Endomorphism from Galois antiautomorphism

Christian Pierre

Introduction

The aim of this paper is to introduce an endomorphism based upon the Eisenstein homology. The Galois homology was previously well worked out and the Eisenstein cohomology was extensively studied, but it seems that the Eisenstein homology was never taken up.

Considering that the Galois homology results from a Galois antiautomorphism, it is proved that every endomorphism, generated from a Galois antiautomorphism, can be decomposed into the direct sum of a Galois homology and of its complementary Galois cohomology.

All developments are initiated from a polynomial ring $A[x_1, ..., x_m]$ in m indeterminates whose specialization is compelled to generate a sequential and graded sheaf of rings θ^m.

Part I gives algebraic basic notions necessary to generating a graded sheaf of rings from a Galois extension, i.e. essentially a specialization, called emergent, from a ring of polynomials $A[x_1, ..., x_m]$ giving rise to a set of compact connected algebraic subgroups which correspond to the different sections of the sheaf of rings θ^m.
Part II refers to the introduction of the Eisenstein homology based upon a Galois antiautomorphism.

Received by the editors March 1994
Communicated by A. Verschoren

Part III gives the conditions of an endomorphism generated from a Galois antiautomorphism.

1 Graded sheaf of rings from Galois extension

Preliminaries.

Let $A[x_1,...,x_m]$ be a ring of polynomials $f_\mu[x_1,...,x_m]$ in m indeterminates over a field K of characteristic zero. The polynomials $f_\mu \in A[x_1,...,x_m]$ are supposed to have the same general form : $f_\mu = \sum_{i=0}^{r} a_i x_1^{k_1}...x_m^{k_m}$, where $k_j \in \mathbb{N}$, $1 \leq j \leq m$, with the same set of k_j, i.e. with the same degree over each variable $x_j^{k_j}$ of the monomials $a_i x_1^{k_1}...x_m^{k_m}$. They thus only differ by the set of coefficients $\{a_i\}$. Note that this hypothesis is only required at first stage from proposition 9 in order to define a homogeneous sequential valuation.

If $f_\mu(\alpha_1,...,\alpha_m)=0$, then $(\alpha_1,...,\alpha_m)$ is said to be a root of the polynomial f_μ and is algebraic over K.

Definition 1. *Let $\alpha=(\alpha_1,...,\alpha_m)$ be a root of a polynomial $f_\mu \in A[x_1,...,x_m]$ supposed to be irreducible.*
The specialization of the ring A is a homomorphism A_0 of A into some ring from the ring $A[x_1,...,x_m]$ with values in a field L which is a finite extension of the field K such that the specialization of the ring A is completely determined by its kernel $p=A_0^{-1}(0)$ which is a prime ideal of A [1] *or a subring of A.*

Definition 2. *The Galois group Γ can be defined* [1] *by means of a homomorphism ν of the group K^* (i.e. the group of non-zero elements of K) into a totally ordered group Γ. Such a homomorphism ν is called a valuation of K^* if it satisfies*

$$\nu(\alpha+\beta) \geq \inf[\nu(\alpha),\nu(\beta)], \quad \forall \alpha,\beta \in K^*.$$

Let $b_1 \subset ... \subset b_n$ be a chain of distinct prime ideals of the discrete valuation ring B obtained under the action of the Galois group Γ.
If b_i, $1 \leq i \leq n$, is a non-zero prime ideal of B and if $p = b_i \cap A$, b_i is said to divide p or to be above p, written $b_i | p$. Then, B is the integral closure of A in L and is a free A-module of rank $n=[L:K]$.

Definition 3. *Let $f_\mu \in A[x_1,...,x_m]$ be a polynomial over a field K. An extension field L_μ of K is said to be a splitting field over K if L_μ is generated by all the roots of f_μ. The Galois group Γ_μ of a polynomial f_μ is the group $\mathrm{Aut}_K L_\mu$. So, we consider the specialization $A_{0\mu}$ of the subring A_μ of the ring A from the subring $f_\mu \in A[x_1,...,x_m]$. We denote by B_μ the integral closure of A_μ into L_μ (i.e. the set of elements of L_μ which are integral over A_μ). Then, the field of fractions of B_μ is L_μ and the subring B_μ is a finitely generated A_μ-module.*
The subring A_μ is a discrete valuation ring because it has a unique non-zero prime ideal p_μ taking its values in L_μ. Then, the subring $B_\mu/p_\mu B_\mu$ is an A_μ/p_μ-algebra of degree $n_\mu=[L_\mu:K]$.

Remark 4. *If we consider the specialization of the subring A_μ, the resulting prime ideal p_μ is not necessarily compact in L^m. For this reason, the specialization A_{T_μ}, called emergent, is introduced.*

Definition 5. *The emergent specialization A_{T_μ} is a specialization such that the prime ideal p_μ^T be generated by the emergent morphism γ_μ into the two step sequence :*

a) *The geometric points corresponding to the roots of the splitting field L_μ are mapped onto the origin of L^m.*

b) *These geometric points are then projected symmetrically from the origin of L^m in an affine connected compact algebraic variety p_μ^T which is a m-dimensional torus T^m* [4].

This emergent morphism γ_μ is completely determined by its representatives $h_{\mu_a} = \{P_a, r(P_a), \gamma_a\}$ defining an abstract compact variety in the sense of A. Weil [2].
A representative h_{μ_a}, where $a \in \mathbb{N}$ labels the set of geometric points, is then defined by :

1) *P_a is a point of the connected algebraic variety p_μ^T*

2) *$r(P_a)$ is the radius of "exjection" of a point P_a from the origin of L^m, called the emergence point of the emergent morphism γ_μ (compare with the injectivity radius of $\Gamma\backslash G/K$ introduced by De George and N. Wallach* [3]*).*

3) *γ_a is a one-to-one correspondence between each point corresponding to a root and its geometrical localization on p_μ^T.*

Lemma 6. *Let G_μ be the affine algebraic group generated from all the automorphisms of the splitting field L_μ and H_μ a subgroup such that $X_\mu^m = G_\mu/H_\mu$ be the symmetric space generated by emergent specialization.*
If P_μ is a maximal connected resoluble subgroup of the algebraic group G_μ, then the space $S_{R_\mu} = P_\mu\backslash G_\mu/H_\mu$ is a compact projective variety.
To the space S_{R_μ} is associated a valuation which corresponds bijectively to the radius of exjection $r(P)$ of a point $P \in S_{R_\mu}$.

Proof : Let K be a field of characteristic zero on which a discrete valuation ν is defined having valuation ring A_μ. Let Z_μ^0 be the centralizer of G_μ^0 (the maximal connected compact subgroup of G_μ) given by diagonal matrices.
Then, $G_\mu(K) = \prod\limits_\nu GL_m(L_{\mu\nu})$ if L_μ is a finite extension of the field K.

For the algebraic subgroup H_μ of G_μ, we get : $H_\mu = \prod\limits_\nu H_\nu$ with :

$$\begin{aligned} H_\nu &= SO(m,R)\cdot Z_0(\mathbb{R}) = SO(m,\mathbb{R})\cdot \mathbb{R}^* \quad \text{if } L_\nu = \mathbb{R} \\ H_\nu &= U(m)\cdot Z_0(\mathbb{C}) = U(m)\cdot \mathbb{C}^* \text{ if } L_\nu \simeq \mathbb{C} \end{aligned}$$

and $X_\mu^m = G_\mu/H_\mu = \prod\limits_\nu GL_m(L_\nu)/H_\nu$.

Let $Z_\mu^0(L_\mu)$ be the centralizer of L_μ in G_μ^0 such that L_μ be the maximal K-split torus

of the radical of $Z^0_\mu(L_\mu)$.
Let U_μ be the unipotent radical of the parabolic subgroup P^0_μ given by the unitrigonal group $UT_m(L_\mu)$. Then, we have the Levi decomposition $P^0_\mu = Z^0_\mu(L_\mu)\cdot U_\mu$ [5].

If G_μ is an algebraic group formed by unipotent matrices, G_μ is nilpotent and can be triangularized. Furthermore, a connected resoluble group is direct product of a maximal torus by the subgroup of its unipotent matrices, its maximal tori being conjugated by interior automorphism. Consequently, P_μ is a resoluble subgroup of G_μ.

Let $(z_1,\dots,z_m)$ be the set of coordinates of a point $P\in S_{R_\mu}$ and $||z||_\nu = |z|_\nu^{N_\nu}$ where N_ν is the order of valuation. Then, the height of a point P, $H(P)$, given by the formula $H(P)=\prod\limits_{\nu}\sup_i||z_i||_\nu$ [6], corresponds bijectively to the radius of exjection $r(P)$ of this point P. Indeed, the height $H(P)$ and the radius of exjection $r(P)$ result directly from the emergent specialization A_{T_μ} and from the automorphism group Γ_μ. Let us note that the symmetric space S_{R_μ} is also generated by an emergent specialization since a valuation ring is always a specialization ring according to the theorem on the extension of specializations in algebraic geometry [1].

Definition 7. *Under the emergent specialization A_{T_μ}, each polynomial f_μ of the ring $A[x_1,\dots,x_m]$ generates a section s_μ of a sheaf of ring θ^m on the space $S_{R_\mu} = P_\mu(A_{L_\mu})\backslash G_\mu(A_{L_\mu})/H_\mu$ where A_{L_μ} is the ring of adeles.*
As the subrings A_μ are discrete valuation rings, the sheaf of rings θ^m is quasi-compact [7] *and finitely generated according to the Mordell-Weil theorem* [6].
θ^m is then called spec$[A]$ and the locally ringed space (D_{T^m},θ^m) with domain D_{T^m} is a variety scheme or a group scheme.

Remark 8. *It is possible to define a graded algebra on the set of sections s_μ of the sheaf of rings θ^m under the action of the Galois group Γ. Indeed, consider that the valuation refers sequentially and gradually to each section s_μ as follows :*

Proposition 9. *Let $\{s_1,\dots,s_\mu,\dots,s_q\}$, $1\le\mu\le q$, be the set of sections of the sheaf of rings θ^m on which a sequential valuation will be introduced.*
Let $b^1_\mu\subset\dots\subset b^n_\mu$ be a chain of distinct ideals dividing the prime ideal p^T_μ and denoted $b_\mu\big|p^T_\mu$.

Let $n_{b_\mu}\in\mathbb{N}$ be the residue degree of b_μ under the Galois extension : $n_{b_\mu} = [B_\mu/b_\mu; A_\mu/p^T_\mu]$. In the unramified case, i.e. when the ramification index $e_{b_\mu}=1$, B_μ/b_μ is separable over A_μ/p^T_μ and the degree $n_\mu=[L_\mu:K]$ of the A_μ/p^T_μ algebra is :

$$n_\mu = \prod_{b_\mu|p^T_\mu} B_\mu/b_\mu^{e_{b_\mu}} = \sum_{b_\mu|p^T_\mu} n_{b_\mu}.$$

Consider that the valuation operates sequentially and gradually on each section of the sheaf of rings θ^m in such a way that :
1) There is a graduation on the residue degrees n_{b_μ} of each section s_μ such that if n_{b_μ} is the residue degree of the section s_μ and $n_{b_{\mu+1}}$ is the residue degree of the section

$s_{\mu+1}$, then :

$$n_{b_{\mu+1}} > n_{b_\mu}.$$

2) There is topological embedding between sequential sections, i.e. that $s_1 \subset s_2 \subset \dots \subset s_\mu \subset \dots \subset s_q$.

Sketch of proof : If one has that $n_{b_{\mu+1}} > n_{b_\mu}$, then $s_{\mu+1} \supset s_\mu$. Indeed, if $r(P_{\mu+1})$ denotes the radius of exjection of a point $P \in s_{\mu+1}$ and if $r(P_\mu)$ denotes the radius of exjection of the same point $P \in s_\mu$, it is evident that $r(P_{\mu+1}) > r(P_\mu)$ with respect to the sequential valuation given by the residue degrees $n_{b_{\mu+1}}$ and n_{b_μ} corresponding respectively to the sections $s_{\mu+1}$ and s_μ.

Corollary 10. *Let $\{s_1, \dots, s_\mu, \dots, s_q\}$, $1 \le \mu \le q$, be the set of sections of the sheaf of rings θ^m on which a sequential valuation has been defined. Then, the q sections of the sheaf of rings θ^m are generated by emergent specialization A_{T^m} if and only if $n_{b_q} > 0$.*

Corollary 11. *Let n_{b_μ} be the residue degree of a section $s_\mu \in \theta^m$. Then, the sequential residue degree of the graded sequence $\{s_1, \dots, s_q\}$ of sections of the sheaf of rings θ^m is given by the set $n_{\theta^m} = \{n_{b_1} \dots, n_{b_\mu} \dots, n_{b_q}\}$ of residue degrees n_{b_μ} referring to the sections $s_\mu \in \theta^m$.*
θ^m can then be considered as a free module with graded rank $n_{\theta^m} = \{n_{b_1} \dots, n_{b_q}\}$.

2 Eisenstein homology

Definition 12. *Let $P(A)$ be the set of parabolic subgroups $P(A) = \{P_1(A_{L_1}), \dots, P_\mu(A_{L_\mu}), \dots, P_q(A_{L_q})\}$ corresponding to the q sections of the sheaf of rings θ^m. Let $G(A) = \{G_1(A_{L_1}), \dots, G_q(A_{L_q})\}$ be the set of q algebraic groups corresponding to the q sections of θ^m and $H = \{H_1(A_{L_1}), \dots, H_q(A_{L_q})\}$ the set of q subgroups of G as introduced in lemma 6.*
The cohomology group $H^*(S_R, \theta^m)$ is the direct sum of the cohomology groups of its connected components [8] given by $H^*(\Gamma \backslash X^m)$ where $X^m = G(A)/H$ is the symmetric space, $S_R = P(A) \backslash G(A)/H$ and $\Gamma = \{\Gamma_1, \dots, \Gamma_\mu, \dots, \Gamma_q\}$ is a set of q arithmetic subgroups given by $P(A)$.

The boundary $\partial \bar{S}_R$ of the Borel-Serre compactification [9] $i : S_R \to \bar{S}_R$ is stratified by proper parabolic subgroups $P(A)$ such that $\partial_p S_R$ be a sequential module defined over K and that $\partial \bar{S}_R = \bigcup_P \partial_p S_R$. As developped by G. Harder in [10], let Ξ_{H_f} be the finite set of double cosets $P(A_f) \backslash G(A_f)/H_f$ for any place f in the adele ring A. Then, we have the following decomposition for the Eisenstein Cohomology [11] :

$$H^*(\partial_p S_R, \theta^m) = H^*(P(A) \backslash G(A)/H_f, \theta^m) = \bigoplus_{\xi \in \Xi_{H_f}} H^*\Big(S^{\theta^m}_{H(\xi)}, H^*(\tilde{u}_p, \theta^m)\Big)$$

where $H^*(\tilde{u}_p, \theta^m)$ is the Lie algebra cohomology referring to unipotent algebraic group U/K with U given as usual by the set $U = \{U_1, \dots, U_\mu, \dots, U_q\}$.

Considering that the sheaf of rings θ^m is a module for the algebraic group Gx_KL, after passing to the limit, it becomes a $\pi_0(G_\infty)xG(A_f)$ module, and we get :

$$H^*(\partial_P S,\theta^m)=\mathrm{Ind}^{\pi_0(G_\infty)xG(A_f)}_{\pi_0(P_\infty)xP(A_f)}H^*(S^{\theta^m},H^*(\tilde{u}_p,\theta^m))$$

for all induced representations of the cohomology.

Definition 13. *Let* $\lambda_{\mu\nu}=(\lambda_{\mu 1},...,\lambda_{\mu n})_{\nu:K\to L}$ *be an element in* $X(T^m_\mu)=\mathrm{Hom}(T^m_\mu x_K L, GL1)=\bigoplus\limits_{\nu:K\to L}X(T^m_{0\mu})$ *referring to the section* s_μ *of* θ^m *which is a* m*-dimensional torus* T^m_μ.
Similarly, let $\lambda_\nu=\{\lambda_{1\nu},...,\lambda_{\mu\nu},...,\lambda_{q\nu}\}$ *be an element in* $X(\theta^m)$.

Considering the irreducible representation $Gx_KL=\prod\limits_{\nu:K\to L}GL_mxL=\prod\limits_{\nu:K\to L}GL_m$, the sheaf of rings θ^m is a tensor product $\theta^m=\bigotimes\limits_{\nu:K\to L}\theta^m_\nu=\theta^m(\lambda_\nu)$ where θ^m_ν is determinated by the sequential weight λ_ν.

Let $\omega_\nu=\{\omega_{1\nu},...,\omega_{\mu\nu},...,\omega_{q\nu}\}$ be the action of the Weylgroup on $\lambda_\nu\in X(\theta^m)$ and $\Phi_\nu=\omega_\nu\lambda_\nu$ be the corresponding algebraic Hecke character [12] on θ^m.
Then, the Eisenstein cohomology $H^*(\partial_P S,\theta^m)$ decomposes into one dimensional eigenspaces according to a theorem of Kostant [10] :

$$H^*(\partial_P S,\theta^m)=\bigoplus_{\omega_\nu}\bigoplus_{\Phi_\nu}\mathrm{Ind}^{\pi_0(G_\infty)XG(A_f)}_{\pi_0(P_\infty)XP(A_f)}H^*(S^{\theta^m},H^*(\tilde{u}_P,\theta^m)(\omega_\nu\cdot\lambda_\nu)).$$

Definition 14. *Let K be a field of characteristic zero, L an algebraic extension of K and $G(L/K)$ the corresponding Galois group. This group $G(L/K)=\mathrm{Aut}_K L$ acts transitively on the right on the set of prime ideals b of B dividing a given prime ideal p of the discrete valuation ring A (see definitions* 2 *and* 3, [13] *and* [14]*). The ring B/pB is an A/p-algebra of degree $n=[L:K]$.*

From the Galois automorphic group $G(L/K)$, it is possible to define a Galois antiautomorphic group $G^*(L/K)=\widetilde{\mathrm{Aut}}_K L$ acting transitively on the left on the set of prime ideals b of B. A subring $B'\subset B$ is then characterized by a descending chain of distinct prime ideals :

$$b_n\supset b_{n-1}\supset ...\supset b_{n-\rho},\qquad \rho<n$$

such that $(n-\rho)$ be the retro-valuation degree corresponding to the considered Galois antiautomorphism, noted antiaut.

Proposition 15. *Let L be an algebraic extension of a field K of characteristic zero and $G^*(L/K)$ a Galois antiautomorphic group. Then, the space ∂S^*_R, associated to this Galois antiaut., is given by $\partial S^*_R=P^*\backslash G^*/H^*$ where G^* is the algebraic group submitted to a Galois antiaut., P^* is a connected resoluble subgroup and H^* is a subgroup of G^* referring to a decreasing valuation.*

Proof : The antiautomorphism given by the Galois antiaut. $G^*(L/K)$ leads to consider a descending chain of ideals $b_n\supset b_{n-1}\supset ...\supset b_{\eta-\rho}$ as described in definition 14.

This implies that inverse elements of the connected resoluble subgroup P have to be considered such that $P^{0*} = Z^{0*}(L) \cdot U^*$ where $Z^{0*}(L)$ is the centralizer of L in G^{0*} given by the diagonal group $D(L)$ and where U^* is the inverse of the unipotent radical U of the Galois extension L/K and is represented by the unitrigonal group $UT(M)^{-1}$.

According to lemma 6, one has furthermore that :

$$\begin{array}{ll} H^* = SO(m;\mathbb{R}) \cdot Z^{0*}(\mathbb{R}_{\nu-\nu^*}) & \text{if } L = \mathbb{R}_{\nu-\nu^*} \\ H^* = U(m) \cdot Z^{0*}(\mathbb{C}_{\nu-\nu^*}) & \text{if } L = \mathbb{C}_{\nu-\nu^*} \end{array}$$

where $(\nu - \nu^*)$ refers to a decreasing valuation associated to a Galois antiaut.

Definition 16. *Consider that the retro-valuation, corresponding to a Galois antiaut., operates sequentially and gradually on the sheaf of rings θ^m such that if $(n-\rho)_\mu$ is the residue degree of the Galois antiaut. of the section s_μ and if $(n-\rho)_{\mu+1}$ is the residue degree of Galois antiaut. of the sections $s_{\mu+1}$ of θ^m, then $(n-\rho)_{\mu+1} > (n-\rho)_\mu$. The sheaf of rings θ^{*m} is then characterized under a Galois group $G^*(L/K)$ by the set $(n-\rho)_{\theta^m} = (n_1-\rho_1), ..., (n_\mu-\rho_\mu), ..., (n_q-\rho_q)\}$ of q difference of residue degrees corresponding to the q sections of θ^m, where n_μ refers to the residue degree of the section $s_\mu \in \theta^m$ under a Galois extension $G(L/K)$.*

Proposition 17. *Let $G^*(A)$ be the algebraic group generated by a Galois group $G^*(L/K)$ and given by the set of algebraic subgroups : $G^*(A) = \{G_1^*(A), ..., G_\mu^*(A), ..., G_q^*(A)\}$ corresponding to the q sections of the sheaf of rings θ^{*m}.*
Let $P^* = \{P_1^*, ..., P_\mu^*, ..., P_q^*\}$ be the set of q connected resoluble subgroups of the q sections of θ^{*m} corresponding to a Galois antiaut. and defined in proposition 15.
Let ∂S_R^* be given by : $\partial S_R^* = P^*(A)\backslash G^*(A)/H^*$.
Then, the homology group $H_*(\partial S_R^*, \theta^{*m})$ is the direct sum of the homology groups of its connected components given by $H_*(\Gamma^*\backslash X^{*m})$.
Let $\Xi^*_{H^*f}$ be the finite set of double cosets $P^*(A_f)\backslash G^*(A_f)/H_f^*$ where $P^*(A_f)$ is a parabolic subgroup corresponding to a Galois antiaut.

Then, the Eisenstein homology, corresponding to a Galois antiaut., is given by (in analogy with the Eisenstein cohomology) :

$$\begin{aligned} H_*(\partial_{P^*} S_R^*, \theta^{*m}) &= H_*(P^*(A)\backslash G^*(A)/H_f^*, \theta^{*m}) \\ &= \bigoplus_{\xi \in \Xi^*_{H_f^*}} H_*(S^{\theta^{*m}}_{H^*(\xi)}, H_*(\tilde{u}_{P^*}, \theta^{*m})) \end{aligned}$$

where $H_*(\tilde{u}_{P^*}, \theta^{*m})$ is the Lie algebra homology referring to unipotent algebraic group U^*/K.

<u>Proof</u> : The proof is obvious by considering preceeding sections.

Corollary 18. *Let $\lambda^*_{\mu(\nu-\nu^*)} = (\lambda^*_{\mu n}, ..., \lambda^*_{\mu(n-\rho)})_{\nu-\nu^*:K\to L}$ be an element in $X^*(T^m_\mu) =$* $\mathrm{Hom}(T^m_\mu x_K L, GL1) = \bigoplus_{\nu-\nu^*:K\to L} X^*(T^m_{0\mu})$ *referring to the section s_μ of θ^{*m}. Let $\lambda^*_{(\nu-\nu^*)} = \{\lambda^*_{1(\nu_1-\nu_1^*)}, ..., \lambda^*_{\mu(\nu_\mu-\nu_\mu^*)}, ..., \lambda^*_{q(\nu_q-\nu_q^*)}\}$ be an element in $X^*(\theta^{*m})$.*

The sheaf of rings θ^{*m} is then a tensor product : $\theta^{*m} = \bigotimes_{\nu-\nu^*:K\to L} \theta^{*m}_{\nu-\nu^*} = \theta^{*m}(\lambda^*_{(\nu-\nu^*)})$, where $\lambda^*_{(\nu-\nu^*)}$ is a decreasing sequential weight.
Let $\omega^*_{(\nu-\nu^*)} = \{\omega^*_{1(\nu_1-\nu_1^*)}, ..., \omega^*_{\mu(\nu_\mu-\nu_\mu^*)}, ..., \omega^*_{q(\nu_q-\nu_q^*)}\}$ be the action of the Weylgroup on $\lambda^*_{(\nu-\nu^*)} \in X^*(\theta^{*m})$ defining the corresponding algebraic Hecke character $\Phi^*_{(\nu-\nu^*)} = \omega^*_{(\nu-\nu^*)} \cdot \lambda^*_{(\nu-\nu^*)}$ on θ^{*m}. Then, the Eisenstein homology $H_*(\partial_{P^*}S^*, \theta^{*m})$ decomposes into one dimensional eigenspaces :

$$H_*(\partial_{P^*}S^*, \theta^{*m}) = \bigoplus_{\omega^*_{(\nu-\nu^*)}} \bigoplus_{\Phi^*_{(\nu-\nu^*)}} \mathrm{Ind}^{\pi_0(G^*_\infty)XG^*(A_f)}_{\pi_0(P^*_\infty)XP^*(A_f)} H_*(S^{*\theta^{*m}}, H_*(\tilde{u}_{P^*}, \theta^{*m})...$$

$$...(\omega^*_{(\nu-\nu^*)} \cdot \lambda^*_{(\nu-\nu^*)})).$$

3 Endomorphism from Galois antiautomorphism group

Definition 19. *Let $\partial S^*_R = P^*(A)\backslash G^*(A)/H^*$ be the space generated by the Galois group $G^*(L/K)$ with retrovaluation residue degree $(n-\rho)_{\theta^{*m}}$ applied to the sheaf of rings θ^{*m} generated from an initial sheaf of rings θ^m of residue degree n_{θ^m}. The subspace H^* of $G^*(A)$ will be defined for $L = \mathbb{R}_{(\nu-\nu^*)}$ by $H^* = SO(m,\mathbb{R}) \cdot Z^{0*}(\mathbb{R}_{(\nu-\nu^*)})$. The corresponding Eisenstein homology $H_*(\partial_{P^*}S^*, \theta^{*m})$ decomposes into one dimensional eigenspaces as described in corollary 18 from algebraic Hecke characters $\Phi^*_{(n-\rho)} = \omega^*_{(n-\rho)} \cdot \lambda^*_{(n-\rho)}$.*

Let $\partial S_{R(I)} = P_I(A)\backslash G_I(A)/H_I$ be the space complementary of ∂S^*_R and corresponding to a Galois group $G_I(K/L)$ conjugated to the Galois group $G^*(L/K)$, as resulting from the definition 14 of a Galois antiautomorphism.
The complementary automorphic space $\partial S_{R(I)}$ will be characterized by a sequential residue degree ρ_I counted from the sequential retrovaluation residue degree $(n-\rho)$.
The subspace H_I of $G_I(A)$ will parallely be defined for $L = \mathbb{R}$ by $H_I = SO(m,\mathbb{R}) \cdot Z^0_I(\mathbb{R})$.
The Eisenstein cohomology $H^*(\partial_P S_I, \theta^m_I)$ corresponding to the generation of this complementary space $\partial S_{R(I)}$ will also decompose into one dimensional eigenspaces as described in definition 13 and will be characterized by complementary algebraic Hecke characters $\Phi_{\rho_I} = \omega_{\rho_I} \cdot \lambda_{\rho_I}$.

Proposition 20. *Every smooth endomorphism of an algebraic group $G(A)$ on an automorphic space ∂S_R of Galois group $G(L/K)$ with residue degree n can be generated by means of a Galois antiautomorphism and can decompose into the direct sum of two connected algebraic groups $G^*(A)$, corresponding to a Galois antiaut. of group $G^*(L/K)$, and $G_I(A)$, which is a complementary algebraic group of $G^*(A)$ and subgroup of $G(A)$ given by a Galois extension of group $G_I(L/K)$.*

We then have the following decomposition for the smooth endomorphism $E_t[G(A)]$:

$$E_t[G(A)] = G^*(A) \oplus G_I(A)_t$$

satisfying the conditions:
1) The subgroups $H^* \subset G^*(A)$ and $H_I \subset G_I(A)$ are such that the groups $SO(m, \mathbb{R})$, from which they are defined, must have the same witt index (i.e., the same rank) but different orders.
2) The Eisenstein homology $H_*(\partial_{P^*} S^*, \theta^{*m})$ and cohomology $H^*(\partial_P S_I, \theta_I^m)$, associated respectively to $G^*(A)$ and $G_I(A)$, are varying oppositely in such a way that $H_*(\partial_{P^*} S^*, \theta^{*m})$ "generates" $H^*(\partial_P S_I, \theta_I^m)$ according to the following equality on algebraic Hecke characters :

$$\Phi^*_{(n-\rho)} = C_H(\rho) \cdot \Phi_{\rho_I}$$

where $C_H(\rho)$ is a set of q parameters $C_H(\rho) = \{C_{H1}(\rho_1), ..., C_{H\mu}(\rho_\mu)... C_{Hq}(\rho_q)\}$ depending on the proper valuation ρ and measuring the variation in sequential degrees of Φ_{ρ_I}, related to $H^*(\partial_P S_I, \theta_I^m)$, in function of $\Phi^*_{(n-\rho)}$, related to $H_*(\partial_P S^*, \theta^{*m})$.
<u>Proof</u> :
1) The fact that every endomorphism $E_t[G(A)]$ (compare with the standard definitions of endomorphisms [15]) of an algebraic group $G(A)$ can decompose into the direct sum of an algebraic group $G^*(A)$, corresponding to a Galois antiaut. $G^*(L/K)$ and of its complementary algebraic group $G_I(A)_t$, which is a subgroup of $G(A)$ and thus an ideal of the affine variety given by $G(A)$, results from the definition of a Galois antiaut. given in definition 14.

2) The groups $SO(2p, \mathbb{R}) \in H^* \subset G^*(A)$ and $SO(2p+1, \mathbb{R}) \in H_I \subset G_I(A)_t$ have the same rank p but different orders given by $m = 2p$ or $m = 2p+1$ respectively.
Indeed, if the groups $SO(m, \mathbb{R})$ had the same order, this should mean that the map $E_t[G(A)]$ would be the identity map and that $(G^*(A) \oplus G_I(A)_t)$ would not be more the direct sum of two distinct connected algebraic groups $G^*(A)$ and $G_I(A)_t$ but would define a unique connected algebraic group $G(A)$.

3) The fact that the spaces ∂S_R^* and $\partial S_R(I)$ are defined respectively by means of the groups $SO(2p, \mathbb{R}) \in H^*$ and $SO(2p+1, \mathbb{R}) \in H_I$ with different orders results directly from the reduction steps of Hilbert irreducibility theorem [16] applied to the ring of polynomials $A[x_1, ..., x_m]$. Indeed, via the Kronecker's specialization, this theorem tells us that a ring of irreducible polynomials in m variables can be reduced to a ring of irreducible polynomials in n variables where $n < m$.

4) According to the Hilbert's irreducibility theorem, it is obvious that the groups $SO(2p, \mathbb{R}) \in H^*$ and $SO(2p+1, \mathbb{R}) \in H_I$ need not have peremptorily the same rank but this is a prerequisite in order that there is not a breakdown of the endormophism E_t which is then called "smooth".

5) The equality $\Phi^*_{(n-\rho)} = C_H(\rho) \cdot \Phi_{\rho_I}$ results directly from the definition of the algebraic Hecke characters $\Phi^*_{(n-\rho)}$ and Φ_{ρ_I} and from the definition of the endomorphism $E_t[G(A)]$.
Note that the set of parameters $C_H(\rho)$ is the most closed to the unity $\{1, ..., 1, ..., 1\}$ when the degree of retrovaluation on the sheaf of rings θ_m^* is equal to the degree of valuation of its complementary sheaf θ_I^m.

Corollary 21. *If the reduced algebraic group $G^*(A)$ and its complementary algebraic group $G_I(A)$, obtained from the algebraic group $G(A)$ by smooth endomorphism, are not connected, the smooth endomorphism $E[G(A)]$ will be introduced :*

$$E[G(A)] = G^*(A) \oplus G_I(A)$$

such that the subgroups $SO(m,\mathbb{R}) \in H^* \subset G^*(A)$ and $SO(m,\mathbb{R}) \in H_I \subset G_I(A)$ have the same ranks and the same orders. Then, there exists a unique emergent morphism γ which maps the sheaf of rings θ_I^m from the space $\delta S_{R(I)} = P_I(A)\backslash G_I(A)/H_I$ into its orthogonal complement $\delta S^{\perp}_{R(I)}$ on which is defined the resulting complementary orthogonal sheaf $\theta_I^{m\perp}$.

Proof : As the origin O_S is the unique point of intersection between the space $\Sigma^*_R = \partial S^*_R \cup O_S$ and its orthogonal complement $\Sigma^{\perp}_R = \partial S^{\perp}_R \cup O_S$, i.e. that $O_S = \Sigma^*_R \cap \Sigma^{\perp}_R$, the envisaged morphism γ from $\partial S_{R(I)}$ to $\partial S^{\perp}_{R(I)}$ may be decomposed in the two step sequence :

1) mapping of the sheaf of rings θ_I^m onto O_S, called the emergence point of the morphism γ.

2) Projection of θ_I^m from O_S into $\partial S^{\perp}_{R(I)}$ such that $\rho_{\theta_I^{m\perp}} = \{\rho_1, ..., \rho_\mu, ..., \rho_q\}$ of the q sections of the complementary orthogonal sheaf $\theta_I^{m\perp}$ be conserved, i.e. that $\rho_{\theta_I^{m\perp}} = \rho_{\theta_I^m}$. This morphism γ is an emergent morphism according to definition 5, and thus, is unique.

References

[1] A. Weil, Arithmetic on algebraic varieties, Ann. of Math. **53** (1951) 412-444.

[2] A. Weil, Foundations of algebraic geometry, Amer. Math. Soc., Colloq. Publ. **29** (1946).

[3] D. De George, N. Wallach, Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$, Ann. of Math. **107** (1978) 133-150.
N. Wallach, Limit multiplicities in $L^2(\Gamma\backslash G)$, Lect. Not. Math. **1447** (1990) 31-56.

[4] A. Borel, J. Tits, Groupes réductifs, Publ. Math. IHES **27** (1965) 54-151.

[5] R. Langlands, On the functional equations satisfied by Eisenstein series, Lect. Not. Math. **544** (1976) 1-320.
Harish-Chandra, Automorphic forms on semi-simple Lie groups, Lect. Not. Math. **62** (1968) 1-130.

[6] S. Lang, A. Néron, Rational points of abelian varieties over function fields, Amer. J. Math. **81** (1959) 95-118.

[7] J.P. Serre, Faisceaux algébriques cohérents, Ann. of Math. **61** (1955) 197-278.

[8] A. Borel, Stable real cohomology of arithmetic groups, Ann. Scient. école. norm. sup. **7** (1974) 235-272.

[9] A. Borel, J.P. Serre, Corners and arithmetic groups, Comment. Math. Helv. **48** (1973) 436-491.

[10] G. Harder, Eisenstein cohomology of arithmetic groups, the case GL_2, Invent. Math. **89** (1987) 37-118.
G. Harder, Some results on the Eisenstein cohomology of arithmetic subgroups of GL_n, Lect. Not. Math. **1447** (1990) 85-153.

[11] J. Schwermer, Cohomology of arithmetic groups, automorphic forms and L-functions, Lect. Not. Math. **1447** (1990) 1-29.

[12] L. Clozel, Représentations Galoisiennes associées aux représentations automorphes autoduales de $GL(n)$, Publ. Math. IHES **73** (1991) 97-145.

[13] A. Douady, Cohomologie des groupes compacts totalement discontinus, Sém. Bourbaki **189** (1959-60) 1-12.

[14] J.P. Serre, Local fields, Grad. texts in Math. **67** (1979) 1-237.

[15] A. Corner, On endomorphism rings of primary abelian groups, Quart. J. Math. Oxford **20** (1969) 277-296.
M. Dugas, R. Göbel, On endomorphism rings of primary abelian groups, Math. Annal. **261** (1982) 359-382.

[16] D. Hilbert, Über die Irreduzibilität ganzer rationaler Funktionen mit ganzähligen Koefficienten, J. Reine Angew. Math. **110** (1892) 104-129.

Christian PIERRE
Clos des Gaulois 15
B1340 OTTIGNIES