Well-posed Solutions of the Third Order Benjamin–Ono Equation in Weighted Sobolev Spaces

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Abstract

Here we continue the study of the initial value problem for the third order Benjamin-Ono equation in the weighted Sobolev spaces $H^s_\gamma = H^s \bigcap L^2_\gamma$, where $s > 3,\ \ \gamma \geq 0$. The result is the proof of well-posedness of the afore mentioned problem in $H^s_\gamma,\ \ s > 3,\ \ \gamma \in [0, 1]$. The proof involves the use of parabolic regularization, the Riesz-Thorin interpolation theorem and the construction technique of auxiliary functions.

1 Introduction

In recent years, the initial value problem for Benjamin-Ono(BO) equation of deep water,

$$\partial_t u + \partial_x(2Hu_x + u^2) = 0$$

has been investigated by many authors [1, 4, 5, 7, 8]. Iorio[5] established the well-posedness of the above BO equation in H^2_γ, where $\gamma \in [0, 1]$, by using Kato's theory of linear evolution equations of "hyperbolic type". Ponce [7] proved the global well-posedness of the BO equation in $H^s,\ \ s \geq 3/2$. In [3], the authors obtained the

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global existence and uniqueness both in the usual Sobolev space H^s and weighted Sobolev spaces H^s_γ for the third order BO equation

$$\partial_t u = -\partial_x(u^3 + 3uHu_x + 3H(uu_x) - 4\partial_x^2 u), \quad t > 0, \quad x \in R \tag{1.1}$$

$$u(x,0) = \phi(x) \tag{1.2}$$

which can be considered as the high order approximation of the BO equation. Here $u = u(x,t)$ is real valued, $\partial_x^k = \frac{\partial^k}{\partial x^k}$ and the subscripts appended to u stand for partial differentiations, and H denotes the Hilbert transform

$$(Hf)(x) = P.V. \int \frac{f(y)}{\pi(y-x)} dy$$

In this paper, $H^s_\gamma = H^s \bigcap L^2_\gamma$ with the following norm

$$||u||^2_{\gamma,s} = ||f||^2_s + |\omega_\gamma f|^2_2, \quad \omega_\gamma(x) = (1+x^2)^\gamma$$

where $||\bullet||_s$ and $|\bullet|_2$ are the usual norms of the Sobolev spaces $H^s(R)$ and Lebesgue space $L^2(R)$, respectively. Throughout the paper, $|\bullet|_p$ will denote the usual norm of the Lebesgue spaces $L^p(R), \ \ p \in [1,\infty]$. If X and Y are Banach spaces we denote the set of all bounded operators from X to Y by $B(X,Y)$. If $X = Y$ we write $B(X)$ instead of $B(X,Y)$. As usual $[A,B]$ will indicate the commutator of two linear operators(not necessarily continuous).

The BO equation arises in the study of unidirectional propagation of nonlinear dispersive waves or long nonlinear sausage-wave propagation in a magnetic slab in an incompressible plasma of the solar atmosphere, and presents the interesting fact that operrators modelling the dispersive effect in the BO equation or the effect of derivative nonlinearity in (1.1) are nonlocal. In [3, 5], the authors have shown that $u \in C([0,T]; H^4_{\frac{3}{2}})$ or $u \in C([0,T]; H^4_2)$ solves equation (1.1) or the BO equation if and only if $u \equiv 0$, and pointed out in [3] that this phenomena are caused by the nonlocal terms modelling the equations. The papers [1, 7] are devoted to the proof of the well-posedness of the global solutions to the initial value problem of the BO equation in H^s, for $s \geq \frac{3}{2}$. Many physicists such as Ruderman (see [8] and references therein) have used this equation to study the wave propagation in a magnetically structured atmosphere and found its solutions in the form of periodic waves of permanent shape numerically. As the high order approximation of the BO equation, the third order BO equation (1.1) is interesting in its own rights and therefore it is of importance to study its well-posedness whether mathematically or physically considered. I would here like to apologize if I miss citing other papers on the BO equation.

Our aim here is to investigate the global well-posedness of problem (1.1), (1.2) in the weighted Sobolev spaces H^s_γ as done in [5] for the BO equation.

Our main result is as follows:

Theorem 1.1. Let $\phi \in H^s_\gamma, \ \ s \geq 4$ and $\gamma \in [0,1]$. Then for all fixed $\mu \geq 0$ there exists a unique $u_\mu \in C([0,T]; H^s)$ for any given $T > 0$ such that $\partial_t u_\mu \in C([0,T]; H^{s-4})$ and $\omega'_\gamma \partial_x^k u_\mu \in C([0,T]; L^2)$ for $k \leq s/2$ and the following (2.6), (2.7) are satisfied. Moreover,

$$|\omega'_\gamma \partial_x^k u|_2 \leq C(\gamma)||u||_{\gamma,s}$$

Furthermore, the map $\phi \mapsto u_\mu$ is continuous in the sense that if $\phi_n \to \phi$ in H_γ^{s-2} and u_γ^n, u_μ are the corresponding solutions then

$$\lim_{n\to\infty} sup_{[0,T]}||u_\mu^n(t) - u_\mu(t)||_{\gamma,s-2} = 0$$

where $s \geq 6$.

In fact, we shall show that the map $\phi \mapsto u$ is globally Lipschitz. In our cases, it is difficult to apply Kato's theory of linear evolution equations of "hyperbolic type" because of the appearance of the derivative nonlinear nonsmooth terms. The proof in this paper is very technical and long. In the following two sections we shall finish the proof of Theorem 1.1 by presenting a series of statements.

2 Preliminary results

In this part, a series of lemmas are given for the purpose of proving Theorem 1.1. The lemmas below are established following the same lines as those of [5].

Lemma 2.1. Suppose $a(x)$ is defined on R and $|a(x)-a(y)| \leq M|x-y|, \;\; x, \;\; y \in R$. Let $T_\epsilon(f) = \int_{|x-y|\geq\epsilon} \left\{\frac{a(x)-a(y)}{(x-y)^2}\right\} f(y)dy$. Then

$$|T_\epsilon(f)|_p \leq A_p|f|_p, \quad \text{for all} \quad f \in L^p(R)$$

with A_p independent of ϵ, $\;1 < p < \infty$.

For details and further results of this kind see Calderon [2] and P.238 of Stein [9]. The following well-known Gagliardo-Nirenberg's inequality will be unexplainedly used throughout the paper.

Lemma 2.2. For $f \in H^m(R)$ and $p \geq 2$, then we have

$$|\partial_x^d f|_p \leq 2^{\frac{(p-2)}{2p}} |\partial_x^k u|_2^\lambda |u|^{1-\lambda} \tag{2.1}$$

where $d, \;\; m, \;\; k \in N\bigcup\{0\}, \;\; \lambda = (d + \frac{1}{2} - \frac{1}{p})/k$, and $0 \leq d < k \leq m$. More precisely, for $f \in H^{d+1}(R)$ we have

$$|\partial_x^d f|_\infty \leq |f|_2^{\frac{1}{2(d+1)}} |\partial_x^{d+1} f|_2^{\frac{2d+1}{2(d+1)}} \tag{2.2}$$

Proof. (2.1) has been well-known. To prove (2.2), noting that

$$|f(x)|^2 = 2\int_{-\infty}^x ff_x dx \leq 2\int_{-\infty}^x |ff_x|dx$$

$$|f(x)|^2 = -2\int_x^\infty ff_x dx \leq 2\int_x^\infty |ff_x|dx$$

We get by using the Cauchy-Schwartz inequality

$$|f|_\infty^2 \leq \int |ff_x|dx \leq |f|_2|f_x|_2 \tag{2.3}$$

which gives (2.2) for the case d=0. By using (2.1) and (2.3) we get

$$|\partial_x^d u|_\infty^2 \leq |\partial_x^d u|_2 |\partial_x^{d+1} u|_2 \leq |\partial_x^{d+1} u|_2^{\frac{2d+1}{d+1}} |u|_2^{\frac{1}{d+1}}$$

which finishes the proof of (2.2).

It should be pointed out that inequality (2.2) can be used to improve the uniqueness result for the smallness case of the initial data in [3].

Lemma 2.3. Let $\lambda \in [0, \infty)$, $\gamma \in [0, 1]$, $\mu > 0$, and $Q_\mu = -\mu\partial_x^4 + 4\partial_x^3$. Then $\mathcal{S}_\mu(t) = exp(tQ_\mu) \in B(H_\gamma^s, H_\gamma^{s+\lambda})$ for all $t > 0$, $s \geq 0$ and satisfies the estimate

$$||\mathcal{S}_\mu(t)\phi||_{\gamma,s} \leq G(t, \mu, \lambda, \gamma)||\phi||_{\gamma,s} \tag{2.4}$$

for all $\phi \in H_\gamma^s$, where $G(t, \mu, \lambda, \gamma)$ is locally integrable with respect to t. Moreover the map $t \in (0, \infty) \mapsto \mathcal{S}_\mu(t)\phi$ is continuous with respect to the topology of $H_\gamma^{s+\lambda}$.

Proof. Due to [3], where the Lemma is established in the case of $H_0^s = H^s$, $s \in R$, it suffices to examine $\mathcal{S}_\mu(t)$ as an operator from H_γ^s into $H_\gamma^0 = L_\gamma^2$. If $\gamma = 0$ we obtain $H_0^0 = L^2$ so that $||\mathcal{S}_\mu(t)\phi||_{0,0} \leq ||\phi||_{0,0}$. Next, if $\gamma = 1$ and $\phi \in H_1^s = H^s \bigcap L_1^2$ we have

$$(\omega_1 \mathcal{S}_\mu(t)\phi)^\wedge(\xi) = E_\mu(\xi, t)(1 - \partial_\xi^2)\hat{\phi} - (\partial_\xi^2 E_\mu(\xi, t))\hat{\phi} - 2(\partial_\xi E_\mu(\xi, t))\partial_\xi\hat{\phi} \tag{2.5}$$

where $E_\mu(\xi, t) = exp[-(\mu\xi^4 + 4i\xi^3)t]$. The result then follows by combining (2.5) with the formulas

$$\partial_\xi E_\mu(\xi, t) = -(4\mu\xi^3 + 12i\xi^2)tE_\mu(\xi, t)$$

$$\partial_\xi^2 E_\mu(\xi, t) = -12t(\mu\xi^2 + 2i\xi)E_\mu(\xi, t) + 16t^2\xi^4(\mu\xi + 3i)E_\mu(\xi, t)$$

and the estimate

$$0 \leq \xi^{2\lambda} exp(-\mu t\xi^4) \leq \lambda^\lambda exp(\frac{\lambda}{2})(2\mu t)^{-\lambda/2}$$

The statement in the Lemma is now an easy consequence of the Riesz-Thorin interpolation theorem.

We are now in position to establish a local existence for the following parabolic regularized problem

$$\partial_t u_\mu = -\mu\partial_x^4 u_\mu - \partial_x(u_\mu^3 + 3u_\mu H\partial_x u_\mu + 3H(u_\mu\partial_x u_\mu) - 4\partial_x^2 u_\mu) \tag{2.6}$$

$$u_\mu(x, 0) = \phi(x) \tag{2.7}$$

in H_γ^s, $\gamma \in [0, 1]$ and $\mu > 0$. Let T be a positive number and consider the set

$$X_{\gamma,s}(T) = \Big\{f \in C([0, T]; H_\gamma^s);\ \ ||f(t) - \mathcal{S}_\mu(t)\phi||_{\gamma,s} \leq ||\phi||_{\gamma,s},\ \ t \in [0, T]\Big\}$$

which becomes a complete metric space when provided with the distance obtained from the sup norm. Note that if $f \in X_{\gamma,s}(T)$, then by Lemma 2.3 it is not difficult to show that one can choose $T > 0$ such that the map

$$(Af)(t) = \mathcal{S}_\mu(t)\phi - \int_0^t \mathcal{S}_\mu(t - \tau)\partial_x\left(f^3 + 3fHf_x + 3H(ff_x)\right)d\tau$$

defines a contraction in $X_{\gamma,s}(T)$. In view of the uniqueness and regularity results established in [3] we have

Corollary 2.4. Let $\phi \in H_\gamma^s$, $s \geq 4$, $\gamma \in [0, 1]$, $\mu > 0$. Then there exist a $T > 0$ and a unique $u_\mu \in C([0, T]; H_\gamma^s)$ such that $\partial_t u_\mu \in C([0, T]; H^{s-4})$ satisfies problem (2.6), (2.7). Moreover, $u_\mu \in C((0, T]; H_\gamma^q)$ for all $q \in R$.

Let $\mu > 0$ be fixed. From now on, unless otherwise specifed, we write $u = u_\mu$ for simplicity. The next step is to establish global existence in H^s_γ, $s \geq 4$ and $\gamma \in [0, 1]$. This has already been done in [3] in the cases $\gamma = 0$ and $\gamma = 1$ (the result for $\gamma = 1$ will be re-obtained below from a slightly different point of view). In order to obtain global estimates for the H^s_γ norm of the solution it suffices to study what happens in L^2_γ since the H^s result was proved in [3]. Now, by Corollary 2.4 and integration by parts, at least formally we have

$$\begin{aligned}\partial_t||u(t)||^2_{\gamma,0} + 2\mu\int_0^t\int \omega^2_\gamma u^2_{2x}dx\\ = -4\mu\int uu_{2x}\omega_\gamma\omega''_\gamma dx - 4\mu\int uu_{2x}(\omega'_\gamma)^2dx - 8\mu\int u_xu_{2x}\omega_\gamma\omega''_\gamma dx\\ -8\int\omega^2_\gamma u\partial_x(u^3)\,dx - 3\int\omega^2_\gamma u^2H(\partial^2_x u)dx + 6\int\omega_\gamma\omega'_\gamma u^2Hu_xdx\\ +2\int(\omega^2_\gamma)'''u^2dx - 24\int\omega_\gamma\omega'_\gamma uu_{2x}\,dx - 3\int\omega^2_\gamma uH(\partial^2_x(u^2))dx \qquad (2.8)\end{aligned}$$

In order to extend the local solution to the whole time interval $[0, T]$, for any fixed $T > 0$ we must estimate the right-hand side of (2.8). For this purpose we need the following Lemmas.

Lemma 2.5. Let $u \in H^4_\gamma$, $\gamma \in [0, 1]$. Then $\omega_{\gamma/2}u_{2x} \in L^2$ and satisfies

$$|\omega_{\frac{\gamma}{2}}u_{2x}|_2 \leq C||u||^{\frac{1}{2}}_4|\omega_\gamma u|^{\frac{1}{2}}_2 \leq C||u||_{\gamma,4} \qquad (2.9)$$

where C is a generic constant.

Proof. By definition, for $u \in S(R)$(the Schwartz space) we have

$$\begin{aligned}|\omega_{\frac{\gamma}{2}}u_{2x}|^2_2 = \int\omega_\gamma u^2_{2x}dx = \int\omega''_\gamma uu_{2x}\,dx + 2\int\omega'_\gamma uu_{3x}dx + \int\omega_\gamma uu_{4x}dx\\ \leq |\omega''_\gamma u|_2|u_{2x}|_2 + 2|\omega'_\gamma u|_2|u_{3x}|_2 + |\omega_\gamma u|_2|u_{4x}|_2 \leq C||u||_4|\omega_\gamma u|_2 \qquad (2.10)\end{aligned}$$

which combines a limiting argument to yield the result.

Lemma 2.6. Let $f \in H^s_\gamma$, $\gamma \in [0, 1]$. Then $x(1 + x^2)^\gamma\partial^k_x f \in L^2$ and

$$|x(1 + x^2)^{(r-1)}\partial^k_x f|_2 \leq C(r)||f||_{\gamma,s},\quad 2k \leq s \qquad (2.11)$$

where $C(\gamma)$ is a constant depending only on γ. The same results are true if $x(x^2 + 1)^{\gamma-1}$ is replaced by $\omega'_\gamma(x)$.

Proof. A simple limiting argument shows that it suffices to prove (2.11) for $f \in S(R)$. Assume therefore that this is the case and integrate by parts to obtain for $k = 1$

$$|x(1 + x^2)^{\gamma-1}f_x|^2_2 = -\sum_{j=1}^3 I_j$$

where

$$I_1 = 2\int(1 + x^2)^\gamma f(x)[x(1 + x^2)^{\gamma-2}]f_xdx$$

$$I_2 = 2(2\gamma - 2)\int(1 + x^2)^\gamma f(x)[x^3(1 + x^2)^{\gamma-3}]f_xdx$$

$$I_3 = \int (1+x^3)^\gamma f(x)[x^2(1+x^2)^{\gamma-2}]\partial_x^2 f(x)dx$$

It is easy to verify that the expressions inside the square brackets in all the three ingegrals are bounded of x. By this remark and the Cauchy-Schwartz inequality we obtain

$$|I_1| \leq 2|f_x|_2|\omega_\gamma|_2, \quad |I_2| \leq |4\gamma - 4||f_x|_2|\omega_\gamma f|_2, \quad |I_3| \leq |\partial_x^2 f|_2|\omega_\gamma f|_2$$

Now, we have

$$|x(1+x^2)^{\gamma-1} f_x|_2 \leq \{(2+|4\gamma-4|)|f_x|_2 + |f_{2x}|_2\} |\omega_\gamma f|_2 \leq C(\gamma)||f||_2|\omega_\gamma f|_2$$

which finishes the proof. For the general case, a similar argument yields

$$|x(1+x^2)^{\gamma-1}\partial_x^k f|_2^2 = \sum_{j=k}^{2k} \int (1+x^2)^\gamma f(x) p_j(x) \partial_x^j f(x)dx$$

where $p_j(x)$ is bounded in x. So the result follows.

Corollary 2.7. Let $f \in H_\gamma^4$, $\gamma \in [0,1]$. Then $\omega_{\frac{\gamma}{2}} f_x \in L^2$, $\omega_{\frac{\gamma}{2}} f_x \in L^\infty$ and satisfy the estimates

$$|\omega_{\frac{\gamma}{2}} f_x|_2 \leq C(\gamma)||f||_2^{\frac{1}{2}} |\omega_\gamma f|_2^{\frac{1}{2}} \tag{2.12}$$

$$|\omega_{\frac{\gamma}{2}} f_x|_\infty \leq C(\gamma)||f||_4^{\frac{1}{4}} |\omega_\gamma f|_2^{\frac{1}{2}} \tag{2.13}$$

where $C(\gamma) > 0$ depends only on γ.

Proof. Similar to the proof of (2.9), it suffices to prove (2.12) for $f \in S(R)$. If this is the case, then we have

$$|\omega_{\frac{\gamma}{2}} f_x|_2^2 = \int \omega_\gamma f_x f_x dx = -\int \omega_\gamma f f_{2x} dx - \int \omega_\gamma' f f_x dx$$

$$\leq \left(|f_x|_2|\omega_\gamma' f|_2 + |f_{2x}|_2|\omega_\gamma f|_2\right) \leq C(\gamma)||f||_2|\omega_\gamma f|_2$$

which gives (2.12). By (2.2), (2.9), (2.11) and (2.12) we obtain

$$|\omega_{\frac{\gamma}{2}} f_x|_\infty^2 \leq |\omega_{\frac{\gamma}{2}} f_x|_2 |\omega_{\frac{\gamma}{2}}' f_x + \omega_{\frac{\gamma}{2}} f_{2x}|_2 \leq C(\gamma)||f||_4|\omega_\gamma f|_2$$

This finishes (2.13).

Lemma 2.8. Let $f \in H_\gamma^s$, $\gamma \in [0,1]$. Then $[\omega_{\frac{\gamma}{2}}, H\partial_x]f \in L^2$ and

$$|[\omega_{\frac{\gamma}{2}}, H\partial_x]f|_2 \leq C(\gamma)|f|_2 \tag{2.14}$$

where $C(\gamma)$ depends only on γ.

Proof. First observe that it suffices to prove (2.14) in case of $f \in S(R)$. Now

$$[\omega_{\frac{\gamma}{2}}, H\partial_x]f = [\omega_{\frac{\gamma}{2}}, H]\partial_x f - H[\partial_x, \omega_{\frac{\gamma}{2}}]f \tag{2.15}$$

For the second commutator in (2.15) we have

$$H[\partial_x, \omega_{\frac{\gamma}{2}}]f = H(\omega_{\frac{\gamma}{2}}' f) \tag{2.16}$$

For the first commutator in (2.15) we integrate by parts once to obtain

$$[\omega_{\frac{\gamma}{2}}, H]\partial_x f = P.V.\frac{1}{\pi}\int \frac{\omega_{\frac{\gamma}{2}}(x) - \omega_{\frac{\gamma}{2}}(y)}{(x-y)^2} f(y)dy - H(\omega'_{\frac{\gamma}{2}} f) \tag{2.17}$$

From (2.15)-(2.17) we know that

$$[\omega_{\frac{\gamma}{2}}, H\partial_x] f = P.V.\frac{1}{\pi}\int \frac{\omega_{\frac{\gamma}{2}}(x) - \omega_{\frac{\gamma}{2}}(y)}{(x-y)^2} f(y)dy - 2H(\omega'_{\frac{\gamma}{2}} f) \tag{2.18}$$

Since $\omega'_{\frac{\gamma}{2}}$ is bounded and $\omega_{\frac{\gamma}{2}}$ is Lipschitz, by Lemma 2.1 and the fact that $H \in B(L^2)$ it follows from (2.18) that

$$|[\omega_\gamma, H\partial_x^2]f|_2 \leq C(\gamma)|f|_2$$

which finishes the proof.

From the proof above we see that (2.14) holds for all $f \in L^2$.

Lemma 2.9. Let $f \in H^2_\gamma, \gamma \in [0,1]$. Then $[\omega_\gamma, H\partial_x^2]f \in L^2$ and

$$|[\omega_\gamma, H\partial_x^2]f|_2 \leq C(\gamma)||f||_{\gamma,2} \tag{2.19}$$

where $C(\gamma) \geq 0$ depends only on γ.

Proof. First note that it suffices to prove (2.19) for $f \in S(R)$. Evidently

$$[\omega_\gamma, H\partial_x^2]f = [\omega_\gamma, H]\partial_x^2 f - H[\partial_x^2, \omega_\gamma]f \tag{2.20}$$

So it suffices to establish (2.19) with $[\omega_\gamma, H\partial_x^2]f$ replaced by $H[\partial_x^2, \omega_\gamma]f$ and $[\omega_\gamma, H]\partial_x^2 f$. The result then follows for the first of these functions by combining the formula

$$H[\partial_x^2, \omega_\gamma]f = H(\omega''_\gamma f + 2\omega'_\gamma \partial_x f) \tag{2.21}$$

with Lemma 2.6 and the facts that $\omega''_\gamma \in L^\infty$ and $H \in B(L^2)$. In order to handle $[\omega_\gamma, H]\partial_x^2 f$ we integrate by parts once to obtain

$$[\omega_\gamma, H]\partial_x^2 f = H(\omega'_\gamma \partial_x f) - P.V.\frac{1}{\pi}\int \frac{\omega_\gamma(x) - \omega_\gamma(y)}{(x-y)^2} f(y)dy \tag{2.22}$$

In view of Lemma 2.6 the first term on the right-hand side of (2.22) satisfies $|H(\omega'_\gamma \partial_x f)|_2 \leq C(\gamma)||f||_{\gamma,2}$, so it remains to bound the second. Since ω'_γ is bounded if and only if $\gamma \in [0, \frac{1}{2}]$, in this case ω_γ is Lipschitz. Hence, by Lemma 2.1 we have

$$\left| P.V.\frac{1}{\pi}\int \frac{\omega_\gamma(x) - \omega_\gamma(y)}{(x-y)^2} f(y)dy \right| \leq A|f|_2 \tag{2.23}$$

where $A > 0$ is a generic constant, $\gamma \in [0, \frac{1}{2}]$. In the case of $\gamma \in (\frac{1}{2}, 1]$ we integrate by parts once in (2.22) to get

$$[\omega_\gamma, H]\partial_x^2 f = -P.V.\frac{1}{\pi}\int K(x,y)g(y)dy \tag{2.24}$$

$$K(x,y)=\frac{2(\omega_\gamma(x)-\omega_\gamma(y))-2\omega_\gamma'(y)(x-y)-\omega_\gamma''(y)(x-y)^2}{(x-y)^3(1+y^2)^\gamma} \tag{2.25}$$

where $g(y)=(1+y^2)^\gamma f(y)\in L^2$. If $\gamma\in(\frac{1}{2},1]$, it is not difficult to verify that $K(x,y)$ is a Hilbert-Schmidt kernel. Thus

$$\left|P.V.\frac{1}{\pi}\int K(x,y)\omega_\gamma(y)f(y)dy\right|\le C|\omega_\gamma f|_2 \tag{2.26}$$

where $C>0$ is a generic constant, $\gamma\in(\frac{1}{2},1]$. Now the result of the lemma follows from (2.20)-(2.26).

Now we are in position to estimate each term in the right-hand side of (2.8) as follows:

$$R_1)\quad 4\mu\left|\int uu_{2x}\omega_\gamma\omega_\gamma''dx\right|\le\frac{\mu}{3}|\omega_\gamma u_{2x}|_2^2+C\mu|\omega_\gamma''u|_2\le\frac{\mu}{3}|\omega_\gamma u_{2x}|_2^2+C\mu|u|_2^2$$

$$R_2)\quad 4\mu\left|\int uu_{2x}(\omega_\gamma')^2dx\right|\le\frac{\mu}{3}|\omega_\gamma u_{2x}|_2^2+C\mu|\omega_\gamma u|_2^2$$

$$R_3)\quad 8\mu\left|\int u_xu_{2x}\omega_\gamma\omega_\gamma''dx\right|\le\frac{\mu}{3}|\omega_\gamma u_{2x}|_2^2+C\mu|u_x|_2^2$$

$$R_4)\quad 8\left|\int(\omega_\gamma)^2u\partial_x(u^3)dx\right|\le C(||u||_2)|\omega_\gamma u|_2^2$$

$$R_5)\quad 3\left|\int(\omega_\gamma)^2u^2H(\partial_x^2u)dx\right|\le C(||u||_3)|\omega_\gamma u|_2^2$$

$$R_6)\quad 6\left|\int\omega_\gamma'\omega_\gamma u^2Hu_xdx\right|\le C(||u||_2)|\omega_\gamma u|_2^2$$

$$R_7)\quad 2\left|\int(\omega_\gamma)'''u^2dx\right|\le|\omega_\gamma u|_2^2$$

$$R_8)\quad 24\left|\int\omega_\gamma\omega_\gamma'uu_{2x}dx\right|\le 24|\omega_\gamma u|_2|\omega_\gamma'u_{2x}|_2\le C|\omega_\gamma u|_2|\omega_{\frac{\gamma}{2}}u_{2x}|_2\le C(||u||_4)|\omega_\gamma u|_2^2$$

$$\begin{aligned}R_9)\quad &\int(\omega_\gamma)^2uH\partial_x^2(u^2)dx=\int\omega_\gamma u[\omega_\gamma,H\partial_x^2]u^2dx+\int\omega_\gamma uH\partial_x^2(\omega_\gamma u^2)dx\\
&=\int\omega_\gamma u[\omega_\gamma,H\partial_x^2]u^2dx+\int\omega_\gamma uH[\omega_\gamma''u^2+2\omega_\gamma'uu_x+2\omega_\gamma u_x^2+2\omega_\gamma uu_{2x})dx\\
&\le C|\omega_\gamma u|_2\left\{|[\omega_\gamma,H\partial_x^2]u^2|_2+|\omega_\gamma''u^2|_2|\omega_\gamma'uu_x|_2+|\omega_\gamma u_x^2|_2+|\omega_\gamma uu_{2x}|_2\right\}\\
&\le C(\gamma)|\omega_\gamma u|_2\left\{||u^2||_{\gamma,2}+|u^2|_2+|\omega_\gamma'uu_x|_2+|\omega_{\frac{\gamma}{2}}u_x|_\infty|\omega_{\frac{\gamma}{2}}u_x|_2+|u_{2x}|_\infty|\omega_\gamma u|_2\right\}\\
&\le C(\gamma,||u||_4)|\omega_\gamma u|_2\end{aligned}$$

Considering R_1–R_9) and (2.8) we obtain

$$\frac{d}{dt}||u(t)||_{\gamma,0}^2+\mu\int(\omega_\gamma)^2u_{2x}^2dx\le C(\mu,\gamma,||u||_4)|\omega_\gamma u|_2^2 \tag{2.8$'$}$$

By [3] we know that $||u(t)||_s$ is bounded uniformly for t in any fixed interval $[0,T]$ and μ varying in any bounded interval. That is,

$$||u(t)||_s\le C(\mu,||\phi||_s),\quad t\in[0,T] \tag{2.27}$$

where $C(\mu, ||\phi||_s) > 0$ depends in an nondecreasing way on μ and $||\phi||_s$, respectively. If μ remains in a bounded set $[0, \mu_0]$, one can choose $C(\mu, ||\phi||_s)$ depending only on $||\phi||_s$. With this remark in mind, apply Gronwall's inequality to $(2.8)'$ to obtain

$$||u(t)||_{\gamma,0} \leq C(\mu, \gamma, T, ||\phi||_{\gamma,4}), \quad t \in [0, T] \tag{2.28}$$

where $C(\mu, \gamma, T, ||\phi||_{\gamma,4})$ has the same property as that of (2.27).

Now we return to the global existence result. If $\mu > 0$, the global existence result follows from (2.27) and (2.28).The existence result in the case $\mu = 0$ can be proved by the standard limiting argument.

Proof of uniqueness. Here the proof is reproduced from [3]. Let u, v be two solutions to problem (2.6), (2.7) with the initial data $\phi(x)$, $\psi(x) \in H^s$, respectively. Putting $w = u - v$, then w satisfies the following equation

$$w_t = -\mu\partial_x^4 w + 4\partial_x^3 w - \partial_x \left[(u^3 - v^3) + 3wHu_x + 3vHw_x + \frac{3}{2}H\partial_x(u^2 - v^2) \right] \tag{2.29}$$

$$w(x, 0) = \phi(x) - \psi(x) \tag{2.30}$$

For $\mu > 0$ fixed, the uniqueness result follows from the standard energy estimate. So in the following we only give the proof for $\mu = 0$.

From (2.29), a direct calculation of (2.29) and use of Lemma 2.2 yield

$$\begin{aligned} \frac{d}{dt}\int w^2 dx = 2\int \Big\{ &-\frac{1}{2}w^2(u^2 + uv + v^2)_x + 3ww_xHu_x + 3vw_xHw_x \\ &- \frac{3}{2}(u + v)w_xHw_x - \frac{3}{2}(u_x + v_x)wHw_x \Big\} dx \end{aligned} \tag{2.31}$$

Now it is not difficult to find that

$$|w|_2^2 \leq |w(0)|_2 + C(||\phi||_2, ||\psi||_2) \int_0^t ||w(t)||_1^2 dt \tag{2.32}$$

Since $s \geq 4$, we know that $|u_t|_{L^\infty([0,T]\times R)}$, $|v_t|_{L^\infty([0,T]\times R)}$, $|\partial_x^k u(x,t)|_{L^\infty([0,T]\times R)}$ and $|\partial_x^k v(x,t)|_{L^\infty([0,T]\times R)}$ are bounded for $0 \leq k \leq s - 1$. Multiply equation (2.29) by $K_4(u) - K_4(v)$ to know

$$\int w_t \left(K_4(u) - K_4(v)\right) dx = 0 \tag{2.33}$$

where $K_4(u) = u^3 + 3uHu_x + 3H(uu_x) - 4u_{2x}$.

The bound for the term in the right-hand side of (2.33) is carried out as follows.

$$1) \quad \int_0^t \int w_t(u^3 - v^3) dx dt$$

$$\begin{aligned} &= \frac{1}{2}\int w^2(u^2 + uv + v^2)dx\Big|_0^t - \frac{1}{2}\int_0^t \int w^2(u^2 + uv + v^2)_t dx dt \\ &\leq C(|w(t)|_2^2 + |w(0)|_2^2 + \int_0^t |w(\tau)|_2^2 d\tau) \end{aligned} \tag{2.34}$$

$$2)\quad \int_0^t\int w_t[3uHw_x+3wHv_x+\frac{3}{2}H(u^2-v^2)_x]dxdt$$

$$=\left[\frac{3}{2}\int w^2Hv_x\,dx+3\int uwHw_xdx\right]\Big|_0^t-3\int_0^t\int u_t[wHw_x+H(ww_x)]dxdt$$

$$\leq \delta|w_x|_2^2+C(\delta)|w|_2^2+C||w(0)||_1^2+C\int_0^t||w(\tau)||_1^2d\tau \tag{2.35}$$

$$3)\quad -4\int_0^t\int w_tw_{2x}dxdt=2|w_x|_2^2-2|w_x(0)|_2^2 \tag{2.36}$$

Adjusting the value of δ, from (2.32)-(2.36) we derive

$$||w(t)||_1^2\leq C(||\phi||_1,||\psi||_1)||w(0)||_1^2+C(||\phi||_4,||\psi||_4)\int_0^t||w(\tau)||_1^2d\tau \tag{2.37}$$

Applying Gronwall's inequality to (2.37) to yield

$$||w(t)||_1^2\leq C(T,||\phi||_4,||\psi||_4)||w(0)||_1^2,\quad t\in[0,T] \tag{2.38}$$

which implies the uniqueness.

Up to now we have proved the following

Corollary 2.10. Let $\phi\in H_\gamma^s,\quad s\geq 4$ and $\gamma\in[0,1]$. Then for all fixed $\mu\geq 0$ there exists a unique $u_\mu\in C([0,T];H_\gamma^s)$ for any given $T>0$ such that $\partial_tu_\mu\in C([0,T];H^{s-4})$ and $\omega_\gamma'\partial_x^ku_\mu\in C([0,T];L^2)$ for $2k\leq s$ and (2.6), (2.7) are satisfied.

3 Proof of Theorem 1.1 Continued

In this section we continue the proof of Theorem 1.1.

Lemma 3.1. Let $\phi,\quad \psi\in H_\gamma^s,\quad s\geq 4$ and $\gamma\in[0,1],\quad \mu\geq 0$. Let $u_\mu,\quad v_\mu$ be the solutions(obtained in Corollary 2.10) to (2.6) with $\phi,\quad \psi$ as the initial data, respectively. Suppose $w=u_\mu-v_\mu$. Then we have

$$||w(t)||_{\gamma,0}^2\leq ||w(0)||_{\gamma,4}^2+C(\mu,T,||\phi||_{\gamma,4},||\psi||_{\gamma,4})\int_0^t||w(t)||_{\gamma,4}^2dt,\quad t\in[0,T] \tag{3.1}$$

where C depends in an nondecreasing way on its arguments.

Proof. By the assumptions of the Lemma we know that w satisfies problem (2.29), (2.30). Multiply (2.29) by ω_γ^2w and integrate by parts to obtain

$$\frac{d}{dt}\int\omega_\gamma^2w^2\,dx+2\mu\int\omega_\gamma^2(\partial_x^2w)^2dx$$

$$=\left\{-4\mu\int ww_{2x}\omega_\gamma\omega_\gamma''dx-4\mu\int ww_{2x}(\omega_\gamma')^2dx-8\mu\int w_xw_{2x}\omega_\gamma\omega_\gamma''dx\right\}$$

$$+\left\{-8\int\omega_\gamma^2w\partial_x(u^3-v^3)dx+2\int(\omega_\gamma^2)'''w^2dx\right\}-24\int\omega_\gamma\omega_\gamma'ww_{2x}dx$$

$$+\left\{-3\int\omega_\gamma^2w^2Hv_{2x}dx+3\int(\omega_\gamma^2)'w^2Hv_xdx\right\}-6\int\omega_\gamma^2wuHw_{2x}dx$$

$$-6\int \omega_\gamma^2 w u_x H w_x dx - 3\int \omega_\gamma^2 w H\partial_x^2(u^2-v^2)dx \equiv \sum_{j=1}^{7} B_j \tag{3.2}$$

The estimate for each B_j is as follows. By using (2.9), (2.11), (2.12), (2.13) and (2.14), we obtain

$$|B_1| \le \mu|\omega_\gamma w_{2x}|_2^2 + \mu C|\omega_\gamma w|_2^2, \quad |B_2| \le C(||\phi||_2, ||\psi||_2)|\omega_\gamma w|_2^2$$

$$|B_3| \le 24\left|\int \omega_\gamma \omega_\gamma' w w_{2x} dx\right| \le 24|\omega_\gamma w|_2 |\omega_\gamma' w_{2x}|_2 \le C|\omega_\gamma w|_2 |\omega_{\frac{\gamma}{2}} w_{2x}|_2 \le C||w||_{\gamma,4}^2$$

$$|B_4| \le C(T, ||\phi||_3, ||\psi||_3)|\omega_\gamma w|_2^2, \quad |B_5| \le C|\omega_\gamma u|_2 ||w||_{\gamma,2}^2$$

$$B_6 = 6\int \omega_\gamma^2 w u_x H w_x dx = 6\int \omega_\gamma w \omega_{\frac{\gamma}{2}} u_x [\omega_{\frac{\gamma}{2}}, H\partial_x] w dx + 6\int \omega_\gamma w \omega_{\frac{\gamma}{2}} u_x H\partial_x(\omega_{\frac{\gamma}{2}} w) dx$$

$$\le 6|\omega_{\frac{\gamma}{2}} u_x|_\infty \left(|\omega_\gamma w|_2 |[\omega_{\frac{\gamma}{2}}, H\partial_x] w|_2 + |\omega_\gamma w|_2 |\partial_x(\omega_{\frac{\gamma}{2}} w)|_2\right) \le C(||u||_{\gamma,4})||w||_{\gamma,4}^2$$

$$\begin{aligned} B_7 &= 3\int \omega_\gamma^2 w H\partial_x^2(u^2-v^2)dx \\ &= 3\int \omega_\gamma w[\omega_\gamma, H\partial_x^2](u^2-v^2)dx + 3\int \omega_\gamma w H\partial_x^2(\omega_\gamma w(u+v))dx \\ &\le C|\omega_\gamma w|_2 \left(||u^2-v^2||_{\gamma,2} + |\partial_x^2(\omega_\gamma(u+v)w)|_2\right) \\ &\le C(T, ||\phi||_{\gamma,4}, ||\psi||_{\gamma,4})||w||_{\gamma,4}^2 \end{aligned}$$

Considering (3.1) and (3.2) we know that

$$\frac{d}{dt}|\omega_\gamma w|_2^2 + \mu\int (\omega_\gamma)^2(\partial_x^2 w)^2 dx \le C(\mu, T, ||\phi||_{\gamma,4}, ||\psi||_{\gamma,4})||w||_{\gamma,4}^2 \tag{3.3}$$

Integrate (3.3) with respect to t to yield the result.

Lemma 3.2. Let the assumptions in Lemma 3.1 be satisfied and $s \ge 6$. Then we have

$$\begin{aligned} &\frac{d}{dt}\left(|\partial_x^{s-1} w|_2^2 + \frac{2s-3}{4}\int (u+v)\partial_x^{s-2} w H\partial_x^{s-2} w dx\right) \\ &\qquad \le C(\mu, T, ||\phi||_s, ||\psi||_s)||w(t)||_{s-2}^2, \quad t \in [0,T] \end{aligned} \tag{3.4}$$

where C depends in an increasing way on its arguments and remains bounded whenever its arguments stay in a bounded set.

Proof. The proof of (3.4) for $\mu > 0$ is simple and is omitted. But the proof for $\mu = 0$ is technical. For $s = 6$, a complicated calculation yields

$$\frac{d}{dt}|\partial_x^{s-2} w|_2^2 \le -27\int (u+v)_x \partial_x^4 w H\partial_x^5 w dx + C(T, ||\phi||_6, ||\psi||_6)||w(t)||_4^2$$

and

$$\frac{d}{dt}\int (u+v)\partial_x^{s-3} w H\partial_x^{s-2} w dx \le 12\int (u+v)_x \partial_x^4 w H\partial_x^5 w dx + C(T, ||\phi||_6, ||\psi||_6)||w(t)||_4^2$$

These two inequalities mean that the derivative term in the left hand side equals the first term in the right hand side plus a term less than the second term in the right hand side. Hence, (3.4) holds for $s = 6$. For general s we can prove (3.4) by induction on s. The details are omitted.

The Completion of Proof of Theorem 1.1. By Lemma 2.2 we have

$$\frac{2s-3}{4}\left|\int (u+v)\partial_x^{s-3}wH\partial_x^{s-2}wdx\right| \le C(|\phi|_2,|\psi|_2)|\partial_x^{s-3}w|_\infty|\partial_x^{s-2}w|_2$$

$$\le C|w|^{\frac{1}{2(s-2)}}|\partial_x^{s-2}w|_2^{\frac{2s-5}{2(s-2)}+1} \le \frac{1}{2}|\partial_x^{s-2}w|_2^2 + C|w|_2^2 \tag{3.5}$$

From (2.38), (3.4) and (3.5) there appears that

$$|\partial_x^{s-2}w|_2^2 \le C||w(0)||_{s-2}^2 + C\int_0^t ||w(t)||_{s-2}^2 dt \tag{3.6}$$

Combining (2.38), (3.1), (3.6) with Gronwall's inequality we can obtain

$$||w(t)||_{\gamma,s-2}^2 \le C(T,||\phi||_{\gamma,s},||\psi||_{\gamma,s})||w(0)||_{\gamma,s-2}^2, \quad t\in[0,T]$$

which implies the last statement of Theorem 1.1.

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