate  $(k+1)$ -times by parts in (24) to obtain

$$\begin{aligned} s_n &= \frac{(-1)^k}{2^{k+1}k!} \left\{ [P_{n+k}(3, t) + P_{n+1+k}(3, t)] G_A^{(k)}(t) \Big|_{-1}^1 \right. \\ &\quad \left. - \int_{-1}^1 [P_{n+k}(3, t) + P_{n+1+k}(3, t)] G_A^{(k+1)}(t) dt \right\} \\ &= f(\xi) - \frac{(-1)^k}{2^{k+1}k!} \int_{-1}^1 [P_{n+k}(3, t) + P_{n+1+k}(3, t)] G_A^{(k+1)}(t) dt. \end{aligned} \quad (27)$$

In the last line we have used (6), (7) and (16), (25). The assertion of the theorem therefore follows once we can show

$$\int_{-1}^1 (1-t^2)^{-\frac{1}{2}} |P_m(3, t)| dt \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

To see this, we split the integral as follows,

$$\int_{-1}^1 = \int_{-1}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^1. \quad (28)$$

Given  $\epsilon > 0$ , there is a  $\delta \in (0, 1)$  such that the first and the last integral are in modulus less than  $\frac{\epsilon}{3}$ , irrespective of  $m \in \mathbb{N}$ . This is an immediate consequence of (10). From (11) and (5) we see

$$\int_{-1}^1 |P_m(3, t)|^2 dt = \frac{2}{2m+1} \quad (m \in \mathbb{N}).$$

Hence

$$\int_{-1}^1 |P_m(3, t)| dt \leq \frac{2}{\sqrt{2m + 1}} \quad (m \in \mathbb{N}) \tag{29}$$

by the Cauchy–Schwarz inequality. An appropriate choice of  $m$  therefore makes the modulus of the middle term in (28) smaller than  $\frac{\epsilon}{3}$ .

Remark 2. If  $D = 3$  and so  $k = 0$ , one can immediately proceed from (24) to (27), so that the above proof becomes very short and reduces to that given in Kellogg’s book [20,p.259]. Kellogg, however, does not use the parametrization (1) that makes the estimate (26) with  $k = 0$  particularly transparent. It appears that Kellogg was the first to replace the more subtle pointwise bounds for the Legendre polynomials which appear in the earlier proofs by the simple relationship (29). ([47,p.319] replaces (26), again for  $k = 0$ , with the requirement that  $F'_A$  be bounded, which entails a stronger differentiability condition on  $f$ .) However, if one does use a pointwise bound on  $P_m(3, t)$ , one can relax our regularity assumption to  $f \in C^{[\frac{D-2}{2}]}(S^{D-1})$ , provided the modulus of continuity of  $f([\frac{D-2}{2}]), \omega(f([\frac{D-2}{2}]), h)$ , is required to be  $o(h^{\frac{1}{2}})$  as  $h \rightarrow 0$ . It is in this form that Theorem 1 is given for odd dimensions in [36]. For  $D = 3$  this refinement is due to Gronwall [13].

Remark 3. Assume again  $D = 3$  and so  $k = 0$ . There are many proofs which try to avoid integration by parts in (24), but it is not very clear what assumptions have to be imposed on  $f$  itself to render this possible. Such proofs can be found, for example, in the books of Hobson [16,p.344f.] and Jordan [18,p.294ff.]. Jordan’s argument, unaltered throughout the three editions (which extend over a period of 76 years), was justly criticized in [5] but the incriminated points disappear when Jordan’s unfortunate order of integration is reversed.

Remark 4. It was shown by Darboux in 1878 that the integrable function  $f(\Theta, \phi) := (1 - \cos \Theta)^{-\frac{4}{5}}$  ( $\Theta \in (0, \pi], \phi \in [0, 2\pi)$ ) with a singularity only at the north pole has a Laplace series which diverges everywhere on  $S^2$  (cf.[10]). Hence there is no Riemann localisation principle [51,p.52f.] in odd dimensions.

We pass on to the second case the prototype of which is the expansion in a Fourier series.

b) **D even.** Let  $D = 2k + 2$  with a suitable number  $k \in \mathbb{N}$ . In this case we find

$$\frac{(m + D - 3)!}{(D - 2)!m!} P'_m(D, t) = \frac{1}{2^k k!(m + k)} P_{m+k}^{(k+1)}(2, t).$$

Using the doubling formula for the Gamma function [25,p.3], we write (23) as

$$\frac{\sigma_{D-1}}{\sigma_D} = \frac{k!}{\sqrt{\pi}\Gamma(k + \frac{1}{2})} = \frac{2^k k!}{\pi(2k - 1)\dots 3 \cdot 1}$$

and get

$$s_n = \frac{1}{\pi(2k-1)\dots 3 \cdot 1} \int_{-1}^1 \left[ \frac{1}{n+k} P_{n+k}^{(k+1)}(2, t) + \frac{1}{n+1+k} P_{n+1+k}^{(k+1)}(2, t) \right] G_A(t) dt \quad (30)$$

for the partial sum  $s_n$ . For  $l \in \{0, \dots, k-1\}$   $G_A^{(l)}$  permits a continuous extension to the whole interval  $[-1, 1]$  with  $G_A^{(l)}(\pm 1) = 0$ . This follows again from Lemma 1a). For  $t \in (-1, 1)$  we conclude from Lemma 1a,c)

$$\begin{aligned} G_A^{(k)}(t) &= \left(\frac{d}{dt}\right)^k (1-t^2)^{k-\frac{1}{2}} F_A(t) \\ &= F_A(t) \left(\frac{d}{dt}\right)^k (1-t^2)^{k-\frac{1}{2}} + \sum_{j=1}^k \binom{k}{j} F_A^{(j)}(t) \left(\frac{d}{dt}\right)^{k-j} (1-t^2)^{k-\frac{1}{2}} \\ &= (-1)^k (2k-1)\dots 3 \cdot 1 \cdot (1-t^2)^{-\frac{1}{2}} \left\{ \left[ t^k + (1-t^2)P_{k-1}(t) \right] F_A(t) \right. \\ &\quad \left. + \sum_{j=1}^k (1-t^2)^j F_A^{(j)}(t) P_{k-j,k}(t) \right\} \end{aligned}$$

where the  $P_{l,k}$  are polynomials of degree  $\leq l$  ( $l \in \{0, \dots, k-1\}$ ) and  $P_{k-1}$  is a polynomial of degree  $\leq k-1$ . We write the expression in curly brackets as

$$\{\dots\} = H_A(t) + (1-t^2)^{\frac{1}{2}} R_A(t)$$

where

$$H_A(t) := t^k F_A(t),$$

$$R_A(t) := (1-t^2)^{\frac{1}{2}} F_A(t) P_{k-1}(t) + \sum_{j=1}^k (1-t^2)^{j-\frac{1}{2}} F_A^{(j)}(t) P_{k-j,k}(t).$$

Integrating by parts  $k$  times, we can transform (30) into

$$\begin{aligned} s_n &= -\frac{1}{\pi} \int_{-1}^1 \left[ \frac{1}{n+k} P'_{n+k}(2, t) + \frac{1}{n+1+k} P'_{n+1+k}(2, t) \right] \\ &\quad \cdot \left[ H_A(t) + (1-t^2)^{\frac{1}{2}} R_A(t) \right] (1-t^2)^{-\frac{1}{2}} dt \\ &= \frac{1}{\pi} \int_0^\pi K_{n+k}(2, \cos \Theta) [H_A(\cos \Theta) + \sin \Theta R_A(\cos \Theta)] d\Theta. \end{aligned}$$

In the last line we have used (20). On account of (19) we finally arrive at

$$\begin{aligned} s_n &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin(2n+2k+1)s}{\sin s} H_A(\cos 2s) ds \\ &\quad + \frac{4}{\pi} \int_0^{\frac{\pi}{2}} R_A(\cos 2s) \sin(2n+2k+1)s ds. \end{aligned} \quad (31)$$

By Lemma 2  $R_A$  is a bounded function with a bound that does not depend on  $A$ . As a consequence of the Riemann–Lebesgue Lemma [21,p.260f.] the second term in (31) tends to zero as  $n \rightarrow \infty$ , uniformly in  $\xi$ . We claim that there is a number  $M > 0$ , independent of  $A$ , such that

$$|H_A(\cos 2s)| \leq M, \quad |H_A(\cos 2s) - H_A(1)| \leq Ms \quad (s \in [0, \frac{\pi}{2}]). \quad (32)$$

Accepting this for a moment, a standard result in the theory of Fourier series tells us that the first term in (31) tends to

$$H_A(1) = F_A(1) = f(\xi)$$

as  $n \rightarrow \infty$ , uniformly with respect to  $\xi$ .

This leaves us with a proof of (32). The first estimate is obvious. As to the second, it suffices to establish it for  $F_A$ . Let  $s \in (0, \frac{\pi}{2}]$ . Then there is a  $\sigma \in (0, s)$  such that

$$\begin{aligned} F_A(\cos 2s) - F_A(1) &= 2(1 - \cos 2s)F'_A(\cos 2\sigma) \sin 2\sigma \\ &= 4\sin^2 s C_{1,A}(\cos 2\sigma). \end{aligned}$$

By dint of Lemma 2 the continuous function  $C_{1,A}$  can be estimated independently of  $A$ .

## 4 Expansion in a uniformly absolutely convergent Laplace series

The prerequisites that are required in this section are those mentioned under a) and b) of §2, but the more complicated results (20) and (21) listed under d) are not needed any more. As far as c) is concerned, we note that the addition theorem (9), combined with (4) and (10), implies the existence of a number  $C_D > 0$  such that

$$\sum_{j=1}^{N(D,l)} |S_{l_j}(D, \xi)|^2 \leq C_D l^{D-2} \quad (\xi \in S^{D-1}, l \in \mathbb{N}_0). \quad (33)$$

Another property of the spherical harmonics  $S_{l_j}(D, \cdot)$  that did not come into play previously is that they are eigenfunctions of the negative Laplace–Beltrami operator  $-\Delta_S$ , a symmetric operator defined say on the subspace  $C^2(S^{D-1})$  of the Hilbert space  $L^2(S^{D-1})$ , the eigenvalues being

$$\lambda_l := l(l + D - 2)$$

[31,p.39].

The following result and its simple proof are due to Rellich [38,p.58ff.].

**Theorem 2.** Let  $D \geq 2, d := 2 \lfloor \frac{D+3}{4} \rfloor, f \in C^d(S^{D-1})$  and  $c_{l_j}(f)$  the Fourier coefficients of  $f$  with respect to an orthonormal basis  $(S_{l_j}(D, \cdot))$  of spherical harmonics of degree  $l$  in dimension  $D$  ( $j \in \{1, \dots, N(D, l)\}, l \in \mathbb{N}_0$ ). Then

$$\sum_{l=0}^{\infty} \sum_{j=1}^{N(D,l)} c_{l_j}(f) S_{l_j}(D, \xi) = f(\xi),$$

the series being uniformly absolutely convergent with respect to  $\xi \in S^{D-1}$ .

*Proof.* Let  $m, n, p \in \mathbb{N}, m < n$ , and  $\xi \in S^{D-1}$ . Then

$$\sum_{l=m}^n \sum_{j=1}^{N(D,L)} 2 \left| \lambda_l^p c_{l_j}(f) \lambda_l^{-p} S_{l_j}(D, \xi) \right| \leq \sum_{l=m}^n \lambda_l^{-2p} \sum_{j=1}^{N(D,l)} \left| S_{l_j}(D, \xi) \right|^2 + \sum_{l=m}^n \sum_{j=1}^{N(D,l)} \left| \lambda_l^p c_{l_j}(f) \right|^2.$$

In view of (33), the first term can be dominated by

$$\text{const.} \sum_{l=m}^n \frac{1}{l^{4p+2-D}}$$

and thus be made arbitrarily small if  $p > \frac{(D-1)}{4}$ . The Fourier coefficients of the function  $g := (-\Delta_s)^p f$  can be written as (see (12))

$$c_{l_j}(g) = \langle (-\Delta_s)^p f, S_{l_j}(D, \cdot) \rangle = \langle f, (-\Delta_s)^p S_{l_j}(D, \cdot) \rangle = \lambda_l^p c_{l_j}(f).$$

Since  $g \in L^2(S^{D-1})$ ,

$$\sum_{l=0}^{\infty} \sum_{j=1}^{N(D,l)} \left| c_{l_j}(g) \right|^2 < \infty$$

by Bessel's inequality. This proves that the Laplace series is uniformly absolutely convergent. To show that it represents  $f$  it is probably quickest to invoke Abel summability,

$$\lim_{r \uparrow 1} \sum_{l=0}^{\infty} \sum_{j=1}^{N(D,l)} r^l c_{l_j}(f) S_{l_j}(D, \xi) = f(\xi),$$

which holds for every  $f \in C^0(S^{D-1})$  and is very easy to prove [31,p.42f.].

Rellich himself argues differently at this point. He demonstrates first that the spherical harmonics form a dense set in  $C^0(S^{D-1})$  with respect to the supremum norm and then uses the minimality property of the Fourier coefficients in a mean-square approximation by an orthonormal system of functions to show that the Laplace series of every  $f \in L^2(S^{D-1})$  converges to  $f$  in the norm of this space<sup>4</sup>.

**Remark 5.** Since the eigenfunctions  $(S_{l_j}(D, \cdot))$  form a complete orthonormal system in  $L^2(S^{D-1})$ ,  $-\Delta_S$  is an essentially self-adjoint operator. Denoting its closure by  $B$ ,  $B^\alpha$  exists for every  $\alpha > 0$  and its domain of definition is the Sobolev space  $W^{2\alpha,2}(S^{D-1})$ . If one relies on this, the exponent  $p$  in the proof of Theorem 2 need no longer be an integer, which improves the number  $d$  to  $\left[ \frac{D+1}{2} \right]$ .

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<sup>4</sup>In this context the following theorem of Bonami-Clerc is remarkable. Given  $1 \leq p < \infty, p \neq 2$ , there exists an  $f \in L^p(S^{D-1})$  such that the partial sums of the Laplace series for  $f$  do not converge in the  $L^p$ -norm [3,p.248].

## 5 Appendix

We begin by listing three results which were needed in §3 and which admit of a very simple proof by induction. (Note that the null function is, by definition, a polynomial of degree  $-\infty$ .)

**Lemma 1.** Let  $m \in \mathbb{N}_0$ .

a) Given  $r \in \mathbb{R}$ , there is a polynomial  $p_{m,r}$  of degree  $\leq m$  such that

$$\left(\frac{d}{dt}\right)^m (1-t^2)^r = (1-t^2)^{r-m} p_{m,r}(t) \quad (t \in (-1, 1)).$$

b)

$$\left(\frac{d}{dt}\right)^m (1-t^2)^m \Big|_{t=1} = (-1)^m 2^m m!$$

c) There is a polynomial  $p_{m-1}$  of degree  $\leq m-1$  such that

$$\left(\frac{d}{dt}\right)^m (1-t^2)^{m-\frac{1}{2}} = (-1)^m (2m-1)\dots 3 \cdot 1 \cdot (1-t^2)^{-\frac{1}{2}} [t^m + (1-t^2)p_{m-1}(t)]$$

for  $t \in (-1, 1)$ .

Next let  $m \in \mathbb{N}$  and  $f \in C^m(S^{D-1})$ . We wish to differentiate the function defined by (15)  $m$  times and estimate its derivatives. This requires an  $m$ -fold application of the chain rule. To this end, let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be a multi-index, i.e. an  $n$ -dimensional vector with components in  $\mathbb{N}_0$ , and  $|\alpha| := \sum_{j=1}^n \alpha_j$  its length. As is usual, we put

$$x^\alpha := \prod_{j=1}^n x_j^{\alpha_j} \quad (x = (x_1, \dots, x_n) \in \mathbb{R}^n), \quad \text{and} \quad \alpha! := \prod_{j=1}^n \alpha_j!$$

$$\partial^\alpha := \prod_{j=1}^n \partial_j^{\alpha_j} \quad \left(\partial_j = \frac{\partial}{\partial x_j}\right)$$

If  $\alpha$  and  $\beta$  are two multi-indices, it is convenient to write  $\beta \leq \alpha$  for  $\beta_j \leq \alpha_j (j \in \{1, \dots, n\})$  and define

$$\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

Let  $I \subset \mathbb{R}$  be an interval,  $h : I \rightarrow \mathbb{R}^n$   $m$  times continuously differentiable and  $g$   $m$  times continuously differentiable on the range of  $h$ . Then

$$(g \circ h)^{(m)}(t) = \sum_{|\alpha| \leq m} \frac{(\partial^\alpha g)(h(t))}{\alpha!} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} [-h(t)]^{\alpha-\beta} \left(\frac{d}{dt}\right)^m [h(t)]^\beta \quad (34)$$

for  $t \in I$ .

**Remark 6.** This formula may be called Hoppe's formula after Reinhold Hoppe who found it (for the decisive case  $n = 1$ ) in 1845 [17]. (34) could be verified by induction on  $m$ , but a more instructive proof is to compare  $\frac{1}{m!}(g \circ h)^{(m)}(t)$ , the coefficient of  $s^m$  in the formal Taylor series for  $(g \circ h)(t+s)$ , with that of the  $n$ -dimensional formal Taylor series for  $g(h(t+s))$ , using the binomial theorem for  $[h(t+s) - h(t)]^\alpha$ . Applying the multinomial theorem to the Taylor expansion of  $h(s+t) - h(t)$  in the case  $n = 1$ , one would arrive at a formula for  $(g \circ h)^{(m)}(t)$  which is named after Faà di Bruno and which is frequently used in combinatorics [6,p.137ff.]. It appears that J.F.Français was the first to derive it in 1815. It was frequently rediscovered (for a choice of references cf. [6,p.137], [44,p.87f.], [49,p.14]), amongst others by Faà di Bruno in 1855, who was presumably the first mathematician to be beatified (in 1988; see [49]). As compared with Hoppe's formula, there are fewer terms to calculate in the Faà di Bruno formula, but it is notationally slightly more complicated to generalise the latter to  $n > 1$ .

Let us supply a proof of our last ancillary result.  $O(D)$  denotes as before the group of all  $D$ -dimensional orthogonal matrices.

**Lemma 2.** Let  $D \geq 3$ ,  $A \in O(D)$ ,  $m \in \mathbb{N}$ ,  $f \in C^m(S^{D-1})$ ,

$$\zeta(t, \mu) := t \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sqrt{1-t^2} \begin{pmatrix} 0 \\ \mu \end{pmatrix} \quad (t \in [-1, 1], \mu \in S^{D-2})$$

a parametrization of  $S^{D-1}$  and

$$F_A(t) := \frac{1}{\sigma_{D-1}} \int_{|\mu|=1} f(A\zeta(t, \mu)) d\sigma_{D-1}(\mu) \quad (t \in [-1, 1]).$$

Then there exists a function  $C_{m,A}$ , continuous on the interval  $[-1, 1]$  and differentiable in its interior, with

$$F_A^{(m)}(t) = (1-t^2)^{\frac{1}{2}-m} C_{m,A}(t) \quad (t \in (-1, 1)).$$

In addition there is a number  $K_m > 0$  such that

$$|C_{m,A}(t)| \leq K_m \quad (t \in [-1, 1]), \quad |C'_{m,A}(t)| \leq K_m(1-t^2)^{-\frac{1}{2}} \quad (t \in (-1, 1)) \quad (35)$$

for all  $A \in O(D)$ .

*Proof.* Let  $|a| \leq 1$ ,  $|b| \leq 1$ , and  $l \in \mathbb{N}$ ,  $l \leq m$ . Owing to Hoppe's formula (34) it suffices to show that there exists a function  $\gamma_{m,l}(\cdot; a, b)$ , continuous on  $[-1, 1]$  and differentiable in  $(-1, 1)$ , with

$$\left(\frac{d}{dt}\right)^m (at + b\sqrt{1-t^2})^l = (1-t^2)^{\frac{1}{2}-m} \gamma_{m,l}(t; a, b) \quad (t \in (-1, 1)) \quad (36)$$



and admitting bounds as in (35), independent of a,b. On account of Lemma 1a) we have

$$\begin{aligned} & \left(\frac{d}{dt}\right)^m (at + b\sqrt{1-t^2})^l \\ &= \sum_{k=0}^l \sum_{j=0}^m \binom{l}{k} a^{l-k} b^k \binom{m}{j} \left[ \left(\frac{d}{dt}\right)^{m-j} t^{l-k} \right] \left(\frac{d}{dt}\right)^j (1-t^2)^{\frac{k}{2}} \\ &= (1-t^2)^{\frac{1}{2}-m} \sum_{k=0}^l \sum_{j=0}^m (1-t^2)^{m-j+\frac{1}{2}(k-1)} p_{j,k}(t) \binom{l}{k} \binom{m}{j} a^{l-k} b^k \left(\frac{d}{dt}\right)^{m-j} t^{l-k}. \end{aligned}$$

Here  $p_{0,0} = 1$ ; if  $j \in \{1, \dots, m\}$ , then  $p_{j,0} = 0$  while  $p_{j,k} (k \in \{1, \dots, l\})$  is a polynomial of degree  $\leq j$ . This establishes (36) and the desired estimates and thus proves the lemma.

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