The non-Archimedian Laplace Transform

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Abstract

Topological properties of the spaces of analytic test functions and distributions are investigated in the framework of the general theory of non-archimedean locally convex spaces. The Laplace transform, topological isomorphism, is introduced and applied to the differential equations of non-archimedean mathematical physics (Klein-Gordon and Dirac propagators).

Introduction.

Last years a number of quantum models over non-archimedean fields was proposed (quantum mechanics, field and string theory, see for example books [1,2] and references in these books). As usual, new physical formalisms generate new mathematical problems. In particular, a lot of differential equations with partial derivatives were introduced in connection with non-archimedean mathematical physics (Schrödinger, Heisenberg, Klein- Gordon,...), see [1,2]. In the ordinary real and complex analysis, one of the most powerful tools to investigate equations with constant coefficients are the Fourier and Laplace transforms. It is not a simple problem to introduce these transforms in the non- archimedean case, see [3 - 7]. There is a number of different approaches and the main problem is always that the Fourier and Laplace transforms are not isomorphisms. There exist non-zero functions with zero Fourier or Laplace transform. A new approach to this problem was proposed in [2, 8] on the basis of the non-archimedean theory of analytical distributions.

In this paper we study the properties of the non-archimedean locally convex spaces of distributions and test functions. The main result in this direction is that

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these spaces are reflexive. Then it is very important for applications that the strong topology $\beta(\mathbf{A}', \mathbf{A})$ on the space of distributions $\mathbf{A}'$ coincides with the natural inductive limit topology induced by the space $\mathbf{A}_0$ of functions analytic at zero. The basis of our investigations are the results in the general theory of locally convex spaces [9 - 14]. Then we have proved that the Laplace transform is a topological isomorphism (this fact is very useful to prove that solutions of differential equations depend continuously on initial data). The results on the Laplace transform are applied to differential equations with constant coefficients : fundamental solutions of differential operators and the Cauchy problem. As one of the possible applications to non-archimedean mathematical physics, we consider the Klein- Gordon equation of a scalar Boronic field. As a consequence of the general theorem we get the existence of the unique fundamental solution $\xi(x) \in \mathbf{A}'$ of the Klein- Gordon operator and with the aid of the fundamental solution we construct the solution of the Klein-Gordon equation for an arbitrary entire analytic source. Our ideology with respect to the equations of quantum mechanics and field theory is the following. We consider the variables of these equations as formal variables $x = (x_0, x_1, \ldots, x_n)$. Then we study formal power series solutions with rational coefficients. Sometimes, there is the possibility to realize these formal series as $\mathbb{R}$ or $\mathbb{C}$- analytic functions of variables $x \in \mathbb{R}^n$ (or $\mathbb{C}^n$), but sometimes there is only the possibility to realize these solutions as $\mathbb{Q}_p$ (or $\mathbb{C}_p$) valued functions of variables $x \in \mathbb{Q}_p^n$ (or $\mathbb{C}_p^n$). From the usual real point of view these solutions have infinitely large values. As the common domain of the ordinary real mathematical physics and the p-adic one we consider the field of rational numbers $\mathbb{Q}$, which is dense in $\mathbb{R}$ as well as in $\mathbb{Q}_p$ for every p. Thus we can say that non-archimedean fields give us the possibility to introduce new solutions of the equations of mathematical physics. In order to simplify our considerations we propose all proofs and constructions for the one-dimensional case. But, as the generalization to n dimensions is obvious, we consider also examples of differential equations with partial derivatives.

Throughout this paper K is a field with a non-trivial, non-archimedean valuation $| \, . \, |$, for which it is complete. Sequences $(\alpha_0, \alpha_1, \alpha_2, \dots)$ of elements of K will be denoted by (α_n) and e_n is the sequence $(0, 0, ..., 1, 0, 0, ...)$ with the 1 at place $n+1$. Also, if not specified otherwise, $\sum$ will always denote a sum for $n = 0$ to ∞ . A sequence space Λ is a vector subspace of $K^{\mathbb{N}}$ containing the finitely non-zero sequences. The Köthe-dual Λ^* of Λ is defined by $\Lambda^* = \{(\beta_n); \lim_n \alpha_n \beta_n = 0\}$ and Λ is called a perfect sequence space if $\Lambda^{**} = (\Lambda^*)^* = \Lambda$.

On Λ there is a natural locally convex Hausdorff topology $n(\Lambda, \Lambda^*)$ which is determined by the semi-norms $p_b((\alpha_n)) = \max_n |\alpha_n \beta_n|$, $(\alpha_n) \in \Lambda, b = (\beta_n) \in \Lambda^*$. For this topology Λ is complete, the sequences $e_n, n = 0, 1, 2, \dots$, form a Schauder basis for Λ and its topological dual space Λ' is isomorphic as a vector space to Λ^*. If Λ is perfect the same holds for the topology $n(\Lambda^*, \Lambda)$ on Λ^*.

For more information on locally convex sequence spaces we refer to [9,10].

For the general theory of locally convex spaces over K we refer to [14] if K is spherically complete and to [13] if K is not spherically complete. However we recall the basic notions used in this paper.

Let E, F denote locally convex Hausdorff spaces over K. A subset B of E is called compactoid if for every zero-neighbourhood V in E there exists a finite subset S of E such that $B \subset C_0(S) + V$, where $C_0(S)$ is the absolutely convex hull of S. A linear map T from E to F is called compact if there exists a zero-neigbourhood V in E such that $T(V)$ is compactoid in F.

If p is a continuous semi-norm on E we denote by E_p the vector space $E/Kerp$, and by $\pi_p : E \to E_p$ the canonical surjection. The space E_p is normed by $||\pi_p(x)||_p = p(x)$, $\forall x \in E$. If q is a continuous semi-norm on E with $p \leq q$ there exists a unique linear map $\phi_{pq} : Eq \to E_p$ satisfying $\phi_{pq} \circ \pi_q = \pi_p$. The space E is called nuclear if for every continuous semi-norm p on E there exists a continuous semi-norm q on E with $p \leq q$ such that the map ϕ_{pq} is compact. For basic properties of nuclear spaces we refer to [11] and for the nuclearity of sequence spaces to [10].

1 The space of entire functions.

1.1 Definitions :

For $R \in |K| \setminus \{0\}, R = |\rho|$, fixed we define the sequence spaces

$$B(R) = \{(\alpha_n); sup_n|\alpha_n|R^n < \infty\}$$

and

$$A(R) = \{(\alpha_n); \lim_n |\alpha_n|R^n = 0\}.$$

Obviously $B(R) \supset A(R)$.

The following is easily proved.

1.2 Proposition : $A(R)$ and $B(R)$ are perfect sequence spaces. More concretely :

1) $B(R)^* = \{(\beta_n); \lim_n |\beta_n|/R^n = 0\} = A(1/R)$
2) $B(R)^{**} = B(R)$
3) $A(R)^* = \{(\beta_n); sup_n|\beta_n|/R^n < \infty\} = B(1/R)$
4) $A(R)^{**} = A(R)$

1.3 Definitions :

Since $R_2 > R_1$ implies $B(R_1) \supset B(R_2)$ and $A(R_1) \supset A(R_2)$, it is reasonable to define the sequence spaces

$B = \cap\{B(R); R \in |K| \setminus \{0\}\}$ and $A = \cap\{A(R); R \in |K| \setminus \{0\}\}$.

Note that also, for any sequence (R_k) in $|K| \setminus \{0\}$ which increases to infinity, we have $B = \cap\{B(R_k); k = 1, 2, 3, \dots\}$ and $A = \cap\{A(R_k); k = 1, 2, 3, \dots\}$.

1.4 Proposition :

A is a perfect sequence space and $A = B$.

Proof :

We only prove $A \supset B$, the rest is straightforward.

Choose R and take any $R_1 > R$. It is then easy to see that $A(R) \supset B(R_1)$.

Then, for $R_2 > R_1$ etc... we obtain $B(R) \supset A(R) \supset B(R_1) \supset A(R_1) \supset B(R_2) \supset A(R_2) \dots$, from which the desired conclusion follows.

1.5 The topology on A :

On $A(R)$ there is a natural norm $||.||_R$, defined by $||(\alpha_n)||_R = \max_n |\alpha_n| R^n$ and it is easy to see that for this norm $A(R)$ is a Banach space. Moreover, if $R_1 > R$, the canonical injection $A(R_1) \to A(R)$ is continuous. Hence the natural topology on $A(R)$ is the projective limit topology, denoted by T_p. I.e. the coarsest locally convex topology on A making all the canonical injections $A \to A(R)$ continuous. This topology is determined by the family of norms $\{||.||_R; R \in |K| \setminus \{0\}\}$. In fact this family can be reduced to a sequence of norms $(||.||_{R_k})$ (see 1.3). Therefore the space A, T_p is metrizable.

From now on , if not specified, A will always be equipped with the projective limit topology T_p.

1.6 Proposition :

1) A is a nuclear Frechet space.
2) The elements $e_n, n = 0, 1, 2, \ldots$ form a Schauder basis for A.
3) The topology T_p on A is the same as the normal topology $n(A, A^*)$.

Proof :

1) See [10] 3.6.
2) Is straightforward.
3) See [10] 2.2

1.7 Definitions :

For $R \in |K| \setminus \{0\}$, put $U_R = \{\delta \in K; |\delta| \leq R\}$.

A function $f \ : U_R \to K$ is said to be *analytic on* U_R if it can be written as $f(x) = \sum \alpha_n x^n$, where the series converges for all $x \in U_R$. I.e. when $\lim_n |\alpha_n| R^n = 0$.

We denote by $\mathbf{A}(R)$ the vector space of the functions wich are analytic on U_R. Obviously the map $\mathbf{A}(R) \to A(R) \ : f = \sum \alpha_n x^n \mapsto (\alpha_n)$ is an isomorphism of vector spaces and we transfer the norm on $A(R)$, see 1.5, to the space $\mathbf{A}(R)$.

A function f is called an entire function if it is analytic on U_R for all $R \in |K| \setminus \{0\}$. We denote by $\mathbf{A}$ the vector space of the entire functions. I.e. $\mathbf{A} = \cap\{\mathbf{A}(R); R \in |K| \setminus \{0\}\}$.

Obviously $\mathbf{A}$ is algebraically isomorphic to the space A (1.3) and we transfer the topology T_p on A to the space $\mathbf{A}$.

Every polynomial is of course an entire function. But, in contrast to the complex case, the elementary functions $e^x, \sin x, \cos x$ etc... are not entire functions (see section 2). On the other hand, many "exotic" looking functions (from the real point of view) such as $f \ : \mathbb{Q}_p \to \mathbb{Q}_p \ : f(x) = \sum p^{p^n} x^n$, are entire functions.

2 The space of the functions which are analytic at zero.

2.1 Definitions :
For $R \in |K| \setminus \{0\}$ fixed we define the sequence spaces

$$B_0(R) = \{(\alpha_n); \sup_n |\alpha_n|/R^n < \infty\} = B(1/R)$$

and

$$A_0(R) = \{(\alpha_n); \lim_n |\alpha_n|/R^n = 0\} = A(1/R).$$

Obviously $B_0(R) \supset A_0(R)$.

The following is a direct consequence of 1.2.

2.2 Proposition :
$A_0(R)$ and $B_0(R)$ are perfect sequence spaces. More concretely

$$B_0(R)^* = A(R), B_0(R)^{**} = B_0(R), A_0(R)^* = B(R), A_0(R)^{**} = A_0(R).$$

2.3 Definitions :
Since $R_2 > R_1$ implies $B_0(R_2) \supset B_0(R_1)$ and $A_0(R_2) \supset A_0(R_1)$, it is reasonable to define the sequence spaces

$$B_0 = \cup\{B_0(R); R \in |K| \setminus \{0\}\} \text{ and } A_0 = \cup\{A_0(R); R \in |K| \setminus \{0\}\}.$$

Note that for any sequence $(R_k) in |K| \setminus \{0\}$ which increases to infinity we have

$$B_0 = \cup\{B_0(R_k); k = 1, 2, 3, \dots\} \text{ and } A_0 = \cup\{A_0(R_k); k = 1, 2, 3, \dots\}.$$

The proof of the following is analogous to 1.4.

2.4 Proposition :
A_0 is a perfect sequence space (more concretely : $A_0^* = A, A^* = A_0$) and $A_0 = B_0$.

2.5 The topology on A_0.
On $A_0(R)$ there is a natural norm $||.||_R$, defined by $||(\alpha_n)||_R = sup_n|\alpha_n|/R^n$ and it is easy to see that for this norm $A_0(R)$ is a Banach space.

Moreover, if $R_2 > R_1$, the canonical injection $A_0(R_1) \to A_0(R_2)$ is continuous.

Hence the natural topology on A_0 is the inductive limit topology. I.e. the finest locally convex topology on A_0 making all the canonical injections $A_0(R) \to A_0$ continuous. The inductive limit topology on A_0 will be denoted by T_i and, if not specified, the space A_0 will always carry this topology.

We have now immediately :

2.6 Proposition :
The vectors $e_n, n = 0, 1, 2, \dots$ form a Schauder basis for A_0, T_i.

Since $A_0 = A^*$ there is an other natural topology on A_0, namely the normal topology $n(A^*, A)$. We first compare the topologies T_i and $n(A^*, A)$.

2.7 Proposition :

The topology T_i is finer than the topology on A_0 induced by $n(A^*, A)$.

Proof :

It is sufficient to show that for each R the canonical injection $A_0(R) \to A_0, n(A^*, A)$ is continuous. So fix R and take $a \in A, a = (\alpha_n)$. Then for the continuous semi-norm p_a on $A_0, n(A^*, A)$ we have for $b = (\beta_n) \in A_0$:

$p_a(b) = \max_n |\alpha_n \beta_n| \leq \max_n |\alpha_n| R^n . \max_n |\beta_n|/R^n = ||b||_R.C$, where $C = \max_n |\alpha_n| R^n < \infty$, and the desired conclusion follows.

2.8 Lemma :

Let Λ be a perfect sequence space and T a locally convex Hausdorff topology on Λ for which the vectors $e_n, n = 0, 1, 2, \ldots$ form a Schauder basis. Then the topology T is coarser than the normal topology $n(\Lambda, \Lambda^*)$.

Proof :

Let p be a continuous semi-norm on Λ, T. We construct $b = (\beta_n) \in \Lambda^*$ such that $p((\alpha_n)) \leq p_b((\alpha_n))$ for all $(\alpha_n) \in \Lambda$. (*)

If $p(e_n) = 0$ take $\beta_n = 0$. For $p(e_n) \neq 0$ first note that there exists a real number $r > 1$ such that for every $n (n = 0, 1, 2, \ldots)$ one can find an element γ_n in K with $|\gamma_n| = r^n$. ([14] p. 251). Hence, for every n there exists a k_n such that $r^{k_n - 1} \leq p(e_n) < r^{k_n}$. Then choose $\beta_n \in K$ with $|\beta_n| = r^{k_n}$. For $(\alpha_n) \in \Lambda$, we have $|\alpha_n \beta_n| = |\alpha_n| r^{k_n} \leq |\alpha_n| p(e_n) r$.

But since $e_0, e_1, e_2, \ldots$ is a Schauder basis for Λ, T we have $\lim_n p(\alpha_n e_n) = 0$. Hence $\lim_n |\alpha_n \beta_n| = 0$ for all $(\alpha_n) \in \Lambda$ and therefore $b = (\beta_n) \in \Lambda^*$. Obviously the continuous semi-norm $p_b on \Lambda$ satisfies (*).

From 2.6, 2.7 and 2.8 we now obtain immediately the next result, which is crucial for the rest of the paper..

2.9 Proposition :

On A_0 the topologies T_i and $n(A^*, A)$ coincide.

2.10 Proposition :

The space A_0 is nuclear and complete.

Proof : The completeness follows from 2.9 and the nuclearity from 2.9 and [10] 5.4

2.11 Definitions :

A function f is called analytic at zero if there exists an $R \in |K| \setminus \{0\}$ such that f is analytic on U_R (see 1.7). We denote by $\mathbf{A}_0$ the vector space of the functions which are analytic at zero. I.e. $\mathbf{A}_0 = \cup \{\mathbf{A}_0(R); R \in |K| \setminus \{0\}\}$.

Obviously the map $\mathbf{A}_0 \to A_0 \ : \sum \alpha_n t^n \to (\alpha_n)$ is an algebraic isomorphism (see 2.3) and we transfer the topology T_i (see 2.5) on A_0 to $\mathbf{A}_0$.

2.12 Examples : ([12] 25.7)

Suppose char$(K) = 0$ and define :

1) $e^x = \sum x^n/n!, \sin x = \sum (-1) n x^{2n+1}/(2n+1)!, \cos x = \sum (-1)^n x^{2n}/(2n)!$, $\sinh x = \sum x^{2n+1}/(2n+1)!, \cosh x = \sum x^{2n}/(2n)!$.

2) $\log(1 + x) = \sum (-1)^{n+1} x^n/n$, here n starts with 1.

The series in 1) converge for $x \in E = \{x \in K; |x| < p^{1/p-1}\}$ where $p = \mathrm{char}(k)$, with k the residue class field of K.

The series in 2) converges for $|x| < 1$. So all these functions are analytic at zero.

3 The dual spaces. Spaces of distributions.

The function spaces $\mathbf{A}$ and $\mathbf{A}_0$ will be the spaces of test functions. Their topological dual spaces $\mathbf{A}'$ and $\mathbf{A}_0'$ are called - as usual - the spaces of (analytical) distributions.

3.1 Proposition : The strong topologies.

1) $(\mathbf{A}_0, T_i)'$ is algebraically isomorphic to $\mathbf{A}$. Hence the space of distributions $\mathbf{A}_0'$ can be interpreted as a function space.

2) When $(\mathbf{A}_0, T_i)'$ and $\mathbf{A}$ are identified as vector spaces the strong topology $\beta(\mathbf{A}_0', \mathbf{A}_0)$ coincides with the projective limit topology T_p on $\mathbf{A}$.

3) $(\mathbf{A}, T_p)'$ is algebraically isomorphic to $\mathbf{A}_0$. Hence the space of distributions $\mathbf{A}'$ can be interpreted as a function space.

4) When $(\mathbf{A}, T_p)'$ and $\mathbf{A}_0$ are identified as vector spaces the strong topology $\beta(\mathbf{A}', \mathbf{A})$ coincides with the inductive limit topology T_i on $\mathbf{A}_0$.

Proof : 1) Identify $\mathbf{A}_0, T_i$ with the space A_0, $n(A^*, A) = A_0$, $n(A_0, A_0^*)$ (see 2.9). Then $A_0' = A_0^* = A$ and now identify A with $\mathbf{A}$.

2) After the necessary identifications (as in 1)) we have to prove that $\beta(A, A^*) = n(A, A^*)$. This follows from the fact that $A, n(A, A^*)$ is a polar Frechet space, hence polarly barelled.([13] 6.5).

3) Analogous to 1).

4) Again, making the necessary identifications, we have to prove that $\beta(A_0, A_0^*) = n(A_0, A_0^*)$. This follows from the fact that the space $A_0^*, n(A_0^*, A_0) = A, n(A, A^*)$ is nuclear (see 1.6) and the following lemma.

3.2 Lemma :

Let Λ be a perfect sequence space. If $\Lambda^*, n(\Lambda^*, \Lambda)$ is nuclear then $n(\Lambda, \Lambda^*) = \beta(\Lambda, \Lambda^*)$.

Proof :

The topology $n(\Lambda, \Lambda^*)$ is the topology of uniform convergence on finite unions of the sets of the form $B_a = \{(\beta_n) \in \Lambda^*; |\beta_n| \leq |\alpha_n| \text{for all } n\}$, where $a = (\alpha_n) \in \Lambda^*$ ([9] prop.3). Let now B be a $\sigma(\Lambda^*, \Lambda)$- bounded (i.e. weakly bounded) subset of Λ^*. Then B is $n(\Lambda^*, \Lambda)$- bounded ([13] 7.5).By the nuclearity B is then $n(\Lambda^*, \Lambda)$-compactoid ([11] 5.1). Finally, with the same proof as in [9] prop.15, B is then contained in a set B_a for some $a \in \Lambda^*$. Hence the topology $\beta(\Lambda, \Lambda^*)$ is coarser than $n(\Lambda, \Lambda^*)$ and we are done.

3.3 Proposition :

The spaces $\mathbf{A}, T_p$ and $\mathbf{A}_0, T_i$ are reflexive.

Proof :

The reflexivity of $\mathbf{A}, T_p$ follows from [14] p.286 if K is spherically complete and from [13] 9.9 if K is not spherically complete.

The reflexivity of $\mathbf{A}_0, T_i$ follows from 3.1.3) and [13] 9.4.

3.4 Remark :

Let $g \in \mathbf{A}_0$' and denote by $< h, g >$ the value of g in $h \in \mathbf{A}_0$, $h(t) = \sum \alpha_n t^n$. Identify (see 2.11) h with $(\alpha_n) \in A_0$. Then we obtain a continuous linear map $g_1 \ : A_0, T_i \to K \ : (\alpha_n) \mapsto < h, g >$ and therefore (by 2.9) there exists a unique $(\gamma_n) \in A_0^* = A$ such that $< (\alpha_n), g_1 >= \sum \alpha_n \gamma_n, (\alpha_n) \in \mathbf{A}_0$. With these notations we obtain an algebraic isomorphism $\mathbf{A}_0' \to A \ : g \mapsto (\gamma_n)$ and where $< h, g >$ is now given by $\sum \alpha_n \gamma_n$.

As an example - and for later use - let $\mathrm{char}(K) = 0$ and define, for $y \in K$ the function $\exp y. \ : x \mapsto e^{xy}, x \in K$. Then $\exp y. \in \mathbf{A}_0$ (see 2.12) and it is identified with $(y_n/n!) \in A_0$. Now, for $g \in \mathbf{A}_0'$, identified with (γ_n) as above we have $< \exp y., g >= \sum \gamma_n y^n/n!$.

4 The Laplace transform (compare [8] 1.2)

From now on we assume that $\mathrm{char}(K) = 0$ and that the function $1/|n!|$ has exponential growth.
I.e.there exists a $C \in \mathbb{R}, b \in |K| \setminus \{0\}, b = |\beta| > 1$ with $1/|n!| < Cb^n$, for all n. (*)

Note that this is the case in $\mathbb{Q}_p$ and in any extension of $\mathbb{Q}_p$.

4.1 Definition :

For $g \in \mathbf{A}_0'$ and $y \in K$ define

$$\mathbf{L}(g)(y) =< \exp y., g > .$$

The function $L(g)$ is then - as in the complex case - called the Laplace transform of the function g.

4.2 Proposition :

The Laplace transform $\mathbf{L}$ is a linear homeomorphism from the space $\mathbf{A}_0', \beta(\mathbf{A}_0', \mathbf{A}_0)$ onto $\mathbf{A}, T_p$.

Proof :

By 3.4, 3.1 and 1.6 it is sufficient to prove that the linear map $L \ : A_0^* \to A \ : (\gamma_n) \mapsto (\gamma_n/n!)$ is an homeomorphism, when the topology on $A_0^* = A$ is the topology T_p on A which is also the topology $n(A, A^*)$. Note that by (*) L maps indeed A_0^* into A.

Obviously L is injective and the surjectivity of L follows from the fact that $|n!| \leq 1$ for all n.

L is also continuous.

Indeed, take a continuous seminorm p_d on $A, n(A, A^*)$; $d = (\delta_n) \in A^* = A_0$. Then, for $g = (\gamma_n) \in A_0^*$, we have : $p_d(L(g)) = p_d((\gamma_n/n!)) = \max_n |\gamma_n \delta_n/n!| \leq C \max_n |\gamma_n||\delta_n \beta^n|$, by (*).

There exists an $R \in |K| \setminus \{0\}$ with $lim_n |\delta_n|/R^n = 0$. Taking $R_1 = bR$ we obtain $\lim_n |\delta_n \beta^n|/R_1^n = 0$, which implies that $a = (\delta_n \beta^n) \in A_0$. Now $p_d(L(g)) \leq C p_a(g)$ and we are done.

Finally L is open because of the open mapping theorem : A_0^*, as well as A, is metrizable and complete.

4.3 Definition :

We denote by $\mathbf{L}^*$ the transposed of the Laplace transform $\mathbf{L}$. I.e. $\mathbf{L}^* : \mathbf{A}_0" \to \mathbf{A}'$ is defined by $< g, \mathbf{L}^*(f) >=< \mathbf{L}(g), f >, g \in \mathbf{A}', f \in \mathbf{A}_0'$. By 3.3, L^* can also be considered as a map from $\mathbf{A}_0$ to $\mathbf{A}'$. After making the usual identifications with the corresponding sequence spaces it is clear that then $\mathbf{L}^*$ corresponds with the transposed map $L^* : A^* \to A_0$, where L is as in the proof of 4.2.

4.4 Proposition :

The map $\mathbf{L}^*$ is a linear homeomorphism from $\mathbf{A}_0, \beta(\mathbf{A}_0, \mathbf{A}_0')$ onto $\mathbf{A}', \beta(\mathbf{A}', \mathbf{A})$.

Proof :

By the remark made above and by 3.1, it suffices to show that the map $L^* : A^*, n(A^*, A) \to A_0, n(A_0, A_0^*)$ is a linear homeomorphism.

We have for $(\alpha_n) \in A^*$ and $(\gamma_n) \in A_0^*$:

$$< (\gamma_n), L^*((\alpha_n)) >=< (\gamma_n/n!), (\alpha_n) >= \sum \gamma_n \alpha_n / n! =< (\gamma_n), (\alpha_n/n!) > .$$

(Note that, by (*), $(\alpha_n/n!) \in A_0$.)

Hence $L^*((\alpha_n)) = (\alpha_n/n!), (\alpha_n) \in A^*$.

As in 4.2 it now follows immediately that L^* is bijective and continuous.

Finally L^* is open.

Indeed, take $g = (\gamma_n) \in A = A_0^*$. Then for the corresponding semi-norm p_g on $A^*, n(A^*, A)$ and $(\alpha_n) \in A^*$ we have $p_g((\alpha_n)) = \max_n |\alpha_n \gamma_n| \leq \max_n |\alpha_n \gamma_n|/|n!|$.

Hence $p_g((\alpha_n)) \leq p_g(L^*((\alpha_n)))$ and we are done.

The rest of this section has to be seen as a preparation to section 5. The results which follow will be used in 5 without further reference.

4.5 Remark :

If δ is, as usual, the distribution defined by $< f, \delta >= f(0)$ then $L^*(\delta) = 1$.

Indeed, $< f, L^*(\delta) >=< L(f), \delta >= L(f)(0) =< f, 1 >$.

4.6 The derivative of a distribution :

First note that if $f \in \mathbf{A}$ (resp. $f \in \mathbf{A}_0$) then f', the derivative of f, is again in $\mathbf{A}$ (resp. $\mathbf{A}_0$).(see [12])

Then for $\phi \in \mathbf{A}'$ (resp $\phi \in \mathbf{A}_0'$) we define, as in the classical case, the derivative ϕ' of ϕ by $< f, \phi' >=< f', \phi >$. Hence ϕ' always exists and belongs to the same space of distributions as ϕ.

Also note that for $y \in K$: $\mathbf{L}(\phi')(y) = y\mathbf{L}(\phi)(y), \phi \in \mathbf{A}_0'$.

Indeed, $\mathbf{L}(\phi')(y) =< e^{xy}, \phi' >=< (e^{xy})', \phi >= y < e^{xy}, \phi >= y\mathbf{L}(\phi)(y)$.

In an analogous way one obtains : $\mathbf{L}^*(\phi')(y) = y\mathbf{L}(\phi)(y), \phi \in \mathbf{A}'$.

4.7 The convolution :

4.7.1. If $f \in \mathbf{A}(resp. f \in \mathbf{A}_0), f(x) = \sum \alpha_n x^n$, then it is easy to see that for every $y \in K$ the function $f(x+y)$, considered as a function in x, is still in $\mathbf{A}$ (resp. $\mathbf{A}_0$).

Let now $\phi \in \mathbf{A}'$ (resp. $\phi \in \mathbf{A}_0'$), then the convolution $\phi \star f$ of ϕ and f is defined - again as in the complex case - by $\phi \star f(x) =< f(x+y), \phi_y >$, where ϕ_y is ϕ acting on $f(x+y)$ with variable y and x as parameter.

Writing $f(x+y)$ as a power series in y, then making the usual identifications and taking into account that the binomial coefficients all have valuation ≤ 1, it is not hard to calculate the representation of $\phi \star f$ as a sequence and to see that this sequence is an element of $B = A$, see 1.4 (resp. $B_0 = A_0$, see 2.4)

Hence $\phi \star f \in \mathbf{A}$ whenever $f \in \mathbf{A}, \phi \in \mathbf{A}'$ and $\phi \star f \in \mathbf{A}_0$ whenever $f \in \mathbf{A}_0, \phi \in \mathbf{A}_0'$.

Note that in particular $\delta \star f(x) =< f(x+y), \delta_y >= f(x)$. Hence $\delta \star f = f$.

4.7.2. If ϕ and ψ are distributions then - again as usual - the convolution $\phi \star \psi$ is defined by $< f, \phi \star \psi >=<< f(x+y), \phi_x >, \psi_y >$ and it is easy to see that

$$\phi \star \psi \in \mathbf{A}' \text{whenever } \phi, \psi \in \mathbf{A}' \text{ and } \phi \star \psi \in \mathbf{A}_0' \text{ whenever } \phi, \psi \in \mathbf{A}_0'.$$

Also, as above, one derives that $\delta \star \phi = \phi \star \delta = \phi$ for all ϕ.

4.7.3.**Proposition :**

i) $\mathbf{L}(\phi \star \psi) = \mathbf{L}\phi.\mathbf{L}\psi$ for all $\phi, \psi \in \mathbf{A}_0'$.

ii) $\mathbf{L}^*(\phi \star \psi) = \mathbf{L}^*\phi.\mathbf{L}^*\psi$ for all $\phi, \psi \in \mathbf{A}'$.

Proof :

$\mathbf{L}(\phi \star \psi)(y) =< \exp y., \phi \star \psi >=<< \exp(x+y)., \phi_x >, \psi_y >= < \exp y. < \exp x.\phi_x >, \psi_y >=< \exp x., \phi_x >< \exp y, \psi_y >$.

An analogous calculation yields ii).

5 Applications.

5.1 Fundamental solutions of differential operators.

Let $b(D) = \sum_{k=0,\dots,m} b_k d^k/dx^k$, $b_k \in K$, be an arbitrary differential operator with constant coefficients. As usual, a solution of the equation

$$b(D)\xi(x) = \delta(x) \qquad (*)$$

is called a fundamental solution of the differential operator $b(D)$.

Theorem : Let $b_0 \neq 0$, then there exists a unique fundamental solution of the differential operator $b(D)$ in the space of distributions $\mathbf{A}'$.

Proof :

With the aid of the Laplace transform $\mathbf{L}^*$, we transform the equation (*) in the following equation in the space $\mathbf{A}_0$:

$$b(y)\mathbf{L}^*(\xi)(y) = 1, \qquad (**)$$

where $b(y) = \sum_{k=0,\dots,m} b_k y^k$ with the same coefficients b_k as in $b(D)$. As $b_0 \neq 0$, the function $g(y) = 1/b(y)$ belongs to the space $\mathbf{A}_0$ and the distribution $\xi = (\mathbf{L}^*)^{-1}(g) \in \mathbf{A}'$ is the solution of the equation (**).

As in the classical theory of differential equations the function

$$u(x) = \xi * f(x), f \in \mathbf{A},$$

is then the solution of the equation

$$b(D)u(x) = f(x). \qquad (***).$$

But here we can consider functions $f(x)$ that look very exotic from the archimedean point of view. For example $f(x) = \sum (n!)^{n!} x^n$.

The case of n variables $x = (x_1, x_2, \ldots, x_n) \in K^n$ is similar to the previous considerations. If $b(D) = \sum_{|a|=0,\ldots,m} b_\alpha \partial^\alpha / \partial x^\alpha$, with $b_\alpha \in K$ and $\partial^\alpha / \partial x^\alpha = \partial^{\alpha_1} / \partial x_1^{\alpha_1} \ldots \partial^{\alpha_n} / \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then the theorem is valid in the space $\mathbf{A}' = \mathbf{A}'(K^n)$.

5.2 Example : (The Klein- Gordon propagator with non-zero mass)
Let us introduce the Klein- Gordon operator.

$$\Box = \partial^2 / \partial x_0^2 - \Delta + m^2, \ m \in K, m \neq 0,$$

where Δ is the Laplace operator $\Delta = \sum_{j=1,\ldots,n} \partial^2 / \partial x_j^2$.

Then in the space $\mathbf{A}'(K^{n+1})$ there exists a unique fundamental solution of the Klein- Gordon operator ($\Box \xi = \delta$) and for every entire function $f(x_0, x) \in \mathbf{A}(K^{n+1})$ there exists the unique entire analytic solution of the equation $\Box u(x_0, x) = f(x_0, x)$.

In quantum field theory the function $f(x_0, x)$ is a field source. Thus with the aid of the non-archimedean field theory [2] we can realize sources which increase very rapidly as well as in time as in space directions. The most interesting case is $K = \mathbb{Q}_p$. Then the source $f(x_0, x)$ is defined by its values in the rational points $X = (x_0, x) \in \mathbb{Q}^{n+1}$ and, as it is in the the theory of analytic functions, then in the case of rational coefficients we can consider this function also valued in $\mathbb{R}$ (this is the object of the usual quantum field theory). But a lot of well defined $\mathbb{Q}_p$-valued functions $f(x_0, x)$ are not defined as $\mathbb{R}$-valued. For example the source $f(x_0, x) = \sum_{|a|=0,\ldots,\infty} (\alpha!)^{\alpha!} X^\alpha, X = (x_0, x)$, is infinitely strong from the real point of view, but we can find the $\mathbb{Q}_p$-analytic solution for this source.

On the basis of the fundamental solution of the Klein- Gordon equation we can develop a variant of the theory of perturbations for the non-archimedean Bosonic quantum field theory.

5.3 Example : (Dirac propagator with non-zero mass)
Let us introduce the Dirac operator $\partial\!\!\!/ + m, \ m \in K, m \neq 0$, where

$$\partial\!\!\!/ = \gamma_0 \partial / \partial x_0 + \gamma_1 \partial / \partial x_1 + \gamma_2 \partial / \partial x_2 + \gamma_3 \partial / \partial x_3,$$

and $\gamma_j, j = 0, \ldots, 3$ are Dirac's matrices.

Dirac's matrices contain $i = (-1)^{1/2}$ as their elements. If i exists in K, then we can restrict our considerations to the case of K-valued functions. If i is not in K we must modify all our constructions to the case of analytic functions valued in the quadratic extension $K(i)$ of the field K.

Using again theorem 5.1, we get the existence of the Dirac propagator $\xi(x)$, $(\partial\!\!\!/ + m)\xi(x) = \delta(x)$, in the space of analytic distributions $\mathbf{A}'(K^{n+1})$ and also the existence of the unique entire analytic solution $\Psi(x)$ of the equation

$$(\partial\!\!\!/ + m)\Psi = f, \ f \in \mathbf{A}(K^{n+1}).$$

Note that we can always consider a rational mass $m \in \mathbb{Q}$, because we can get only rational numbers in every physical experiment.

5.4 Theorem :

Let $b_0 \neq 0$, then the solutions of the differential equation (***) in 5.1 depend continuously on the right hand side $f(x)$ in the spaces of test functions $\mathbf{A}$ and distributions $\mathbf{A}'$.

To prove this theorem, it is sufficient to use the continuity of the Laplace transform operators $\mathbf{L}$ and $\mathbf{L}^*$.

5.5 Definition :

Let us consider a differential operator of infinite order

$$b(D) = \sum b_k d^k/dx^k,\ b_k \in K.$$

The function $b(y) = \sum b_k y^k$ is called the symbol of $b(D)$.

Theorem :

Let the symbol $b(y)$ of the differential operator of infinite order belong to the space $\mathbf{A}_0$ and let $b_0 \neq 0$. Then there exists a unique fundamental solution $\xi(x)$ of the equation (*) in 5.1, belonging to the space of distributions $\mathbf{A}'$.

Proof :

Using $\mathbf{L}^*$, we get again the equation $b(y)g(y) = 1$ in $\mathbf{A}_0$. But $b(y) = b_0 + x\phi(x)$, where $\phi(x) \in \mathbf{A}(R)$ for sufficiently small R.

If $\rho||\phi||\rho/|b_0| < 1$ then the function $1/b(y) \in \mathbf{A}(\rho), \rho \leq R$, and

$$ix(x) = (\mathbf{L}^*)^{-1}(g)(x)$$

is the fundamental solution.

Again $u(x) = \xi \star f(x), f \in \mathbf{A}$, is the solution of the equation (***).

5.1 The Cauchy problem :

Let us consider the Cauchy problem :

$$\partial u(t,x)/\partial t = b(D)u(t,x), \qquad (K_1)$$

$$u(x,0) = \phi(x), \qquad (K_2)$$

where $t, x \in K$.

Theorem :

Let $b(D)$ be a differential operator with symbol $b(y) \in \mathbf{A}_0$ and let $\phi \in \mathbf{A}$. Then there exists a solution $u(t,x), |t|_K \leq \delta, \delta = \delta(b)$ of the Cauchy problem $(K_1),(K_2)$ which is analytic with respect to t and x. The solution depends continuously on the initial data $\phi(x)$.

Remark :

It is an interesting problem to investigate the dependence of the solutions of the Cauchy problem $(K_1),(K_2)$ on the coefficients of the differential operator $b(D)$.

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