

Integral Representations and Coherent States*

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Abstract

A scheme of construction of integral representations on spaces of analytic functions is presented. The scheme gives a general explicit formula for the reproducing kernels. Its construction is based on symmetry groups of systems of differential equations, which define the “analytic” property of functional spaces. Given examples offer a new view on the classical results on this problem

1 Introduction

Integral representations (or reproducing kernels) for analytic¹ functions are an important tool in the theory of holomorphic functions of several complex variables [15], the theory of monogenic functions [4] etc. Thus it seems to be an important problem to obtain and study such integral representations.

The paper presents a new scheme of construction of integral representations. It is proved that, if a space of analytic functions has a symmetry group then one can construct the integral representation as a convolution operator on the symmetry group. Moreover an explicit formula for the kernel of this convolution is found. Naturally, a symmetry group of a space of analytic functions is a symmetry group of differential equations which define the analytic property. Thus we can construct the integral representation starting from the symmetry group of differential equations. The described scheme is based on a notion of *coherent states* which gives new connections with quantum mechanics.

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¹Due to the wide spectrum of possible examples with essentially different terminology, we will speak simply of a space of *analytic* functions defined by a *system of differential operators*.

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The paper has a simple layout. After general considerations in Section 2 we will apply the results obtained to three classical problems in Section 3. They are the Bargmann projector on the Segal–Bargmann (Fock) space on \mathbb{C}^n , the Bergman projector on the space of holomorphic functions on the unit ball \mathbb{B}^n in \mathbb{C}^n and the Cauchy–Szëgo projector on the Hardy space. The representation of the latter projector as a convolution operator on the Heisenberg group was obtained by Gindikin in [8], but representations of the other two projectors as convolutions seem to be new.

Now we would like to give the basic notions, notations and references, which will be used in Section 3.

The Heisenberg group \mathbb{H}^n (see [18, Chap. 1] or [17, Chap. XII]) is a step 2 nilpotent Lie group. As a C^∞ -manifold it coincides with \mathbb{R}^{2n+1} . If an element of it is given in the form $g = (u, v) \in \mathbb{H}^n$, where $u \in \mathbb{R}$ and $v = (v_1, \dots, v_n) \in \mathbb{C}^n$, then the group law on \mathbb{H}^n can be written as

$$(u, v) * (u', v') = \left(u + u' - \frac{1}{2} \operatorname{Im} \sum_1^n v'_k \bar{v}_k, v_1 + v'_1, \dots, v_n + v'_n \right). \quad (1.1)$$

The left and right Haar measure² on the Heisenberg group coincides with the Lebesgue measure. Let us introduce the right π_r and the left π_l regular representations of \mathbb{H}^n on $L_2(\mathbb{H}^n)$:

$$[\pi_l(g)f](h) = f(g^{-1} * h), \quad (1.2)$$

$$[\pi_r(g)f](h) = f(h * g). \quad (1.3)$$

Then the left (right) convolutions on the Heisenberg group are integrals from the shift operators:

$$K_{l(r)} = (2\pi)^{-n-1/2} \int_{\mathbb{H}^n} k(g) \pi_{l(r)}(g) dg, \quad (1.4)$$

Let $L_2(\mathbb{C}^n, d\mu_n)$ be the space of all square-integrable functions on \mathbb{C}^n with respect to the Gaussian measure

$$d\mu_n(z) = \pi^{-n} e^{-z\bar{z}} dv(z),$$

where $dv(z) = dx dy$ is the usual Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. Denote by P_n the orthogonal Bargmann projector of $L_2(\mathbb{C}^n, d\mu_n)$ onto the *Segal–Bargmann* or *Fock space* $F_2(\mathbb{C}^n)$, namely, the subspace of $L_2(\mathbb{C}^n, d\mu_n)$ consisting of all entire functions. The Fock space $F_2(\mathbb{C}^n)$ was introduced by Fock [6] to give an alternative representation of the Heisenberg group in quantum mechanics. The rigorous theory of $F_2(\mathbb{C}^n)$ was developed by Bargmann [1] and Segal [16]. This theory is closely connected with representations of the Heisenberg group (see also [7, 10, 11]).

Let \mathbb{U}^n be the *upper half-space* [17, Chap. XII])

$$\mathbb{U}^n = \left\{ z \in \mathbb{C}^{n+1} \mid \operatorname{Im} z_{n+1} > \sum_{j=1}^n |z_j|^2 \right\},$$

which is a domain of holomorphy of functions of $n+1$ complex variables. Its boundary

$$b\mathbb{U}^n = \left\{ z \in \mathbb{C}^{n+1} \mid \operatorname{Im} z_{n+1} = \sum_{j=1}^n |z_j|^2 \right\}$$

²The Haar measure on a group is a measure which is invariant under the group action.

may be naturally identified with the Heisenberg group \mathbb{H}^n . One can introduce the *Szegő projector* R as the orthogonal projection of $L_2(\mathbb{H}^n)$ onto its subspace $H_2(\mathbb{H}^n)$ (the *Hardy space*) of boundary values of holomorphic functions on the upper half-space \mathbb{U}^n .

2 General Considerations

Coherent states are a useful tool in quantum theory and have a lot of essentially different definitions [14]. Particular, they were described by Berezin in [2, 3, 10] concerning so-called covariant and contravariant (or Wick and anti-Wick) symbols of operators (quantization).

Definition 2.1 We say that the Hilbert space H has a system of coherent states $\{f_\alpha\}, \alpha \in G$ if for any $f \in H$

$$\langle f, f \rangle = \int_G |\langle f, f_\alpha \rangle|^2 d\mu. \quad (2.1)$$

This definition does not take in account that a group structure within coherent states frequently occurs and is always useful [14]. For example, the original consideration of Berezin is connected with the Fock space, where coherent states are functions $e^{za-a\bar{a}/2}$. But the Fock space is an alternative representation for the Heisenberg group [7, 11] thus it has the corresponding group structure. Another type of coherent states with a group structure is given by the vacuum vector and operators of creation and annihilation, which represents the group \mathbb{Z} . Thus we would like to give an alternative definition.

Definition 2.2 We will say that the Hilbert space H has a system of coherent states $\{f_g\}, g \in G$ if

1. There is a representation $T : g \mapsto T_g$ of the group G by unitary operators T_g on H .
2. There is a vector $f_0 \in H$ such that for $f_g = T_g f_0$ and arbitrary $f \in H$ we have

$$\langle f, f \rangle = \int_G |\langle f, f_g \rangle|^2 d\mu, \quad (2.2)$$

where the integration is taken with respect to the Haar measure $d\mu$ on G .

Because this construction independently arose in different contexts, the vector f_0 has various names: the *vacuum vector*, the *ground state*, the *mother wavelet* etc. Modifications of definition 2.2 for other cases are discussed in [14, § 2.1]. Equation (2.2) implies that the vector f_0 is a cyclic vector of the representation G on H .

Lemma 2.3 *Let T be an irreducible unitary representation of a group G in a Hilbert space H . Then there exists $f_0 \in H$ that equality (2.2) holds. Moreover, one can take an arbitrary non-zero vector of H (up to a scalar factor) as the vacuum vector.*

Proof. Let us fix some Haar measure $d\mu$ on G . (Different Haar measures are different on a scalar factor). If the representation T is irreducible, than an arbitrary vector $f \in H$ is cyclic and we may put $f_0 = c^{-1/2}f$, where

$$c = \frac{\int_G |\langle f, T_g f \rangle|^2 d\mu(g)}{\langle f, f \rangle}.$$

It is easy to check, that for the f_0 equality (2.2) holds [14, § 2.3]. ■

By the way, a polarization of (2.2) gives us the equality

$$\langle f_1, f_2 \rangle = \int_G \langle f_1, f_g \rangle \overline{\langle f_2, f_g \rangle} d\mu. \quad (2.3)$$

Thus we have an isometrical embedding $E : H \rightarrow L_2(G, d\mu)$ defined by the formula

$$E : f \mapsto f(g) = \langle f, f_g \rangle = \langle f, T_g f_0 \rangle = \langle T_g^* f, f_0 \rangle = \langle T_{g^{-1}} f, f_0 \rangle. \quad (2.4)$$

We will consider $L_2(G, d\mu)$ both as a linear space of functions and as an operator algebra with respect to the left and right group convolution operations:

$$[f_1 * f_2]_l(h) = \int_G f_1(g) f_2(g^{-1}h) d\mu(g), \quad (2.5)$$

$$[f_1 * f_2]_r(h) = \int_G f_1(g) f_2(hg) d\mu(g). \quad (2.6)$$

For simplicity we will assume that G is unimodular, i.e. that the left and the right Haar measures on G coincide, and that $L_2(G, d\mu)$ is closed under the group convolution. Thus the construction studied may be considered as a natural embedding of the linear space H into the operator algebra $\mathcal{B}(H)$.

Let $H_2(G, d\mu) \subset L_2(G, d\mu)$ denote the image of H under the embedding E . It is clear that $H_2(G, d\mu)$ is a linear subspace of $L_2(G, d\mu)$ which does not coincide with the whole $L_2(G, d\mu)$ in general. One can see that

Lemma 2.4 *The space $H_2(G, d\mu)$ is invariant under left shifts on G .*

Proof. Indeed, for every $f(g) \in H_2(G, d\mu)$ the function

$$f(h^{-1}g) = \langle f, T_{h^{-1}g} f_0 \rangle = \langle f, T_{h^{-1}} T_g f_0 \rangle = \langle T_h f, T_g f_0 \rangle = [T_h f](g)$$

also belongs to $H_2(G, d\mu)$. ■

If $P : L_2(G, d\mu) \rightarrow H_2(G, d\mu)$ is the orthogonal projector on $H_2(G, d\mu)$, then due to Lemma 2.4 it should commute with all left shifts and thus we get immediately

Corollary 2.5 *The projector $P : L_2(G, d\mu) \rightarrow H_2(G, d\mu)$ is a right convolution on G .*

The following Lemma characterizes *linear subspaces* of $L_2(G, d\mu)$ invariant under shifts in terms of the *convolution algebra* $L_2(G, d\mu)$.

Lemma 2.6 *A closed linear subspace H of $L_2(G, d\mu)$ is invariant under left (right) shifts if and only if H is a left (right) ideal of the right group convolution algebra $L_2(G, d\mu)$.*

Proof. Let us consider e.g. the “right-invariance and right-convolution” case. Let H be a closed linear subspace of $L_2(G, d\mu)$ invariant under right shifts and let $k(g) \in H$. We will show that the right convolution

$$[f * k]_r(h) = \int_G f(g)k(hg) d\mu(g) \in H, \tag{2.7}$$

for any $f \in L_2(G, d\mu)$. Indeed, we can treat the integral (2.7) as a limit of sums

$$\sum_{j=1}^N f(g_j)k(hg_j)\Delta_j. \tag{2.8}$$

But the last sum is simple a linear combination of vectors $k(hg_j) \in H$ (by the invariance of H) with coefficients $f(g_j)$. Therefore the sum (2.8) belongs to H and this is also true for the integral (2.7) by the closedness of H .

On the other hand, let H be a right ideal in the group convolution algebra $L_2(G, d\mu)$ and let $\phi_j(g) \in L_2(G, d\mu)$ be an approximate unit of the algebra [5, § 13.2], i. e. for any $f \in L_2(G, d\mu)$ we have

$$[\phi_j * f]_r(h) = \int_G \phi_j(g)f(hg) d\mu(g) \rightarrow f(h), \text{ then } j \rightarrow \infty.$$

Then for $k(g) \in H$ and any $h' \in G$ the right convolution

$$[\phi_j * k]_r(hh') = \int_G \phi_j(g)k(hh'g) d\mu(g) = \int_G \phi_j(h'^{-1}g')k(hg') d\mu(g'), \quad g' = h'g,$$

is tending to $k(hh')$ whence by the closedness of H (as a right ideal), $k(hh') \in H$. ■

Lemma 2.7 (The reproducing property) For any $f(g) \in H_2(G, d\mu)$ we have

$$[f * f_0]_l(g) = f(g) \tag{2.9}$$

$$[\bar{f}_0 * f]_r(g) = f(g), \tag{2.10}$$

where $f_0(g) = \langle f_0, T_g f_0 \rangle$ is the function corresponding to the vacuum vector $f_0 \in H$.

Proof. We again check only the left case. A simple calculation yields :

$$\begin{aligned} [f * f_0]_l(h) &= \int_G f(g) f_0(g^{-1}h) d\mu(g) \\ &= \int_G f(g) \langle f_0, T_{g^{-1}h} f_0 \rangle d\mu(g) \\ &= \int_G \langle f, T_g f_0 \rangle \langle f_0, T_{g^{-1}h} f_0 \rangle d\mu(g) \\ &= \int_G \langle f, T_g f_0 \rangle \langle T_g f_0, T_h f_0 \rangle d\mu(g) \\ &= \int_G \langle f, T_g f_0 \rangle \overline{\langle T_h f_0, T_g f_0 \rangle} d\mu(g) \\ &\stackrel{(*)}{=} \langle f, T_h f_0 \rangle \\ &= f(h). \end{aligned}$$

The transition (*) is based on (2.3) and the unitary property of the representation T . ■

The following general Theorem easily follows from the previous Lemmas.

Theorem 2.8 *The orthogonal projector $P : L_2(G, d\mu) \rightarrow H_2(G, d\mu)$ is a right convolution on G with the kernel $\bar{f}_0(g)$ defined by the vacuum vector.*

Alluding to Archimedes we can say : *let a representation of the group G on H with a cyclic vector f_0 be given, then we may construct the projector $P : L_2(G, d\mu) \rightarrow H_2(G, d\mu) \cong H$.*

Proof. Let P be the operator of right convolution (2.6) with the kernel $\bar{f}_0(g)$. By Lemma 2.4 $H_2(G, d\mu)$ is an invariant linear subspace of $L_2(G, d\mu)$. Thus by Lemma 2.6 it is an ideal under convolution operators. Therefore the convolution operator P with the kernel $\bar{f}_0(g)$ from $H_2(G, d\mu)$ has an image belonging to $H_2(G, d\mu)$. But by Lemma 2.7 $P = I$ on $H_2(G, d\mu)$, so $P^2 = P$ on $L_2(G, d\mu)$, i. e. P is projector on $H_2(G, d\mu)$.

It is easy to see that $f_0(g)$ has the property $f_0(g) = \bar{f}_0(-g)$, thus $P^* = P$, i. e. P is orthogonal. Remark that the orthogonality of the operator P may also be shown in the following way. Let $f(g) \in L_2(G, d\mu)$ be orthogonal to all functions from $H_2(G, d\mu)$. In particular $f(g)$ should be orthogonal to $f_0(h^{-1}g)$ (due to the invariance of $H_2(G, d\mu)$) for any $h \in G$. Then $P(f) = [f * f_0]_l = 0$ and we have shown the orthogonality again. This completes the proof. ■

REMARK 2.9 The stated left invariance of $H_2(G, d\mu)$ and the representation of P as a right group convolution have a useful connection with differential equations. Indeed, let X_j , $j \leq m$ be a left-invariant vector field (i. e. left-invariant differential operators) on G . If $X_j f_0 \equiv 0$ then $X_j f = 0$ for any $f \in H$. Thus the space $H_2(G, d\mu)$ may be characterized as the space of solutions to the system of equations $X_j f = 0$, $1 \leq j \leq m$.

Another way of formulating reads : Think of P as an integral operator with the kernel $K(h, g) = f_0(g^{-1}h) = \overline{f_0(h^{-1}g)} = \langle T_g f_0, T_h f_0 \rangle$. Then the kernel $K(h, g)$ is an *analytic* function of h and *anti-analytic* of g .

Example 2.10 An important class of applications may be treated as follows. Let us have a space of functions defined on a domain $\Omega \in \mathbb{R}^n$ and let us also have a transitive Lie group G of automorphisms of Ω . Then we can construct a unitary representation T of G on $L_2(\Omega)$ by the formula:

$$T_g : f(x) \mapsto f(g(x)) J_g^{1/2}(x), \quad f(x) \in L_2(\Omega), \quad g \in G, \quad (2.11)$$

where $J_g(x)$ is the Jacobian of the transformation defined by g at the point x .

If we fix some point $x_0 \in \Omega$ then we can identify the homogeneous space G/G_{x_0} with Ω (see [13, § I.5]). Then left-invariant vector fields on G may be considered as differential operators on Ω and convolution operators on G as integral operators on Ω . This is a way of obtaining *integral representations for analytic functions*.

3 Classical Results

We would like to show how the abstract Theorem 2.8 and Example 2.10 are connected with classical results on the Bargmann, Bergman and Szegő projectors on, respectively, the Segal–Bargmann (Fock), the Bergman and the Hardy spaces. We will start from a trivial example.

Corollary 3.1 *Let $\{\phi_j\}$, $-\infty < j < \infty$ be an orthonormalized basis of a Hilbert space H . Then*

$$B = \sum_{j=-\infty}^{\infty} |\phi_j\rangle\langle\phi_j| \tag{3.1}$$

is a reproducing operator, namely, $Bf = f$ for any $f \in H$.

Proof. We will construct a unitary representation of the group \mathbb{Z} on H by its action on the basis:

$$T_k\phi_j = \phi_{j+k}, \quad k \in \mathbb{Z}.$$

If we equip \mathbb{Z} with the invariant discrete measure $d\mu(k) = 1$ and select the vacuum vector $f_0 = \phi_0$, then all coherent states are exactly the basis vectors: $f_k = T_k\phi_0 = \phi_k$. Equation (2.3) turns out to be exactly the Plancherel formula

$$\langle f_1, f_2 \rangle = \sum_{j=-\infty}^{\infty} \langle f_1, T_j f_0 \rangle \overline{\langle f_2, T_j f_0 \rangle} = \sum_{j=-\infty}^{\infty} \langle f_1, \phi_j \rangle \overline{\langle f_2, \phi_j \rangle}$$

and we have obtained the usual isomorphism of Hilbert spaces $H \cong l_2(\mathbb{Z})$ by the formula $f(k) = \langle f, \phi_k \rangle$. Our construction gives

$$\begin{aligned} f(k) &= \sum_{j=-\infty}^{\infty} \overline{\langle f_0, T_j f_0 \rangle} \langle f, T_{j+k} f_0 \rangle \\ &= \sum_{j=-\infty}^{\infty} \delta_{0j} \langle f, \phi_{j+k} \rangle \\ &= \langle f, \phi_k \rangle. \end{aligned}$$

Thus the operator B is the identity operator on H . Note that a similar construction may be given in the case of a non orthonormalized frame. ■

In spite of the simplicity of this construction, it was the (almost) unique tool to establish various projectors (see [15, 3.1.4]). The following less trivial Corollaries give a new glance on classical results.

Corollary 3.2 [1] *The Bargmann projector on the Segal–Bargmann space has the kernel*

$$K(z, w) = e^{\bar{w}(z-w)}. \tag{3.2}$$

Proof. Let us define a unitary representation of the Heisenberg group \mathbb{H}^n on \mathbb{R}^n by the formula [18, § 1.1]:

$$g = (t, q, p) : f(x) \rightarrow T_{(t,q,p)}f(x) = e^{i(2t - \sqrt{2}qx + qp)} f(x - \sqrt{2}p).$$

As “vacuum vector” we will select the original *vacuum vector* $f_0(x) = e^{-x^2/2}$. Then the embedding $L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{H}^n)$ is given by the formula

$$\begin{aligned} \tilde{f}(g) &= \langle f, T_g f_0 \rangle \\ &= \pi^{-n/4} \int_{\mathbb{R}^n} f(x) e^{-i(2t - \sqrt{2}qx + qp)} e^{-(x - \sqrt{2}p)^2/2} dx \\ &= e^{-2it - (p^2 + q^2)/2} \pi^{-n/4} \int_{\mathbb{R}^n} f(x) e^{-((p+iq)^2 + x^2)/2 + \sqrt{2}(p+iq)x} dx \\ &= e^{-2it - z\bar{z}/2} \pi^{-n/4} \int_{\mathbb{R}^n} f(x) e^{-(z^2 + x^2)/2 + \sqrt{2}zx} dx, \end{aligned} \tag{3.3}$$

where $z = p + iq$, $g = (t, p, q) = (t, z)$. Then $\tilde{f}(g)$ belongs to $L_2(\mathbb{H}^n, dg)$. It is easy to see that for every fixed t_0 the function $\check{f}(z) = e^{z\bar{z}/2} \tilde{f}(t_0, z)$ belongs to the Segal-Bargmann space i.e. is analytic in z and square-integrable with respect the Gaussian measure $\pi^{-n} e^{-z\bar{z}}$. The integral in (3.3) is the well known Bargmann transform [1]. Then the projector $L_2(\mathbb{H}^n, dg) \rightarrow L_2(\mathbb{R}^n)$ is a convolution on \mathbb{H}^n with kernel

$$\begin{aligned} P(t, q, p) &= \langle f_0, T_g f_0 \rangle \\ &= \pi^{-n/4} \int_{\mathbb{R}^n} e^{-x^2/2} e^{-i(2t - \sqrt{2}qx + qp)} e^{-(x - \sqrt{2}p)^2/2} dx \\ &= \pi^{-n/4} \int_{\mathbb{R}^n} e^{-x^2/2 - i2t + \sqrt{2}iqx - iqp - x^2/2 + \sqrt{2}px - p^2} dx \\ &= e^{-i2t - (p^2 + q^2)/2} \pi^{-n/4} \int_{\mathbb{R}^n} e^{-(x - (p+iq)/\sqrt{2})^2} dx \\ &= \pi^{n/4} e^{-i2t - z\bar{z}/2}. \end{aligned} \tag{3.4}$$

Let us write down the convolution with the obtained kernel explicitly :

$$\begin{aligned} [P_n f](z) &= (2\pi)^{-n-1/2} \int_{\mathbb{R}} \int_{\mathbb{C}^n} 2^{n+1/2} e^{-(t^2 + 1 + \zeta\bar{\zeta})/2} \\ &\quad \times e^{-i(t \cdot 2I + \sum_{k=1}^n (\zeta'_j X_j^{f'} + \zeta''_j X_j^{f''}))} f(z) dt d\zeta \\ &= \pi^{-n-1/2} \int_{\mathbb{R}} e^{-(t^2 + 1 + 2it)} dt \int_{\mathbb{C}^n} e^{-(\zeta\bar{\zeta})/2} e^{-i \sum_{k=1}^n (\zeta'_j X_j^{f'} + \zeta''_j X_j^{f''})} f(z) d\zeta \\ &= \pi^{-n} \int_{\mathbb{C}^n} e^{-\zeta\bar{\zeta}/2} e^{-i \sum_{k=1}^n (\zeta'_j X_j^{f'} + \zeta''_j X_j^{f''})} f(z) d\zeta \\ &= \pi^{-n} \int_{\mathbb{C}^n} e^{-\zeta\bar{\zeta}/2} e^{-\sum_{k=1}^n (\zeta'_j \frac{\partial}{\partial z'_j} + \zeta''_j \frac{\partial}{\partial z''_j} - \zeta_j \bar{z}_j)} f(z) d\zeta \\ &= \pi^{-n} \int_{\mathbb{C}^n} e^{-\zeta\bar{\zeta}/2} e^{-\zeta\bar{z}} e^{\zeta z} e^{-\sum_{k=1}^n (\zeta'_j \frac{\partial}{\partial z'_j} + \zeta''_j \frac{\partial}{\partial z''_j})} f(z) d\zeta \\ &= \pi^{-n} \int_{\mathbb{C}^n} e^{(\bar{z} - \bar{\zeta})\zeta} f(z - \zeta) d\zeta \\ &= \pi^{-n} \int_{\mathbb{C}^n} e^{\bar{w}(z-w)} f(w) dw, \end{aligned} \tag{3.5}$$

where $X_j^{f'}$ and $X_j^{f''}$ are the following vector fields representing the Heisenberg group [11]:

$$X_j^{f'} = \frac{1}{i} \left(\frac{\partial}{\partial z'_j} - z'_j + iz''_j \right), \quad X_j^{f''} = \frac{1}{i} \left(\frac{\partial}{\partial z''_j} - z''_j - iz'_j \right), \tag{3.6}$$

This shows that a convolution with kernel (3.4) defines the Bargmann projector with kernel (3.2). ■

Corollary 3.3 [15, 3.1.2] *The orthogonal Bergman projector on the space of square integrable analytic functions on the unit ball $\mathbb{B} \subset \mathbb{C}^n$ has the kernel*

$$K(\zeta, v) = (1 - \langle \zeta, v \rangle)^{-n-1},$$

where $\langle \zeta, v \rangle = \sum_1^n \zeta_j \bar{v}_j$ is the scalar product on \mathbb{C}^n .

Proof. We will only rewrite material of Chapters 2 and 3 from [15] using our vocabulary. The group of biholomorphic automorphisms $\text{Aut}(\mathbb{B})$ of the unit ball \mathbb{B} acts on \mathbb{B} transitively. For any $\phi \in \text{Aut}(\mathbb{B})$ there is a unitary operator associated by (2.11) and defined by the formula [15, 2.2.6(i)]:

$$[T_\phi f](\zeta) = f(\phi(\zeta)) \left(\frac{\sqrt{1 - |\alpha|^2}}{1 - \langle \zeta, \alpha \rangle} \right)^{n+1}, \tag{3.7}$$

where $\zeta \in \mathbb{B}$, $\alpha = \phi^{-1}(0)$, $f(\zeta) \in L_2(\mathbb{B})$. The operator T_ϕ from (3.7) obviously preserves the space $H_2(G)$ of square-integrable holomorphic functions on \mathbb{B} . The homogeneous space $\text{Aut}(\mathbb{B})/G_0$ may be identified with \mathbb{B} [15, 2.2.5]. To distinguish points of these two sets we will denote points of $B = \text{Aut}(\mathbb{B})/G_0 \cong \mathbb{B}$ by Roman letters (like a, u, z) and points of \mathbb{B} itself by Greek letters (α, v, ζ correspondingly). We also always assume that $a = \alpha, u = v, z = \zeta$ under the mentioned identification.

We select the function $f_0(\zeta) \equiv 1$ as the vacuum vector. The mean value formula [15, 3.1.1(2)] gives us:

$$\begin{aligned} \tilde{f}(a) &= \langle f(\zeta), T_\phi f_0 \rangle \\ &= \langle T_{\phi^{-1}} f(\zeta), f_0 \rangle \\ &= \int_{\mathbb{B}} f(\phi(\zeta)) \left(\frac{\sqrt{1 - |\alpha|^2}}{1 - \langle \zeta, \alpha \rangle} \right)^{n+1} d\nu(\zeta) \\ &= f(\phi(0)) \left(\frac{\sqrt{1 - |\alpha|^2}}{1 - \langle 0, \alpha \rangle} \right)^{n+1} \\ &= f(a) (\sqrt{1 - |\alpha|^2})^{n+1}, \end{aligned} \tag{3.8}$$

where $a = \alpha = \phi(0)$, $\phi \in B$ and $\tilde{f}(a) \in L_2(B)$. In particular

$$\begin{aligned} \tilde{f}_0(a) &= (\sqrt{1 - |\alpha|^2})^{n+1}, \\ \tilde{f}_0(zu) &= \left(\frac{\sqrt{1 - |\zeta|^2} \sqrt{1 - |v|^2}}{1 - \langle v, \zeta \rangle} \right)^{n+1}. \end{aligned} \tag{3.9}$$

An invariant measure on B is given [15, 2.2.6(2)] by the expression:

$$d\mu(z) = \frac{d\nu(\zeta)}{(1 - |\zeta|^2)^{n+1}}, \tag{3.10}$$

where $d\nu(\zeta)$ is the usual Lebesgue measure on $B \cong \mathbb{B}$. We will substitute the expressions from (3.8), (3.9) and (3.10) into the reproducing formula (2.9):

$$\tilde{f}(u) = f(v)(\sqrt{1-|v|^2})^{n+1} \tag{3.11}$$

$$= \int_B \tilde{f}(z)\tilde{f}_0(z^{-1}u) d\mu(z)$$

$$\stackrel{(*)}{=} \int_B \tilde{f}(z)\tilde{f}_0(zu) d\mu(z)$$

$$= \int_{\mathbb{B}} f(\zeta)(\sqrt{1-|\zeta|^2})^{n+1} \left(\frac{\sqrt{1-|\zeta|^2}\sqrt{1-|v|^2}}{1-\langle v, \zeta \rangle} \right)^{n+1} \frac{d\nu(\zeta)}{(1-|\zeta|^2)^{n+1}}$$

$$= \int_{\mathbb{B}} f(\zeta) \left(\frac{\sqrt{1-|v|^2}}{1-\langle v, \zeta \rangle} \right)^{n+1} d\nu(\zeta)$$

$$= (\sqrt{1-|v|^2})^{n+1} \int_{\mathbb{B}} \frac{f(\zeta)}{(1-\langle v, \zeta \rangle)^{n+1}} d\nu(\zeta). \tag{3.12}$$

Here the transition (*) is possible because every element of B is an involution [15, 2.2.2(v)]. It immediately follows from the comparison of (3.11) and (3.12) that

$$f(v) = \int_{\mathbb{B}} \frac{f(\zeta)}{(1-\langle v, \zeta \rangle)^{n+1}} d\nu(\zeta).$$

The last formula is the integral representation with the Bergman kernel for square integrable holomorphic functions on the unit ball in \mathbb{C}^n . ■

Corollary 3.4 [8] *The orthogonal projector Szegö on the boundary $b\mathbb{U}^n$ of the upper half-space in \mathbb{C}^{n+1} has the kernel*

$$S(z, w) = \left(\frac{i}{2}(\bar{w}_{n+1} - z_{n+1}) - \sum_{j=1}^n z_j \bar{w}_j \right)^{-n-1}.$$

Proof. It is well known [8, 9, 17], that there is a unitary representation of the Heisenberg group \mathbb{H}^n as the simply transitive group of shifts acting on $b\mathbb{U}^n$ (see [17, Chap. XII, § 1.4]):

$$(\zeta, t) : (z', z_{n+1}) \mapsto (z' + \zeta, z_{n+1} + t + 2i\langle z', \zeta \rangle + i|\zeta|^2), \tag{3.13}$$

where $(\zeta, t) \in \mathbb{H}^n, z = (z', z_{n+1}) \in \mathbb{C}^{n+1}, \zeta, z' \in \mathbb{C}^n, t \in \mathbb{R}$. We again apply the general scheme from Example 2.10. This gives an identification of \mathbb{H}^n and $b\mathbb{U}^n$ and \mathbb{H}^n acts on $b\mathbb{U}^n \cong \mathbb{H}^n$ by left group shifts. Left invariant vector fields are exactly the *tangential Cauchy-Riemann equations* for holomorphic functions on \mathbb{U}^n . The shifts (3.13) commute with the tangential Cauchy-Riemann equations and thus preserve the Hardy space $H_2(b\mathbb{U}^n)$ of boundary values of functions holomorphic on \mathbb{U}^n .

As vacuum vector we select the function $f_0(z) = (iz_{n+1})^{-n-1} \in H_2(b\mathbb{U}^n)$. Then the Szegö projector $P : l_2(b\mathbb{U}^n) \rightarrow H_2(b\mathbb{U}^n)$ is the right convolution on $\mathbb{H}^n \cong b\mathbb{U}^n$ with $f_0(z)$ and thus should have the kernel (see the group law formula (1.1) for \mathbb{H}^n)

$$S(z, w) = \left(\frac{i}{2}(\bar{w}_{n+1} - z_{n+1}) - \sum_{j=1}^n z_j \bar{w}_j \right)^{-n-1}.$$

The reader may ask *why we have selected this vacuum vector* ? The answer is : for reason of *simplicity*. Indeed, the *Cayley transform* ([17, Chap. XII, § 1.2] and [15, § 2.3])

$$C(z) = i \frac{e_{n+1} + z}{1 - z_{n+1}}, \quad e_{n+1} = (0, \dots, 0, 1) \in \mathbb{C}^{n+1} \quad (3.14)$$

establishes a biholomorphic map from the unit ball $\mathbb{B} \subset \mathbb{C}^{n+1}$ to the domain \mathbb{U}^n . We can construct an isometrical isomorphism of the Hilbert spaces $H_2(\mathbb{S}^{2n+1})$ and $H_2(\mathbb{U}^n)$ based on (3.14)

$$f(z) \mapsto [Cf](z) = f(C(z)) \frac{-2i^{n+1}z_{n+1}}{(1 - z_{n+1})^{n+2}}, \quad f \in H_2(\mathbb{U}^n), \quad [Cf] \in H_2(\mathbb{S}^{2n+1}). \quad (3.15)$$

Then the vacuum vector $f_0 = (iz_{n+1})^{-n-1}$ is the image of the function $\tilde{f}_0(w) = (-2i/(w - i))^{n+2} \in H_2(\mathbb{S}^{2n+1})$ under the transformation (3.15). It seems to be one of the simplest functions from $H_2(\mathbb{S}^{2n+1})$ with singularities on \mathbb{S}^{2n+1} . ■

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