A class of Buekenhout unitals in the Hall plane

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Abstract

Let U be the classical unital in $PG(2,q^2)$ secant to ℓ_∞. By deriving $PG(2,q^2)$ with respect to a derivation set disjoint from U we obtain a new unital U' in the Hall plane of order q^2. We show that this unital contains O'Nan configurations and is not isomorphic to the known unitals of the Hall plane, hence it forms a new class of unitals in the Hall plane.

1 Introduction

A **unital** is a 2-$(q^3+1, q+1, 1)$ design. A unital embedded in a projective plane of order q^2 is a set U of q^3+1 points such that every line of the plane meets U in 1 or $q+1$ points. A line is a **tangent line** or a **secant line** of U if it contains 1 or $q+1$ points of U respectively. A point of U lies on 1 tangent and q^2 secant lines of U. A point not in U lies on $q+1$ tangent lines and q^2-q secant lines of U.

An example of a unital in $PG(2,q^2)$, the Desarguesian projective plane of order q^2, is the **classical unital** which consists of the absolute points and non-absolute lines of a unitary polarity. It is well known that the classical unital contains no **O'Nan configurations**, a configuration of four distinct lines meeting in six distinct points (a quadrilateral). In 1976 Buekenhout [4] proved the existence of unitals in all translation planes of dimension at most 2 over their kernel.

Let U be the classical unital in $PG(2,q^2)$ secant to ℓ_∞. We derive $PG(2,q^2)$ with respect to a derivation set disjoint from U. Let U' be the set of points of $\mathcal{H}(q^2)$,

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the Hall plane of order q^2, that corresponds to the point set of U. We prove the following results about U'.

Theorem 1 *The set U' forms a Buekenhout unital with respect to ℓ'_∞ in $\mathcal{H}(q^2)$.*

Theorem 2 *The unital U' contains no O'Nan configurations with two or three vertices on ℓ'_∞. If U' contains an O'Nan configuration l_1, l_2, l_3, l_4 with one vertex $T = l_1 \cap l_2$ on ℓ'_∞, then the lines $\overline{l_1} = l_1 \cap U'$ and $\overline{l_2} = l_2 \cap U'$ are disjoint in $PG(4, q)$.*

Theorem 3 *If $q > 5$, the unital U' contains O'Nan configurations. If H is a point of $U' \setminus \ell'_\infty$ and l a secant of U' through H that meets the classical derivation set, then there is an O'Nan configuration of U' that contains H and l.*

The only unitals previously investigated in the Hall plane is a class of Buekenhout unitals found by Grüning [5] by deriving $PG(2, q^2)$ with respect to $U \cap \ell_\infty$. We show that the class of unitals U' is not isomorphic to Grüning's unital and so forms a new class of unitals in $\mathcal{H}(q^2)$. In [2] the Buekenhout and Buekenhout-Metz unitals of the Hall plane are studied.

2 Background Results

We will make use of the André [1] and Bruck and Bose [3] representation of a translation plane $\mathcal{P}$ of dimension 2 over its kernel in $PG(4, q)$. The results of this section are discussed in [6, Section 17.7]. Let Σ_∞ be a hyperplane of $PG(4, q)$ and $\mathcal{S}$ a spread of Σ_∞. The affine plane $\mathcal{P} \setminus \ell_\infty$ can be represented by the affine space $PG(4, q) \setminus \Sigma_\infty$ as follows: the points of $\mathcal{P} \setminus \ell_\infty$ are the points of $PG(4, q) \setminus \Sigma_\infty$, the lines of $\mathcal{P} \setminus \ell_\infty$ are the planes of $PG(4, q)$ that meet Σ_∞ in a line of $\mathcal{S}$ and incidence is the natural inclusion. We complete the representation to a projective space by letting points of ℓ_∞ correspond to lines of the spread $\mathcal{S}$. Note that $\mathcal{P}$ is Desarguesian if and only if the spread $\mathcal{S}$ is regular.

We use the phrase **a subspace of** $PG(4, q) \setminus \Sigma_\infty$ to mean a subspace of $PG(4, q)$ that is not contained in Σ_∞.

Under this representation, Baer subplanes of $\mathcal{P}$ secant to ℓ_∞ (that is, meeting ℓ_∞ in $q + 1$ points) correspond to planes of $PG(4, q)$ that are not contained in Σ_∞ and do not contain a line of the spread $\mathcal{S}$. Baer sublines of $\mathcal{P}$ meeting ℓ_∞ in a point T correspond to lines of $PG(4, q)$ that meet Σ_∞ in a point of t, the line of $\mathcal{S}$ that corresponds to T. A Baer subplane tangent to ℓ_∞ at T corresponds to a ruled cubic surface of $PG(4, q)$ that consists of $q + 1$ lines of $PG(4, q) \setminus \Sigma_\infty$, each incident with the line t of $\mathcal{S}$ and such that no two are contained in a plane about t, [9].

Let U be the classical unital in $PG(2, q^2)$ secant to ℓ_∞. Buekenhout [4] showed that the set of points $\mathcal{U}$ in $PG(4, q)$ corresponding to U forms a non-singular quadric that meets the underlying spread in a regulus. If U is a unital of a translation plane $\mathcal{P}$ of dimension at most 2 over its kernel and $\mathcal{U}$ corresponds to a non-singular quadric of $PG(4, q)$ that contains a regulus of the underlying spread, then U is called a **Buekenhout unital** with respect to ℓ_∞. Note that the classical unital is Buekenhout with respect to any secant line.

Let $PG(2, q^2)$ be the Desarguesian plane of order q^2 and let $\mathcal{D} = \{T_0, \ldots T_q\}$ be a derivation set of ℓ_∞. Deriving $PG(2, q^2)$ with respect to $\mathcal{D}$ gives the Hall plane of order q^2, $\mathcal{H}(q^2)$ (see [8]). The affine points of $\mathcal{H}(q^2)$ are the affine points of $PG(2, q^2)$. The affine lines of $\mathcal{H}(q^2)$ are the lines of $PG(2, q^2)$ not meeting $\mathcal{D}$ together with the

Baer subplanes of $PG(2,q^2)$ that contain $\mathcal{D}$. The line at infinity of $\mathcal{H}(q^2)$ consists of the points of $\ell_\infty \setminus \mathcal{D}$ and $q+1$ new points, $\mathcal{D}' = \{R_0,\dots,R_q\}$. If we derive $\mathcal{H}(q^2)$ with respect to $\mathcal{D}'$, we get $PG(2,q^2)$. The Hall plane contains other derivation sets of ℓ'_∞, we call $\mathcal{D}'$ the **classical derivation set of** $\mathcal{H}(q^2)$.

Let Σ_∞ be a hyperplane of $PG(4,q)$ and let $\mathcal{S}$ be the regular spread of Σ_∞ that generates $PG(2,q^2)$. Let $\mathcal{R} = \{t_0,\dots,t_q\}$ be the regulus of $\mathcal{S}$ that corresponds to $\mathcal{D}$. The spread $\mathcal{S}'$ obtained from $\mathcal{S}$ by replacing the regulus $\mathcal{R}$ with its complementary regulus $\mathcal{R}' = \{r_0,\dots,r_q\}$ (that is, $\mathcal{S}' = \mathcal{S}\setminus\mathcal{R}\cup\mathcal{R}'$) generates the Hall plane $\mathcal{H}(q^2)$.

We use the following notation throughout this paper: we denote the lines at infinity of $PG(2,q^2)$ and $\mathcal{H}(q^2)$ by ℓ_∞ and ℓ'_∞ respectively; if $\mathcal{D}$ is the derivation set used to derive $PG(2,q^2)$ to give $\mathcal{H}(q^2)$, then we denote by $\mathcal{D}'$ the classical derivation set of ℓ'_∞; in $PG(4,q)$, we denote the spreads of Σ_∞ that generate $PG(2,q^2)$ and $\mathcal{H}(q^2)$ by $\mathcal{S}$ and $\mathcal{S}'$ respectively. If T is a point of ℓ_∞, we denote the corresponding line of the spread in $PG(4,q)$ by t. Let $\mathcal{U}$ be the non-singular quadric of $PG(4,q)$ that corresponds to U and U'.

3 The Buekenhout unital

Let U be the classical unital in $PG(2,q^2)$ that is secant to ℓ_∞. Derive the plane using a derivation set $\mathcal{D}$ that is disjoint from U. Let U' be the set of points in $\mathcal{H}(q^2)$ that corresponds to the point set of U.

Theorem 1 *The set U' forms a unital in $\mathcal{H}(q^2)$.*

Proof The set U' contains q^3+1 points of $\mathcal{H}(q^2)$. We show that every line of $\mathcal{H}(q^2)$ meets U' in either 1 or $q+1$ points, from which it follows that U' is a unital. Clearly ℓ'_∞ meets U' in $q+1$ points since ℓ_∞ meets U in $q+1$ points.

Let l be a line of $\mathcal{H}(q^2)$ that meets ℓ'_∞ in the point T. If T is not in the derivation set $\mathcal{D}'$, then the points of l lie on a line of $PG(2,q^2)$ and so l contains 1 or $q+1$ points of U. Hence l contains 1 or $q+1$ points of U'.

Suppose T is in the derivation set $\mathcal{D}'$, then $T \notin U'$. Let $\mathcal{U}$ be the non-singular quadric of $PG(4,q)$ that corresponds to U, then $\mathcal{U}$ also corresponds to U'. Let α be the plane that corresponds to the line l, so $\alpha\cap\Sigma_\infty = t$, the line of the spread corresponding to T. Now α meets $\mathcal{U}$ in either a point, a line, a conic or two lines. If $\alpha\cap\mathcal{U}$ contains a line, then t contains a point of $\mathcal{U}$ which is a contradiction as $T\notin U'$. Thus α meets $\mathcal{U}$ in either 1 or $q+1$ points and so l meets U' in either 1 or $q+1$ points. Note that if l is secant to U', then the $q+1$ points $l\cap U'$ are not collinear in $PG(2,q^2)$; they form a conic in the Baer subplane that corresponds to l and a $(q+1)$-arc of $PG(2,q^2)$. □

The proof of this theorem shows in fact that U' is a Buekenhout unital with respect to ℓ'_∞ in $\mathcal{H}(q^2)$. We will show that the designs U and U' are not isomorphic by constructing an O'Nan configuration in U'. We first investigate whether U' can contain an O'Nan configuration with a vertex on ℓ'_∞.

Theorem 2 *The unital U' contains no O'Nan configurations with two or three vertices on ℓ'_∞. If U' contains an O'Nan configuration l_1,l_2,l_3,l_4 with one vertex $T=l_1\cap l_2$ on ℓ'_∞, then the lines $\overline{l_1} = l_1\cap U'$ and $\overline{l_2} = l_2\cap U'$ are disjoint in $PG(4,q)$.*

Proof Suppose U' contains an O'Nan configuration with two or three vertices on ℓ'_∞. Such a configuration consists of four lines that each meet ℓ'_∞ in a point of U'. Since the derivation set is disjoint from the unital, these lines are also secants of U giving an O'Nan configuration in the classical unital U, which is a contradiction. Thus U' cannot contain an O'Nan configuration with two or three vertices on ℓ'_∞.

Suppose U' contains an O'Nan configuration with lines l_1, l_2, l_3, l_4 that has one vertex $T = l_1 \cap l_2$ on ℓ'_∞. Let the vertices of the O'Nan configuration be $A = l_1 \cap l_3, B = l_2 \cap l_3, C = l_3 \cap l_4, X = l_1 \cap l_4, Y = l_2 \cap l_4$, and T.

We use the representation of $\mathcal{H}(q^2)$ in $PG(4,q)$ and let $\mathcal{U}$ be the non-singular quadric corresponding to U'. Since U' is Buekenhout with respect to ℓ'_∞, the sets $\overline{l_1} = l_1 \cap U'$ and $\overline{l_2} = l_2 \cap U'$ are Baer sublines of $\mathcal{H}(q^2)$ and correspond to lines of $PG(4,q)$ that meet Σ_∞ in a point of t (the line of $\mathcal{S}'$ that corresponds to T). These lines either meet in a point of t or they are disjoint in $PG(4,q)$.

Suppose the lines $\overline{l_1}$ and $\overline{l_2}$ meet in a point of t in $PG(4,q)$, then they are contained in a unique plane α of $PG(4,q)$ that does not contain a line of the spread (if the plane contained t, it would meet $\mathcal{U}$ in $3q+1$ points which is impossible). Now $X, Y, A, B \in \alpha$, hence $XY \cap AB = C \in \alpha$, thus $C \notin \mathcal{U}$, as α already contains $2q+1$ points of $\mathcal{U}$. Hence, in $\mathcal{H}(q^2)$, $l_3 \cap l_4 = C \notin U$, a contradiction. Hence $\overline{l_1} = l_1 \cap U'$ and $\overline{l_2} = l_2 \cap U'$ are disjoint in $PG(4,q)$. □

In order to prove the existence of O'Nan configurations in U' we will need several preliminary lemmas.

We need to know what a conic in a Baer subplane $\mathcal{B}$ of $PG(2,q^2)$ looks like in the Bruck and Bose representation in $PG(4,q)$. If $\mathcal{B}$ is secant to ℓ_∞, then $\mathcal{B}$ corresponds to a plane α of $PG(4,q)$ and the points of a conic in $\mathcal{B}$ form a conic of α. If $\mathcal{B}$ is tangent to ℓ_∞, then $\mathcal{B}$ corresponds to a ruled cubic surface in $PG(4,q)$. The following lemma shows that certain conics of these Baer subplanes correspond to $(q+1)$-arcs of $PG(3,q)$ in $PG(4,q)$ (that is, $q+1$ points lying in a three dimensional subspace of $PG(4,q)$, with no four points coplanar).

Lemma A *Let $\mathcal{B}$ be a Baer subplane of $PG(2,q^2)$ that meets ℓ_∞ in the point T. Let $\mathcal{C} = \{T, K_1, \ldots, K_q\}$ be a conic of $\mathcal{B}$. In the Bruck and Bose representation of $PG(2,q^2)$ in $PG(4,q)$, the points $K_1, \ldots, K_q$ form a q-arc of a three dimensional subspace of $PG(4,q)$.*

Proof In $PG(4,q)$, $\mathcal{B}$ corresponds to a ruled cubic surface $\mathcal{V}$ that meets Σ_∞ in the line t of the spread. The points of $\mathcal{V}$ lie on $q+1$ disjoint lines of $\mathcal{V}$, $l_1, \ldots, l_{q+1}$, called **generators**. Each generator meets Σ_∞ in a distinct point of t. The lines of $\mathcal{B}$ not through T correspond to conics of $\mathcal{V}$ in $PG(4,q)$. We label the points of $\mathcal{C}$ so that the point K_i lies on the line l_i, $i = 1, \ldots, q$ (since at most one point of $\mathcal{C} \setminus T$ lies on each l_i).

Suppose that l_1, l_2, l_3 span a three dimensional subspace Σ_1. Let X be a point of $\mathcal{B}$ not incident with l_1, l_2 or l_3, then a line l of $\mathcal{B}$ through X meets each of l_1, l_2, l_3. Now l corresponds to a conic in $PG(4,q)$ with three points in Σ_1, hence all points of l lie in Σ_1. Thus every point of $\mathcal{V}$ lies in Σ_1, a contradiction as $\mathcal{V}$ spans four dimensional space. Hence no three of the l_i lie in a three dimensional subspace.

As a consequence of this, if A, B, C are points of $\mathcal{V}$ lying on different generators l_1, l_2, l_3 of $\mathcal{V}$, then A, B, C are not collinear in $PG(4,q)$. Since, suppose A, B, C lie on a line m of $PG(4,q)$, then m and t span a three dimensional subspace which contains two points of each l_i, $i=1,2,3$ and so contains three generators l_1,l_2,l_3 of $\mathcal{V}$, a contradiction.

We now show that in $PG(4,q)$, no four of the K_i lie in a plane. Suppose K_1, K_2, K_3, K_4 lie in a plane α, then α corresponds to an affine Baer subplane $\mathcal{B}'$ of $PG(2,q^2)$ (since no three of the K_i lie on a line of $PG(2,q^2)$). However K_1,K_2,K_3,K_4 form a quadrangle of $PG(2,q^2)$ and so are contained in a unique Baer subplane of $PG(2,q^2)$. This is a contradiction as $\mathcal{B}\neq\mathcal{B}'$. Hence no four of the K_i are coplanar.

Let the three dimensional subspace spanned by K_1,K_2,K_3,K_4 be Σ. Note that Σ meets t in one point and so can contain at most one of the l_i. Suppose one of l_1,l_2,l_3,l_4 lies in Σ, without loss of generality suppose $l_1\in\Sigma$. Let $L_0=l_1\cap t$, $L_1=K_1$, $L_i=l_i\cap\Sigma$, $i=2,\dots,q+1$ (so $L_i=K_i, i=1,2,3,4$).

Note that by the above, no three of the L_i, $i\geq 1$ are collinear in $PG(4,q)$. Since the only lines of $\mathcal{V}$ meeting t are the generators, no three of the L_i, $i\geq 0$ are collinear in $PG(4,q)$. We show that no three of the L_i are collinear in $PG(2,q^2)$. Suppose that L_i, L_j and L_k, $i,j,k>0$, are collinear in $PG(2,q^2)$, then the line l containing them corresponds to a plane β in $PG(4,q)$ which lies in Σ. Now in $PG(2,q^2)$, l contains three points of $\mathcal{B}$, and so l contains $q+1$ points of $\mathcal{B}$. Hence l contains a point of each l_i. Thus in $PG(4,q)$, β contains a point of each of l_i, hence β contains $L_2,\dots,L_{q+1}$ as these are the only points of $l_2,\dots,l_{q+1}$ respectively in Σ. However, $L_i=K_i$, $i=2,3,4$, so in $PG(2,q^2)$, the points K_2,K_3,K_4 lie on the line l which is a contradiction as the K_i form a conic of $\mathcal{B}$. Therefore no three of the L_i, $i>0$ are collinear in $PG(2,q^2)$.

If L_i, L_j and $L_0=T$ are collinear in $PG(2,q^2)$, then the line containing them has three points in $\mathcal{B}$ and so has $q+1$ points in $\mathcal{B}$. This is a contradiction as the only lines of $\mathcal{B}$ through T are the generators l_i, and the points L_i and L_j lie on different generators. Therefore, no three of the L_i, $i\geq 0$ are collinear in $PG(2,q^2)$.

Suppose that four of the L_i lie in a plane α of $PG(4,q)$, then α corresponds to a line or an affine Baer subplane of $PG(2,q^2)$. If α corresponds to a line of $PG(2,q^2)$, then four of the L_i are collinear in $PG(2,q^2)$ which is not possible by the above. If α corresponds to an affine Baer subplane of $PG(2,q^2)$, then the L_i cannot form a quadrangle of $PG(2,q^2)$ (as a quadrangle is contained in a unique Baer subplane). Hence three of the L_i must be collinear in $PG(2,q^2)$ which again contradicts the above. Therefore no four of the L_i are coplanar in $PG(4,q)$.

Thus $L_0,L_1,\dots,L_{q+1}$ form a set of $q+2$ points of Σ, no four of them lying in a plane. This is impossible as the maximum size of a k-arc in $PG(3,q)$ is $k=q+1$. Hence l_1 cannot lie in Σ. Similarly $l_2,l_3,l_4\notin\Sigma$. Thus if one of the l_i lie in Σ, then $i\neq 1,2,3,4$.

We now let $l_i\cap\Sigma=L_i$ if $l_i\notin\Sigma$ (so $L_i=K_i$, $i=1,2,3,4$). If $l_i\in\Sigma$, we let $L_i=l_i\cap t$. Using the same arguments as above, no three of the L_i are collinear in $PG(2,q^2)$ and consequently no four of the L_i are coplanar in $PG(4,q)$. Hence the set of points $\mathcal{C}'=\{L_1,\dots,L_{q+1}\}$ satisfy the property that no four of them are coplanar and so $\mathcal{C}'$ is a $(q+1)$-arc of Σ.

Now the set $\mathcal{C}'$ corresponds to a set of $q+1$ points of $\mathcal{B}$ with no three of them collinear (since no three of the L_i are collinear in $PG(2,q^2)$). Moreover,

$\mathcal{C} = \{T, K_1, \dots, K_q\}$ and $\mathcal{C}'$ have five points in common, T, K_1, K_2, K_3, K_4, hence $\mathcal{C} = \mathcal{C}'$. Thus $L_i = K_i$, $i = 1, \dots, q$ and $L_{q+1} = l_{q+1} \cap t$. Hence in $PG(4, q)$, the K_i together with L_{q+1} form a $(q+1)$-arc of a three dimensional subspace Σ and the K_i form a q-arc of Σ. □

Let $\mathcal{U}$ be a non-singular quadric in $PG(4, q)$. A **tangent hyperplane** of $\mathcal{U}$ is a hyperplane that meets $\mathcal{U}$ in a conic cone. Let G be the group of automorphisms of $PG(4, q)$ that fixes $\mathcal{U}$. There are $q^4(q^2+1)$ planes of $PG(4, q)$ that meet $\mathcal{U}$ in a conic. By [7, Theorem 22.6.6], the set of conics of $\mathcal{U}$ acted on by G has two orbits. If q is odd, one orbit contains **internal conics** and the other contains **external conics**. There are $\frac{1}{2}q^3(q-1)(q^2+1)$ internal conics and $\frac{1}{2}q^3(q+1)(q^2+1)$ external conics of $\mathcal{U}$ ([7, Theorem 22.9.1]). If q is even, one orbit consists of **nuclear conics** while the other contains **non-nuclear conics**. There are $q^2(q^2+1)$ nuclear conic and $q^2(q^4-1)$ non-nuclear conics of $\mathcal{U}$ ([7, Theorem 22.9.2]). The next lemma describes how many tangent hyperplanes of $\mathcal{U}$ contain a given conic of $\mathcal{U}$.

Lemma B *1. If q is odd, every internal conic of $\mathcal{U}$ is contained in zero tangent hyperplanes of $\mathcal{U}$, and every external conic of $\mathcal{U}$ is contained in two tangent hyperplanes of $\mathcal{U}$.*

2. If q is even, every nuclear conic of $\mathcal{U}$ is contained in $q+1$ tangent hyperplanes of $\mathcal{U}$, and every non-nuclear conic of $\mathcal{U}$ is contained in one tangent hyperplane of $\mathcal{U}$.

Proof There are q^3+q^2+q+1 tangent hyperplanes of $\mathcal{U}$ ([7, Theorem 22.8.2]) and $\mathcal{U}$ contains q^3+q^2+q+1 points. Since G is transitive on the points of $\mathcal{U}$ ([7, Theorem 22.6.4]), each point of $\mathcal{U}$ is the vertex of exactly one tangent hyperplane.

Let $\mathcal{U}$ meet the hyperplane Σ_∞ in a hyperbolic quadric $\mathcal{H}_3$. Every point V of $\mathcal{H}_3$ is the vertex of a conic cone of $\mathcal{U}$ that meets $\mathcal{H}_3$ in the two lines containing V. This accounts for $(q+1)^2$ of the tangent hyperplanes of $\mathcal{U}$, the remaining q^3-q meet Σ_∞ in a plane that contains a conic of $\mathcal{H}_3$. Suppose q is odd. Let x be the number of tangent hyperplanes containing a given internal conic and let y be the number of tangent hyperplanes containing a given external conic. By counting the number of conics of $\mathcal{U}$ in two ways we deduce that:

$$x|\text{internal conics of } \mathcal{U}| + y|\text{external conics of } \mathcal{U}|$$

$$= |\text{tangent hyperplanes of } \mathcal{U}|.|\text{conics of } \mathcal{U} \text{ in a tangent hyperplane}|.$$

Therefore:

$$\begin{aligned} \frac{1}{2}q^3(q-1)(q^2+1)x + \frac{1}{2}q^3(q+1)(q^2+1)y &= q^3(q^3+q^2+q+1) \\ (y+x)q + (y-x) &= 2q+2. \end{aligned}$$

Equating like powers of q implies that $y = 2$ and $x = 0$. Therefore, every external conic in contained in two tangent hyperplanes of $\mathcal{U}$ and every internal conic is contained in zero tangent hyperplanes of $\mathcal{U}$.

Suppose q is even. Let x be the number of tangent hyperplanes containing a given nuclear conic and let y be the number of tangent hyperplanes containing a given non-nuclear conic. By counting the number of conics of $\mathcal{U}$ in two ways as above, we deduce that:

$$\begin{aligned} q^2(q^2+1)x + q^2(q^4-1)y &= q^3(q^3+q^2+q+1) \\ x + y(q^2-1) &= q^2+q. \end{aligned}$$

This has solution $y=1$, $x=q+1$ in the required range $0 \le x, y \le q+1$. Therefore, every nuclear conic is contained in $q+1$ tangent hyperplanes of $\mathcal{U}$ and every non-nuclear conic is contained in one tangent hyperplane of $\mathcal{U}$. Consequently, a nuclear conic is not contained in any hyperbolic quadrics or elliptic quadrics of $\mathcal{U}$, since the $q+1$ hyperplanes containing it are all tangent hyperplanes of $\mathcal{U}$. □

Lemma C *Let U be the classical unital in $PG(2,q^2)$ and let $\mathcal{B}$ be a Baer subplane of $PG(2,q^2)$, then $\mathcal{B}$ meets U in one point, $q+1$ points of a conic or line of $\mathcal{B}$, or in $2q+1$ points of a line pair of $\mathcal{B}$.*

Proof We work in $PG(4,q)$. Recall that the classical unital is Buekenhout with respect to any secant line and Buekenhout-Metz with respect to any tangent line. Let l be a secant line of U, then in $PG(4,q)$ with l as the line at infinity, U corresponds to a non-singular quadric. All Baer subplanes secant to l correspond to planes of $PG(4,q)$, [5] Any plane of $PG(4,q)$ meets a non-singular quadric in one point, $q+1$ points of a conic or line, or $2q+1$ points of two lines. So in $PG(2,q^2)$, a Baer subplane secant to l meets $\mathcal{U}$ in one point, a conic, a line, or two lines.

If, however, we take a tangent line of U to be our line at infinity, and work in $PG(4,q)$, then U corresponds to an orthogonal cone in $PG(4,q)$. A Baer subplane secant to the line at infinity corresponds to a plane of $PG(4,q)$. A plane of $PG(4,q)$ meets an orthogonal cone in either a point, a line, a conic, or two lines. Thus all Baer subplanes of $PG(2,q^2)$ meet the classical unital in one point, $q+1$ points of a conic or a line, or in a line pair. □

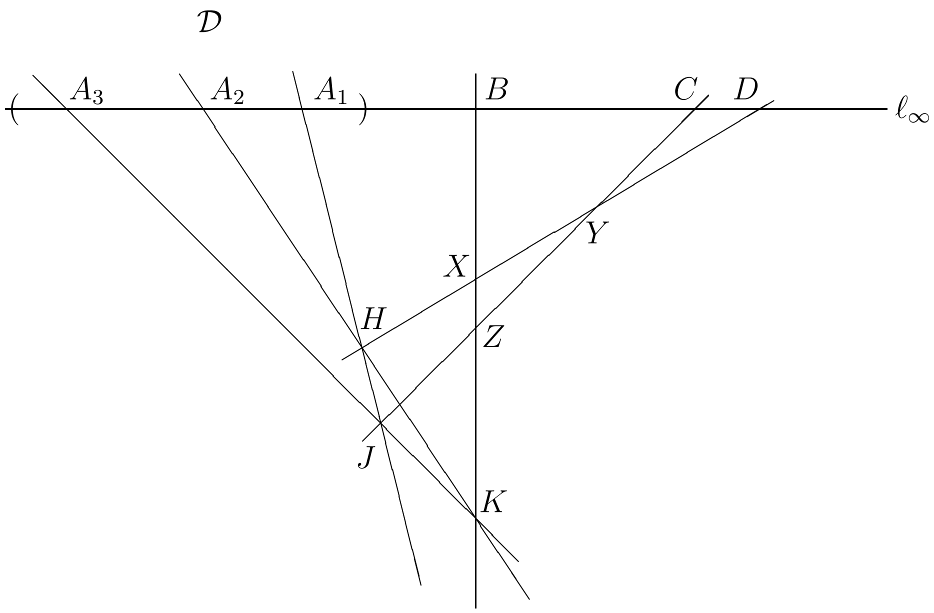
We are now able to show that the unital U' does contain O'Nan configurations. We do this by constructing a configuration in the classical unital that derives to an O'Nan configuration of U'.

Theorem 3 *If $q>5$, the unital U' contains O'Nan configurations. If H is a point of $U' \setminus \ell'_\infty$ and l a secant of U' through H that meets the classical derivation set, then there is an O'Nan configuration of U' that contains H and l.*

Proof We will show that U' contains an O'Nan configuration whose four lines meet ℓ'_∞ in the distinct points A, B, C, D where $A \in \mathcal{D}'$ and $B, C, D \notin \mathcal{D}'$. We prove this by constructing a configuration in the classical unital U in $PG(2,q^2)$ that will derive to an O'Nan configuration of U' in $\mathcal{H}(q^2)$.

The configuration that we construct in U is illustrated in the figure. It consists of six lines l_{A_1}, l_{A_2}, l_{A_3}, l_B, l_C, l_D and six points H, J, K, X, Y, Z of U with intersections as illustrated and such that the line l_* meets ℓ_∞ in the point $*$ where $* \in \{A_1, A_2, A_3, B, C, D\}$ and with $A_1, A_2, A_3 \in \mathcal{D}$ and $B, C, D \notin \mathcal{D}$.

Note that J, K, H, A_1, A_2, A_3 form a quadrangle and so are contained in a unique Baer subplane which contains $\mathcal{D}$ (as A_1, A_2, A_3 are contained in the unique Baer subline $\mathcal{D}$). Hence derivation with respect to $\mathcal{D}$ leaves l_B, l_C and l_D unchanged in $\mathcal{H}(q^2)$ with H, J, K collinear in $\mathcal{H}(q^2)$, giving an O'Nan configuration in U'.



Let U be the classical unital in $PG(2,q^2)$ and $\mathcal{D}$ a derivation set of ℓ_∞ disjoint from U, as above. Let l_{A_1} be a secant line of U that meets $\mathcal{D}$ in the point A_1. Let H and J be two points of U that lie on l_{A_1}. There is a unique Baer subplane $\mathcal{B}$ that contains H, J and $\mathcal{D}$ since a quadrangle is contained in a unique Baer subplane. Now the Baer subline $\mathcal{B} \cap l_{A_1}$ meets U in 0, 1, 2 or $q+1$ points and since $H, J \in U$ and $A_1 \notin U$ we have $\mathcal{B} \cap l_{A_1}$ meets U in two points. By Lemma C, $\mathcal{B}$ meets U in $q+1$ points that form a conic in $\mathcal{B}$, as $\mathcal{D}$ is disjoint from U. Denote the points of the conic $\mathcal{B} \cap U$ by $H, K_1, K_2, \ldots, K_q$ (so $J = K_i$ for some i).

Through H there are q^2 secants of U, let l_D be a secant through H that meets ℓ_∞ in the point $D \notin \mathcal{D}$. Label the points of U on l_D by $H, Y_1, \ldots, Y_q$, then the lines K_jY_i, $i, j = 1, \ldots, q$, each contain two points of U and hence are secant to U.

We want to show that for some $i \neq j$ and $m \neq n$ the secants K_iY_m and K_jY_n meet in a point Z of U with $K_iY_m \cap \ell_\infty \notin \mathcal{D}$ and $K_jY_n \cap \ell_\infty \notin \mathcal{D}$. The configuration containing the points H, K_i, K_j, Y_m, Y_n, Z is the required configuration of U that will derive to an O'Nan configuration of U'.

In order to complete the construction of the configuration we will use the Bruck and Bose representation of $PG(2,q^2)$ in $PG(4,q)$ taking the line $HD = l_D$ as the line at infinity. Recall that the classical unital is Buekenhout with respect to any secant line. Hence in $PG(4,q)$, U corresponds to a non-singular quadric $\mathcal{U}$ that meets the spread of Σ_∞ in the regulus $\mathcal{R} = \{h, y_1, \ldots, y_q\}$. If l is a secant of U that meets $l_D \cap U$, then $\bar{l} = l \cap U$ is a Baer subline of $PG(2,q^2)$ and corresponds to a line of $PG(4,q)$ that meets Σ_∞ in a point of $\mathcal{R}$.

The line l_D is tangent to $\mathcal{B}$ as $D \notin \mathcal{D}$, so in $PG(4,q)$, $\mathcal{B}$ corresponds to a ruled cubic surface. By Lemma A, the K_i form a q-arc of a three dimensional subspace Σ in $PG(4,q)$.

If l is a secant of U, let $\bar{l}$ denote the $q+1$ points of $l \cap U$. In $PG(4,q)$, let $\overline{K_iH} \cap h = H_i$, $i = 1, \ldots, q$, and let $\overline{K_iY_j} \cap y_j = Y_{ji}$, $i, j = 1, \ldots, q$.

We now show that the set of points $\mathcal{C}_1 = \{H_1, Y_{11}, \ldots, Y_{q1}\}$ forms a conic in Σ_∞ and hence that $K_1\mathcal{C}_1$ is a conic cone. If the three points H_1, Y_{11}, Y_{21} are collinear, then the lines $K_1H_1, K_1Y_{11}, K_1Y_{21}$ are contained in a plane of $PG(4,q)$ that meets

$\mathcal{U}$ in $3q+1$ points which is not possible. Thus $\mathcal{C}_1$ is a set of $q+1$ points, no three collinear.

Now consider the three dimensional subspace Σ_1 spanned by the lines K_1H_1, K_1Y_{11}, K_1Y_{21}. It meets $\mathcal{U}$ in either a hyperbolic quadric, an elliptic quadric or a conic cone, hence Σ_1 meets $\mathcal{U}$ in a conic cone whose vertex is K_1. The only lines of $PG(4,q)$ through K_1 that are secant lines of $\mathcal{U}$ are those that meet a line of $\mathcal{R}$. Hence $K_1\mathcal{C}_1$ is a conic cone and $\mathcal{C}_1$ forms a conic of the plane $\Sigma_1 \cap \Sigma_\infty$.

Similarly $\mathcal{C}_i = \{H_i, Y_{1i}, \ldots, Y_{qi}\}$ is a conic for each $i = 1, \ldots, q$, and $K_i\mathcal{C}_i$ forms a conic cone of $\mathcal{U}$. We denote the three dimensional subspace containing the conic cone $K_i\mathcal{C}_i$ by Σ_i, $i = 1, \ldots, q$.

Recall that Σ is the hyperplane of $PG(4,q)$ containing the K_i. Suppose $\Sigma_1 = \Sigma$, then $q-1$ of the lines of the cone $K_1\mathcal{C}_1$ are $K_1K_2, \ldots, K_1K_q$. In $PG(2,q^2)$, this means that $q-1$ of the lines $K_1Y_1, \ldots, K_1Y_q$ meet $\mathcal{D}$.

Let $\mathcal{D} = \{T_0, \ldots, T_q\}$ with $H \in K_1T_0$ and consider the lines $K_1T_1, \ldots, K_1T_q$. Suppose that $HD = l_1$ meets y of these lines $K_1T_1, \ldots, K_1T_y$ in a point of U, that is, y of the lines K_1Y_j meet ℓ_∞ in a point of $\mathcal{D}$. Now any other secant line l_2 of U through H can meet at most one of the lines $K_1T_1, \ldots, K_1T_y$ in a point of U, otherwise we would have an O'Nan configuration in U. So l_2 meets at most $q-y+1$ of the lines $K_1T_1, \ldots, K_1T_q$ in a point of U.

So if we pick any other secant of U through H, we can ensure that $\Sigma_1 \neq \Sigma$. By excluding at most q secants of U through H we can ensure that $\Sigma_i \neq \Sigma$, $i = 1, \ldots, q$. There are q^2-q-1 possibilities for D, as $D \notin \mathcal{D}$, so there are enough choices left for D if $q^2 > 2q+1$; that is, if $q > 2$.

So Σ_1 meets Σ in a plane that contains at most three of the K_i, since no four of the K_i are coplanar. By Lemma B, if q is even, the $\mathcal{C}_i$ are all distinct and if q is odd, a given $\mathcal{C}_i$ is distinct from at least $q-2$ of the $\mathcal{C}_i$'s. Hence if q is even, we can pick $K_i \notin \Sigma_1$ with $\mathcal{C}_i \neq \mathcal{C}_1$ for $i = 2, \ldots, q-2$ (since two of the K_i may lie in Σ_1). If q is odd, we can pick $K_i \notin \Sigma_1$ with $\mathcal{C}_i \neq \mathcal{C}_1$ for $i = 2, \ldots, q-3$ (since two of the K_i may lie in Σ_1 and one of the $\mathcal{C}_i$ may equal $\mathcal{C}_1$). Thus if $q \geq 4$, we can pick Σ_1 and Σ_2 so that $\mathcal{C}_1 \neq \mathcal{C}_2$ and $K_2 \notin \Sigma_1$.

Let α_{12} be the plane $\Sigma_1 \cap \Sigma_2$. We investigate how α_{12} meets the conic cones $K_1\mathcal{C}_1$ and $K_2\mathcal{C}_2$ by looking at how it meets $\mathcal{U}$. Since $\mathcal{U}$ and $K_i\mathcal{C}_i$ are quadrics, a plane must meet them in a quadric; that is, in a point, a line, a conic or two lines. We list the four possibilities explicitly for $\alpha_{12} \cap K_2\mathcal{C}_2$; the same possibilities occur for $\alpha_{12} \cap K_1\mathcal{C}_1$.

(a) α_{12} meets $K_2\mathcal{C}_2$ in the vertex K_2,

(b) α_{12} meets $K_2\mathcal{C}_2$ in a line through K_2,

(c) α_{12} meets $K_2\mathcal{C}_2$ in a conic and $K_2 \notin \alpha_{12}$,

(d) α_{12} meets $K_2\mathcal{C}_2$ in two lines through K_2.

Now since $K_2 \notin \Sigma_1$, possibilities (a), (b) and (d) cannot occur for $\alpha_{12} \cap K_2\mathcal{C}_2$, thus $\alpha_{12} \cap K_2\mathcal{C}_2$ is a conic and α_{12} meets $\mathcal{U}$ in a conic. Hence α_{12} meets $K_1\mathcal{C}_1$ in a conic or the vertex K_1. If $K_1 \in K_2\mathcal{C}_2$, then $K_1 \in K_2Y_{i2}$ for some i. However, there is only one line of $\mathcal{U}$ from K_1 to y_i, so $K_2Y_{i2} \in \alpha_{12}$, a contradiction. Thus α_{12} meets $K_1\mathcal{C}_1$ in the conic $\alpha_{12} \cap \mathcal{U}$.

We have deduced that every line of the cone $K_1\mathcal{C}_1$ meets a line of the cone $K_2\mathcal{C}_2$. At most two of these intersections occur in Σ_∞ since α_{12} meets Σ_∞ in a line which meets $\mathcal{C}_i$ in at most two points, hence $\mathcal{C}_1$ and $\mathcal{C}_2$ have at most two points in common. If $H_1 \neq H_2$, then one of the points of $\mathcal{U} \cap \alpha_{12}$ lies in K_1H_1 and one lies in K_2H_2, since K_1H_1 does not meet K_2H_2. Thus we have at least $q-3$ pairs of lines of $\mathcal{U}$, K_1Y_{n1} and K_2Y_{m2}, $n \neq m$, that meet in a point of $PG(4,q) \setminus \Sigma_\infty$.

The line K_1Y_{n1} is contained in a unique plane γ about y_m. Recall that y_m corresponds to the point Y_m in $PG(2,q^2)$ and γ corresponds to the line of $PG(2,q^2)$ through K_1 and Y_m. Therefore, in $PG(2,q^2)$, we have at least $q-3$ pairs of secants of U, K_1Y_n and K_2Y_m, $n \neq m$, that meet in a point Z of U. In order to complete the proof that we have constructed the required configuration in the classical unital of $PG(2,q^2)$, we need to ensure that for one of these pairs both the lines K_1Y_n and K_2Y_m are disjoint from $\mathcal{D}$.

Suppose that x of the lines $K_1Y_1,\ldots,K_1Y_q$ meet $\mathcal{D}$, that is, $HD = l_1$ meets x of the lines $K_1T_1,\ldots,K_1T_q$ in a point of U (recall that K_1H meets $\mathcal{D}$ in T_0). As before, if l_2 is a different secant line of U through H, then l_1 and l_2 can meet at most one common $K_1T_1,\ldots,K_1T_q$ in a point of U, otherwise we have an O'Nan configuration in U. If l_1 meets none of the $K_1T_1,\ldots,K_1T_q$, then we retain l_1. Otherwise l_1 meets K_1T_i for some i. There are q other secants of U through H that contain a point of $K_1T_i \cap U$, we label them $l_2,\ldots,l_{q+1}$. In the worst case, each l_k meets exactly one of the K_1T_j, $j \neq i$, in point of U. However, there are $q+1$ lines l_i and only $q-1$ lines K_1T_j, $j \neq i$, thus at least one of the l_k meets K_1T_i in a point of U, and no further K_1T_j, $j \neq i$, in a point of U. Thus by excluding at most $q-1$ choices of a secant line through H, we can ensure that at most one of the lines $K_1Y_1,\ldots,K_1Y_q$ meets $\mathcal{D}$. We have already excluded at most q choices for D, there are q^2-q-1 possibilities for D, so if $q^2 > 3q$, that is, $q > 3$, there are enough choices left for D.

In order to ensure that at least one of the above $q-3$ pairs K_1Y_n, K_2Y_m that meet in a point of U are disjoint from $\mathcal{D}$, it suffices to show that at least two of the $q-3$ lines K_2Y_m are disjoint from $\mathcal{D}$ (since at most one of the $q-3$ lines K_1Y_n meets $\mathcal{D}$). Thus we need at least five of the lines $K_2Y_1,\ldots,K_2Y_q$ disjoint from $\mathcal{D}$. Hence we need $q \geq 5$.

Suppose $x < 5$ of the lines $K_2Y_1,\ldots,K_2Y_q$ are disjoint from $\mathcal{D}$, so $q-x$ of them meet $\mathcal{D}$. If $q-3 > 2$, we can pick $K_3 \notin \Sigma_1$ with $\mathcal{C}_3 \neq \mathcal{C}_1$. Repeating the above argument gives $q-3$ pairs K_1Y_j, K_3Y_k, $j \neq k$, that meet in a point of U. Now if K_2Y_i meets $\mathcal{D}$, then K_3Y_i cannot meet $\mathcal{D}$, otherwise $Y_i \in \mathcal{B}$ which is a contradiction. Thus, in the worst case, exactly $\frac{q}{2}$ of the $K_2Y_1,\ldots,K_2Y_q$ meet $\mathcal{D}$ and $\frac{q}{2}$ of the $K_3Y_1,\ldots,K_3Y_q$ meet $\mathcal{D}$. So if $\frac{q}{2} \geq 5$, that is, $q \geq 10$, we have two lines that meet in a point of U and are disjoint from $\mathcal{D}$, hence we have constructed the required configuration in U when $q \geq 10$.

If $q-3 > 3$ with q odd, or $q-2 > 3$ with q even, that is $q \geq 6$, then we can pick $K_4 \notin \Sigma_1$ and $\mathcal{C}_4 \neq \mathcal{C}_1$. The above argument gives $q-3$ pairs K_1Y_j, K_4Y_k, $j \neq k$, that meet in a point of U. In the worst case, exactly $\frac{q}{3}$ of the $K_iY_1,\ldots,K_iY_q$, $i = 2,3,4$ meet $\mathcal{D}$. So if $q - \frac{q}{3} \geq 5$, that is, $q \geq 8$, we have a pair K_1Y_j, K_iY_k that meet in a point of U and are disjoint from $\mathcal{D}$. Hence if $q \geq 8$, we can construct the required configuration in the classical unital.

If $q = 7$, q is not divisible by three, so in the worst case, we can pick two of the $K_2Y_1, \ldots, K_2Y_q$ meeting $\mathcal{D}$ and so $7 - 2 = 5$ of them are disjoint from $\mathcal{D}$, giving the required configuration.

Therefore, if $q > 5$, we have constructed the required configuration in the classical unital that will derive to an O'Nan configuration of U.

We have shown that for any point H of $U \setminus \ell_\infty$ and for any Baer subplane $\mathcal{B}$ of $PG(2, q^2)$ containing $\mathcal{D}$, H and $q + 1$ points of U, we can construct the required configuration in the classical unital through H and two other points of $U \cap \mathcal{B}$. Thus in U', for any point H of $U' \setminus \ell'_\infty$ and secant line l of U' through H that meets the classical derivation set $\mathcal{D}'$, the above configuration in the classical unital derives to an O'Nan configuration of U' containing H and l. □

As U' contains O'Nan configurations and U does not contain O'Nan configurations, we obtain the immediate corollary that the designs U and U' are not isomorphic.

Corollary 4 *The unital U' is not isomorphic to the classical unital U.*

The only unital of the Hall plane examined in detail has been the Buekenhout unital obtained by Grüning [5]. This is constructed by taking the classical unital U in $PG(2, q^2)$ secant to ℓ_∞ and deriving with respect to $U \cap \ell_\infty$. By examining the occurrence of O'Nan configurations in the two unitals, we show that they are non-isomorphic. Therefore the class of unitals investigated here have not previously been studied in detail.

Theorem 5 *The class of Buekenhout unitals U' in $\mathcal{H}(q^2)$, $q > 3$, is not isomorphic to the class of Buekenhout unitals in $\mathcal{H}(q^2)$ found by Grüning [5].*

Proof We show the two unitals are non-isomorphic by examining the frequency distribution of O'Nan configurations in them. Let V be Grüning's unital of $\mathcal{H}(q^2)$ and let $\bar{l} = V \cap \ell'_\infty$. Grüning showed that (i) V contains no O'Nan configurations with two or more points on $\bar{l}$ and (ii) for any point $P \in \bar{l}$, if l_1, l_2 are lines of V through P and l_3 a line of U that meets l_1 and l_2, then there exists an O'Nan configuration of U containing l_1, l_2 and l_3. Let U be a unital and $\bar{l}$ a line of U. We call $\bar{l}$ a **G-O axis** (Grüning-O'Nan axis) of U if it satisfies (i) and (ii). We show that U' does not contain a G-O axis.

There are three possibilities for such an axis in U': $U' \cap \ell'_\infty$, a secant line of U' that meets ℓ'_∞ in a point of U', a secant line of U' that meets ℓ'_∞ in a point not in U'.

Let P be a point of $U' \cap \ell'_\infty$. Let l_1, l_2 be secants of U' through P. In $PG(4, q)$, $\overline{l_1} = l_1 \cap U'$ and $\overline{l_2} = l_2 \cap U'$ are lines of $PG(4, q)$ that meet the line p of the spread $\mathcal{S}'$. Choose l_1 and l_2 such that $\overline{l_1}$ and $\overline{l_2}$ meet in a point of p in $PG(4, q)$. By Theorem 2,, there is no O'Nan configuration that contains l_1 and l_2. This violates (ii), thus $U' \cap \ell'_\infty$ is not a G-O axis of U'. The same example shows that any other line of U' through P cannot be a G-O axis of U'.

Let l be a secant line of U' that meets ℓ'_∞ in a point not in U'. Let Q be a point of U' on l and let l_1 and l_2 be secants of U' through Q that contain a point of $U' \cap \ell'_\infty$. There is no O'Nan configuration of U' containing l_1, l_2 and $U' \cap \ell'_\infty$ as U' contains no O'Nan configurations with three vertices on ℓ'_∞ (Theorem 2). Therefore, l does not satisfy (ii) and cannot be a G-O axis of U'. Hence no line of U' is a G-O axis and so U' is not isomorphic to V. □

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