A Note on Semi-classical Orthogonal Polynomials

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Abstract

We prove that one characterization for the classical orthogonal polynomials sequences (Hermite, Laguerre, Jacobi and Bessel) cannot be extended to the semi-classical ones.

1 Introduction

Recently, in [6] were established new characterizations of the classical monic orthogonal polynomials sequences (MOPS). In that work, the authors consider as a starting point the Pearson's equation in a distributional sense. It is well known that most of this characterizations can be extended for semi-classical MOPS (see [2, 4, 11]). Following another point of view we will prove that:

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- The Proposition 3.3 of [6]:
  - Let $\{P_n\}$ be a MOPS. A necessary and sufficient condition for $\{P_n\}$ belongs to one of the classical families is

$$P_n = \frac{P'_{n+1}}{n+1} + \sum_{k=n-1}^{n} a_{n,k} \frac{P'_k}{k},\ n \geq 2$$

  needs of additional hypothesis on the parameters of the structure formula.

- This result cannot be extended to semi-classical MOPS of class s.

Before proving these results we will study some problems related to them. We will begin by introducing some algebraic concepts that we will use in this work (see [7, 12]). Let $\{p_n\}$ be a MPS, i.e. $p_n = x^n + \ldots,\ n \in \mathbb{N}$. We can define the dual basis, $\{\alpha_n\}$ in $\mathbb{P}^\star$, the algebraic dual space of $\mathbb{P}$, the linear space of polynomials with complex coefficients, as $\langle \alpha_n, p_m \rangle = \delta_{n,m}$, where $\langle ., . \rangle$ means the duality bracket and $\delta_{n,m}$ is the Kronecker symbol. Now, if $v \in \mathbb{P}^\star$, it can be expressed by $v = \sum_{i \in \mathbb{N}} \langle v, p_i \rangle \alpha_i$.

Definition 1 For every polynomial $\phi(x)$ a new linear functional can be introduced from v. This functional is called the *left product of v by ϕ*:

$$\langle \phi(x) v, p(x) \rangle = \langle v, \phi(x) p(x) \rangle,\ \forall p(x) \in \mathbb{P}.$$

Definition 2 The usual *distributional derivative of v* is given by

$$\langle Dv, p(x) \rangle = -\langle v, p'(x) \rangle,\ \forall p(x) \in \mathbb{P}.$$

So, we can state (see [12]):

- If $v \in \mathbb{P}^\star$ is such that $\langle v, p_i \rangle = 0,\ i \geq l$ then

$$v = \sum_{i=0}^{l-1} \langle v, p_i \rangle \alpha_i \tag{1}$$

- If $\{\alpha'_n\}$ is the dual basis associated with the MPS $\{\frac{P'_{n+1}}{n+1}\}$ then

$$D(\alpha'_n) = -(n+1)\alpha_{n+1},\ n \in \mathbb{N} \tag{2}$$

Definition 3 Let $\{P_n\}$ be a MPS; we say that $\{P_n\}$ is *orthogonal* with respect to the *quasi-definite linear functional* u if $\langle u, P_n(x) P_m(x) \rangle = K_n \delta_{n,m}$ with $K_n \neq 0$ for $n, m \in \mathbb{N}$.
We say that u is *positive definite* if $K_n > 0,\ n \in \mathbb{N}$.

Furthermore,

- $\{P_n\}$ satisfies the following three term recurrence relation (TTRR)

$$\begin{array}{l} xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \text{ for } n = 1, 2, \dots \\ P_0(x) = 1,\ P_1(x) = x - \beta_0. \end{array} \tag{3}$$

  where (β_n) and (γ_n) are two sequences of complex numbers with $\gamma_{n+1} \neq 0$ in the quasi-definite case and $\gamma_{n+1} > 0$, $(\beta_n) \subset \mathbb{R}$ in the positive definite case, for $n \in \mathbb{N}$.

- The elements of the dual basis $\{\alpha_n\}$ associated with $\{P_n\}$ can be written as

$$\alpha_n = \frac{P_n u}{\langle u, P_n^2 \rangle},\ n \in \mathbb{N} \tag{4}$$

Now we state the basic definition which will be used along this paper:

Definition 4 Let $\{p_n\}$ be a MPS and u be a quasi-definite linear functional; we say that p_n is *quasi-orthogonal of order* s with respect to u if

$$\begin{array}{r} \langle u, p_m p_n \rangle = 0,\ |n - m| \geq s + 1 \\ \exists r \geq s\ :\ \langle u, p_{r-s} p_r \rangle \neq 0. \end{array}$$

Remark A quasi-orthogonal MPS of order 0 is orthogonal in the above sense. In fact, if $\langle u, P_r^2 \rangle \neq 0$ then $\langle u, P_r^2 \rangle = \gamma_r \langle u, P_{r-1}^2 \rangle$.

The following definition was given by Ronveaux (see [13]) and Maroni (see [11]):

Definition 5 Let $\{P_n\}$ be a MOPS with respect to the quasi-definite linear functional u; we say that $\{P_n\}$ is *semi-classical of class* s if there exists $\phi \in \mathbb{P}_{s+2}$ such that $\{\frac{P'_{n+1}}{n+1}\}$ is quasi-orthogonal of order s with respect to ϕu.
If $s = 0$ we say that $\{P_n\}$ is *classical*.

The canonical expressions of ϕ, $d\mu$: $\langle u, x^n \rangle = \int_I x^n d\mu(x)$, $n \in \mathbb{N}$, where I is a complex contour and $d\mu$ a complex measure, and the coefficients of the TTRR for the classical MOPS (Hermite, H_n, Laguerre, L_n^α, Jacobi, $P_n^{\alpha,\beta}$ and Bessel, B_n^α) are presented in the Tables 1, 2 (see Ismail and al. [9]).

Notation *In* Table *1:*

- Ψ *is the Tricomi function (see [8, Chapter 6]).*
- $S(R) = \{z \in \mathbb{C}\ :\ |z| = R,\ \exp(-R^2) \leq \arg(z) \leq 2\pi - \exp(-R^2)\}$.
- $X^{\alpha,\beta} = [\Gamma(\alpha+\beta+2)(z-1)]^{-1}\, {}_2F_1\left(\begin{array}{c} 1, \alpha+1 \\ \alpha+\beta+2 \end{array} \middle| 2/1 - z \right)$.
- $\{z \in \mathbb{C}\ :\ |z - 1| > 2\} \subset C$.

P_n	ϕ	$d\mu$	I	Restrictions
H_n	1	$\exp(-x^2)$	$\mathbb{R}$	
L_n^α	x	$-\Psi(1,1-\alpha,-z)$	$S(R)$	$\alpha \neq -1,-2,\dots$
$P_n^{\alpha,\beta}$	$1-x^2$	$2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)X^{\alpha,\beta}$	C	$\alpha,\beta \neq -1,-2,\dots$
B_n^α	x^2	$x^\alpha \exp(-2/x)$	unit circle	$\alpha \neq -1,-2,\dots$

Table 1:

P_n	β_n	γ_{n+1}
H_n	0	$\frac{n+1}{2}$
L_n^α	$2n+\alpha+1$	$(n+1)(n+\alpha+1)$
$P_n^{\alpha,\beta}$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$\frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}$
B_n^α	$\frac{-2\alpha}{(2n+\alpha)(2n+\alpha+2)}$	$\frac{-4(n+1)(n+\alpha+1)}{(2n+\alpha+1)(2n+\alpha+2)^2(2n+\alpha+3)}$

Table 2:

2 Classical Case

From Definition 5, $\{P_n\}$ is a classical MOPS if and only if $\{\frac{P'_{n+1}}{n+1}\}$ is a MOPS. In [6] we gave another characterization of these MOPS:

THEOREM 6 *Let $\{P_n\}$ be a MOPS. A necessary and sufficient condition for $\{P_n\}$ belongs to one of the classical families is*

$$P_n = \frac{P'_{n+1}}{n+1} + \sum_{k=n-1}^{n} a_{n,k}\frac{P'_k}{k},\ n \geq 2$$

with $a_{n,n-1} \neq (n-1)\gamma_n$ for $n \geq 2$.

Proof. Since $\{P_n\}$ is a MOPS

$$\begin{aligned} &xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \text{ for } n = 1,2,\dots \\ &P_0(x) = 1,\ P_1(x) = x - \beta_0. \end{aligned}$$

So, we can take derivatives

$$P_n = P'_{n+1} + \beta_n P'_n + \gamma_n P'_{n-1} - xP'_n$$

Now, consider

$$P_n = \frac{P'_{n+1}}{n+1} + \sum_{k=1}^{n} a_{n,k}\frac{P'_k}{k}$$

and put this expression into the above

$$x\frac{P'_n}{n} = \frac{P'_{n+1}}{n+1} + (\beta_n - \frac{a_{n,n}}{n})\frac{P'_n}{n} + \frac{(n-1)\gamma_n - a_{n,n-1}}{n}\frac{P'_{n-1}}{n-1} - \frac{1}{n}\sum_{k=1}^{n-2} a_{n,k}\frac{P'_k}{k}$$

P_n	$a_{n+1,n+1}$	$a_{n+2,n+1}$
H_n	0	0
L_n^α	$n+1$	0
$P_n^{\alpha,\beta}$	$\frac{2(\alpha-\beta)(1+n)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}$	$\frac{4(n+1)(n+2)(n+\alpha+2)(n+\beta+2)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)^2(2n+\alpha+\beta+5)}$
B_n^α	$\frac{4(n+1)}{(2n+\alpha+2)(2n+\alpha+4)}$	$\frac{-4(n+1)(n+2)}{(2n+\alpha+3)(2n+\alpha+4)^2(2n+\alpha+5)}$

Table 3:

Hence, $\{\frac{P'_{n+1}}{n+1}\}$ is orthogonal if and only if

$a_{n,k} = 0$, for $k = 1, 2, \ldots, n-2$

$a_{n,n-1} \neq (n-1)\gamma_n$, for $k = 2, \ldots$

■

Remark This theorem has been established in [6] by the authors without any restrictions on the parameters, $a_{n,k}$, of the structure relation. This condition is only important in the cases of Jacobi and Bessel.

From the last theorem we can state:

Corollary 7 *Let $\{P_n\}$ be a classical MOPS and $(\beta_n), (\gamma_n)$ the coefficients of the TTRR, (3), that this MOPS satisfy. If we denote by $(\beta'_n), (\gamma'_n)$ the coefficients of the TTRR that $\{\frac{P'_{n+1}}{n+1}\}$ satisfy, i.e.*

$$x\frac{P'_{n+1}}{n+1} = \frac{P'_{n+2}}{n+2} + \beta'_n\frac{P'_{n+1}}{n+1} + \gamma'_n\frac{P'_n}{n} \quad \text{for } n = 1, 2, \ldots$$
$$\frac{P'_1}{1} = 1, \ \frac{P'_2}{2} = x - \beta'_0.$$

then

$$a_{n+1,n+1} = (n+1)(\beta_{n+1} - \beta'_{n+1}) \tag{5}$$
$$a_{n+2,n+1} = (n+1)\gamma_{n+2} - (n+2)\gamma'_{n+1} \tag{6}$$

for $n \in \mathbb{N}$.

Now, because $\{\frac{P'_{n+1}}{n+1}\}$ is the MOPS with respect to ϕu, where ϕ is defined in Table 1, we can calculate $a_{n+1,n+1}, a_{n+2,n+1}$ from (5), (6) and Table 2, and the result is sumarized in Table 3.

3 Semi-classical Results

Here, we only state some results of the semi-classical polynomials that are extensions of the well-known characterizations of the classical polynomials. They have been stated by Maroni in [11] (see also Bonan and al. [3] and Branquinho and al. [5] for the last characterization).

THEOREM 8 *Let $\{P_n\}$ be a MOPS with respect to the linear functional u. Then the following statements are equivalent:*

(a) $\{P_n\}$ *is semi-classical of class s;*

(b) $\exists \phi, \psi \in \mathbb{P}$ *with* $\deg \phi \leq s+2$, $\deg \psi \leq s+1$ *such that*

$$\phi P'_{n+1} + \psi P_{n+1} = \sum_{k=n-s}^{n+s+2} b_{n,k} P_k, \; n \geq s$$

and $b_{n,n-s} \neq 0$, $n \geq s$;

(c) $\exists \phi, \psi \in \mathbb{P}$ *with* $\deg \phi \leq s+2$, $\deg \psi \leq s+1$ *such that*

$$D(\phi u) = \psi u$$

i.e. u is a semi-classical functional of class s;

(d) $\{\frac{P'_{n+1}}{n+1}\}$ *is quasi-orthogonal of order s with respect to* ϕu.

(e) There exists a MOPS $\{R_n\}$ *with respect to a linear functional v such that*

$$\phi R'_{n+1} = \sum_{k=n-s}^{n+p} \lambda_{n,k} P_k, \; n \geq s \tag{7}$$

and $\lambda_{n,n-s} \neq 0$, $n \geq s$.

REMARK

- This ϕ, ψ must satisfy the condition

$$\prod_{c \in \mathcal{Z}_\phi} (|r_c| + |\langle \psi_c u, 1 \rangle|) \neq 0$$

where $\mathcal{Z}_\phi$ is the set of zeros of ϕ and

$$\phi(x) = (x-c)\phi_c(x)$$
$$\psi(x) + \phi_c(x) = (x-c)\psi_c(x) + r_c(x)$$

like it was shown in [2].

- In [3] the authors prove that in (7) we can take $R_{n+1}^{(i)}$ with $i \geq 1$ instead of R'_{n+1}. There they want to generalize the semi-classical definition of MOPS.
- In [5] the authors prove that if we have (7), $\{R_n\}$ is also semi-classical and there exists $h \in \mathbb{P}$ such that $\phi(x)u = h(x)v$ with

$$h(x) = \langle u_y, \phi(y) \left[P_1(y) K_{s+2}^{(0,1)}(x,y) - P_1(x) K_{s+1}^{(0,1)}(x,y) \right] \rangle$$

where $K_n^{(r,s)}(x,y) = \displaystyle\sum_{j=0}^{n} \frac{R_j^{(r)}(x) R_j^{(s)}(y)}{\langle v, R_j^2 \rangle}$ and by u_y we mean the action of u over the variable y for polynomials in two variables.

First of all we try to explain why we have conjectured that the Theorem 6 could be generalized to the semi-classical case. From now, we suppose that $s \geq 1$.

THEOREM 9 *If $\{P_n\}$ is a MOPS with respect to the linear functional u and verifies*

$$P_n = \frac{P'_{n+1}}{n+1} + \sum_{k=n-(s+1)}^{n} a_{n,k} \frac{P'_k}{k},\ n \geq s+2 \tag{8}$$

with $a_{n,n-(s+1)} \neq 0$ then there exists $\phi_{s+2} \in \mathbb{P}$ with $\deg \phi_{s+2} = s+2$ such that

$$D(\phi_{s+2} u) = P_1 u \tag{9}$$

i.e. u is semi-classical of class s.

Proof. Let $\{\alpha_n\}$ and $\{\alpha'_n\}$ be the dual bases associated with $\{P_n\}$ and $\{\frac{P'_{n+1}}{n+1}\}$, respectively. We can write

$$\alpha'_n = \sum_{k \geq n} \lambda_{n,k} \alpha_k$$

where

$$\begin{aligned} \lambda_{n,k} &= \langle \alpha'_n, P_k \rangle = \langle \alpha'_n, \frac{P'_{k+1}}{k+1} + \sum_{j=k-(s+1)}^{n} a_{k,j} \frac{P'_j}{j} \rangle \\ &= \begin{cases} 1 & , k = n \\ a_{k,n+1} & , k = n+1, n+2, \ldots, n+s+2 \\ 0 & , k = 0, \ldots, n-1 \end{cases} \end{aligned}$$

Hence, by (1)

$$\alpha'_n = \alpha_n + \sum_{k=1}^{s+2} a_{n+k,n+1} \alpha_{n+k},\ n \in \mathbb{N}$$

Put $n = 0$ in this expression and take derivatives we get after applying (2) and (4)

$$-\frac{P_1}{\langle u, P_1^2 \rangle} u = D\left((\frac{1}{\langle u, 1 \rangle} + \sum_{k=1}^{s+2} a_{k,1} \frac{P_k}{\langle u, P_k^2 \rangle}) u \right)$$

so we have (9) where $\phi_{s+2}(x) = -\frac{\langle u, P_1^2 \rangle}{\langle u, 1 \rangle} \left(1 + \sum_{k=1}^{s+2} \frac{a_{k,1}}{\prod_{j=1}^{k} \gamma_j} P_k \right)$. ■

REMARK • We are tempted to search our MOPS, between the semi-classical MOPS that the corresponding linear functionals verify (9). Belmehdi (see [1]) gave some examples of semi-classical MOPS, $\{P_n\}$ associated with a linear functional, u, which verify (9) with $s=1$. The linear functional u is defined in terms of the classical linear functionals v by

$$(x-c)u = v$$

for some $c \in \mathbb{C}$. In this case $\{P_n\}$ can be written in terms of the MOPS associated with v, $\{R_n\}$, by

$$\begin{aligned} P_{n+1} &= R_{n+1} - a_{n+1}R_n,\ n \in \mathbb{N} \\ P_0 &= R_0 \end{aligned} \tag{10}$$

where $a_{n+1} = \frac{R_{n+1}(c;-u_0^{-1})}{R_n(c;-u_0^{-1})}$, $u_0 = \langle u, 1\rangle$ and $\{R_n(x;d)\}$ is the co-recursive MOPS.

- Belmehdi has shown that in this case $\{R_n\}$ cannot be the Hermite polynomials.
- The cases studied by Belmehdi are particular cases of (10).

Now, we can state the following result:

THEOREM 10 *If $\{R_n\}$ is a classical MOPS, then the MOPS $\{P_n\}$ with respect to u defined by (10) are semi-classical of class ≤ 1 but cannot be expressed by a finite linear combination of consecutives derivatives of elements of this family.*

Proof. The semi-classical character has been proved by Belmehdi in [1].

From theorem 6

$$R_n = \frac{R'_{n+1}}{n+1} + \sum_{k=n-1}^{n} a_{n,k}\frac{R'_k}{k},\ n \geq 2$$

with $a_{n,n-1} \neq (n-1)\gamma_n$ for $n \geq 2$; then, put this into (10) and get after some calculations

$$P_{n+1} = \frac{P'_{n+2}}{n+2} + s_{n+1}\frac{P'_{n+1}}{n+1} + t_{n+1}\frac{P'_n}{n} - (a_{n+1}a_{n,n-1} - \frac{(n-1)t_{n+1}a_n}{n})\frac{R'_{n-1}}{n-1}$$

where

$$\begin{aligned} s_{n+1} &= a_{n+1,n+1} - a_{n+1} + \frac{(n+1)a_{n+2}}{n+2} \\ t_{n+1} &= a_{n+1,n} - a_{n+1}a_{n,n} + \frac{ns_{n+1}a_{n+1}}{n+1} \end{aligned}$$

for $n \in \mathbb{N}$ where a_n is defined by (10) and $a_{n,n}, a_{n,n-1}$ are given in Table 3. ■

Now we can see when we can reduce the class of the semi-classical orthogonal polynomials to the classical ones.

COROLLARY 11 *In the conditions of the last theorem we have that, $\{P_n\}$ is a classical MOPS if and only if*

$$a_{n+3}a_{n+2,n+1} - \tfrac{n+1}{n+2}t_{n+2}a_{n+2} = 0$$
$$t_{n+1} \neq (n+1)\gamma_{n+2}$$

for $n \in \mathbb{N}$.

REMARK Here we have an example of semi-classical MOPS of class one, with respect to a linear functional which verify (9) and cannot be expressed as a linear combination of four consecutive derivatives.

If $\{P_n\}$ is a MOPS with respect to the linear functional u and u verifies (9) then $\{P_n\}$ is a sequence of *Generalized Jacobi* polynomials, as can be seen in the Magnus work [10].

An example of a generalized Jacobi MOPS $\{P_n\}$ such that

$$P_n = \frac{P'_{n+1}}{n+1} + \sum_{k=1}^{n} a_{n,k}\frac{P'_k}{k}$$

with $a_{n,k} \neq 0$ for $k = 1, \ldots, n$ was given by Magnus with a aid of a computer.

From this we can suspect that there aren't MOPS that can be expanded as a linear combination of four consecutives derivatives.

4 Main Problem

Here we will prove that there aren't MOPS that verify (8) and (9) with $a_{n,n-(s+1)} \neq 0$ and $s \geq 1$. We only prove this result for $s = 1$ but the same is true for any $s > 1$. First of all we state the following results:

LEMMA 12 *If $\{P_n\}$ is a MOPS with respect to the linear functional u and verifies the TTRR (3) then*

(a) $\gamma_{n+1} = \dfrac{\langle u, x^{n+1}P_{n+1}\rangle}{\langle u, x^n P_n\rangle}$, $n \in \mathbb{N}$;

(b) $\dfrac{\langle u, x^{n+1}P_n\rangle}{\langle u, x^n P_n\rangle} = \displaystyle\sum_{k=0}^{n} \beta_k$, $n \in \mathbb{N}$, $n \in \mathbb{N}$.

Proof. See Chihara [7].

We know that if $\{P_n\}$ is a MOPS then can be represented by

$$P_n(x) = x^n - \sum_{k=0}^{n-1} \beta_k x^{n-1} + \left(\sum_{0 \leq i < j \leq n-1} \beta_i\beta_j - \sum_{k=1}^{n-1} \gamma_k \right) x^{n-2} + \ldots$$

Now, if we put this expression in $\beta_n = \frac{\langle u, xP_n^2\rangle}{\langle u, P_n^2\rangle}$ we get

$$\begin{aligned}\beta_n &= \frac{\langle u, x(x^n - \sum_{k=0}^{n-1}\beta_k x^{n-1} + \ldots)P_n\rangle}{\langle u, P_n^2\rangle} \\ &= \frac{\langle u, x^{n+1}P_n\rangle}{\langle u, P_n^2\rangle} - \sum_{k=0}^{n-1}\beta_k\end{aligned}$$

i.e. (b). To get (a) we only have to multiply (3) by P_{n-1} and apply u to the resulting equation. ■

LEMMA 13 *Let $\{P_n\}$ is a semi-classical MOPS of class 1 with respect to the linear functional u; if u verifies $D(\phi u) = P_1 u$ where $\phi(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ with $a_0 \neq 0$ then:*

(a) $\langle \phi u, P_{n-1}P'_{n+1}\rangle = -a_0(n-1)\langle u, P_{n+1}^2\rangle$, $n \geq 1$;

(b) $\langle \phi u, P_m P'_{n+1}\rangle = 0$, $0 \leq m \leq n-2$, $n \geq 2$ *or* $m \geq n+4$;

(c) $\langle \phi u, P_n P'_{n+1}\rangle = -(a_0(n(\beta_n + \beta_{n+1}) + \sum_{k=0}^{n-1}\beta_k) + na_1 + 1)\langle u, P_{n+1}^2\rangle$, $n \in \mathbb{N}$.

Proof. If we substitute in the Definition 4, p_n by $\frac{P'_{n+1}}{n+1}$, u by ϕu and s by 1, we obtain

$$\begin{aligned}&\langle \phi u, P'_{m+1}P'_{n+1}\rangle = 0,\ |n-m| \geq 2 \\ &\exists r \geq 1 : \langle \phi u, P'_r P'_{r+1}\rangle \neq 0.\end{aligned}$$

But $\{\frac{P'_{n+1}}{n+1}\}$ is a MPS so we can write these conditions like

$$\langle \phi u, P_m P'_{n+1}\rangle = 0,\ 0 \leq m \leq n-2,\ n \geq 2 \ \text{ or } \ m \geq n+4 \tag{11}$$

$$\exists r \geq 1 : \langle \phi u, P_{r-1}P'_{r+1}\rangle \neq 0 \tag{12}$$

Proof of **(a)**. We know that $P_{r-1}P'_{r+1} = (P_{r-1}P_{r+1})' - P'_{r-1}P_{r+1}$ so if we put this expression in (12) we get

$$\begin{aligned}\langle \phi u, P_{r-1}P'_{r+1}\rangle &= -\langle D(\phi u), P_{r-1}P_{r+1}\rangle - \langle \phi u, P'_{r-1}P_{r+1}\rangle \\ &= -\langle P_1 u, P_{r-1}P_{r+1}\rangle - a_0\langle u, P_{r+1}^2\rangle \\ &= -a_0\langle u, P_{r+1}^2\rangle\end{aligned}$$

Proof of **(c)**. Put $m = n$ in (11), using the same technique and the Lemma 12 we get

$$\begin{aligned}\langle \phi u, P_n P'_{n+1}\rangle &= -\langle u, P_{n+1}^2\rangle - \\ &\quad n\langle u, (a_0x^3 + a_1x^2)(nx^{n-1} - (n-1)\sum_{k=0}^{n-1}\beta_k x^{n-2} + \ldots)P_{n+1}\rangle \\ &= -(a_0(n(\beta_n + \beta_{n+1}) + \sum_{k=0}^{n-1}\beta_k) + na_1 + 1)\langle u, P_{n+1}^2\rangle\end{aligned}$$

Note that (b) coincides with (11). ■

Now, we are able to state:

THEOREM 14 *Let $\{P_n\}$ is a semi-classical MOPS of class 1 with respect to the linear functional u and u verifies $D(\phi u) = P_1 u$ where $\phi(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ with $a_0 \neq 0$; then it admits the following representation in terms of its derivatives*

$$P_n = \frac{P'_{n+1}}{n+1} + \sum_{k=2}^{n} b_{n,k}\frac{P'_k}{k} \tag{13}$$

for $n \in \mathbb{N}$ with $b_{n,2} \neq 0$.

Proof. The procedure that we use for proving this assertion is the following:

- Multiply sucessively (13) by P_j with $j = 0, 1, \ldots, n-4$ and apply ϕu on each sides of the resulting equation.

Hence, for $j = 0$ we get

$$\begin{aligned} 0 &= b_{n,1}\langle \phi u, P'_1\rangle + \frac{b_{n,2}}{2}\langle \phi u, P'_2\rangle + \frac{b_{n,3}}{3}\langle \phi u, P'_3\rangle \\ &= -b_{n,1}\langle u, P_1^2\rangle - \frac{b_{n,2}}{2}\langle u, P_2P_1\rangle - \frac{b_{n,3}}{3}\langle u, P_3P_1\rangle \end{aligned}$$

i.e. $b_{n,1} = 0$.
For $j = 1$, and using the same technique, we get $\frac{b_{n,3}}{3} = -\frac{1+a_1+a_0(\beta_0+\beta_1+\beta_2)}{a_0\gamma_3}\frac{b_{n,2}}{2}$.
Procedure in the same way until $j = n-4$. At that time you will get $b_{n,n-2}$ given in terms of $b_{n,2}$.
Now if you consider $b_{n,2} = 0$ you have that $b_{n,k} = 0$, for $k = 2, \ldots, n-2$, i.e. $\{P_n\}$ is a classical MOPS, in a contradiction with the hypothesis of the theorem. ■

As a conclusion, we can state:

THEOREM 15 *If $\{P_n\}$ is a MPS that verifies (8) with $a_{n,n-(s+1)} \neq 0$ for $n \geq s+2$ and s don't depend on n then $\{P_n\}$ is a MOPS if and only if $s = 0$.*

REMARK If we put the expression (13) in the derivative of (3), like we have done in Theorem 6, we get the following relation for the derivatives

$$x\frac{P'_n}{n} = \frac{P'_{n+1}}{n+1} + (\beta_n - \frac{b_{n,n}}{n})\frac{P'_n}{n} + \frac{(n-1)\gamma_n - b_{n,n-1}}{n}\frac{P'_{n-1}}{n-1} - \sum_{k=2}^{n-2}\frac{b_{n,k}}{n}\frac{P'_k}{k}$$

valid for $n \geq 1$.

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