Frobenius Collineations in Finite Projective Planes

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1 Introduction

Given a finite field $F = GF(q^n)$ of order q^n it is well-known that the map $f : F \to F$, $f : x \mapsto x^q$ is a field automorphism of F of order n, called the *Frobenius automorphism.* If V is an n-dimensional vector space over the finite field $GF(q)$, then V can be considered as the vector space of the field $GF(q^n)$ over $GF(q)$. Therefore the Frobenius automorphism induces a linear map over $GF(q)$

$$\begin{aligned} R &: V \to V \\ R &: x \mapsto x^q \end{aligned}$$

of order n on V. It follows that R induces a projective collineation φ on the $(n-1)$-dimensional projective space $PG(n-1, q)$. We call φ and any projective collineation conjugate to φ a *Frobenius collineation.* In the present paper we shall study the case $n = 3$, that is, the Frobenius collineations of the projective plane $PG(2, q)$.

Let $P = PG(2, q^2)$. Then every Singer cycle σ (see Section 3) of P defines a partition $\mathcal{P}(\sigma)$ of the point set of P into pairwise disjoint Baer subplanes. These partitions are called *linear Baer partitions* or, equivalently, *Singer Baer partitions* [17]. If ϱ is a Frobenius collineation of P, then we define $\mathcal{E}_\varrho$ to be the set of Baer subplanes of P fixed by ϱ. It turns out that for $q \equiv 2 \mod 3$ we have $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| \in \{0, 1, 3\}$ with $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$ if and only if $\varrho \in N_G(<\sigma>)$, where $G = PGL_3(q^2)$ (see 3.5).

Received by the editors March 1996.
Communicated by J. Thas.
1991 *Mathematics Subject Classification* : 51E20, 51E24.
Key words and phrases : diagram geometry, Baer partitions, Frobenius collineations.

Therefore we define a geometry $\mathcal{F}$ of rank 2 as follows: Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P.

- The points of $\mathcal{F}$ are the Baer subplanes of $\mathcal{E}_\varrho$.
- The lines of $\mathcal{F}$ are the sets $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$, where σ is a Singer cycle such that $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$.
- A point $B \in \mathcal{E}_\varrho$ and a line $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$ are incident if and only if $B \in \mathcal{P}(\sigma)$.

Then the geometry $\mathcal{F}$ is called a *Frobenius plane of order* q. Our first result reads as follows:

Theorem 1.1. *Let* $q \equiv 2 \mod 3$, *and let* $\mathcal{F}$ *be a Frobenius plane of order* q.

a) $\mathcal{F}$ *is a partial linear space with* $3(q^2 - 1)$ *points,* $\frac{2}{3}(q^2-1)^2$ *lines, three points on a line and* $\frac{2}{3}(q^2 - 1)$ *lines through a point.*

b) Given a non-incident point-line-pair $(B, \mathcal{G})$*, then there are either one or two lines through* B *intersecting* $\mathcal{G}$.

c) Let d_0*,* d_1 *and* g *be the 0-diameter*[1]*, the 1-diameter and the gonality of* $\mathcal{F}$*, respectively.*

If $q = 2$*, then* $d_0 = d_1 = g = 4$*. Actually,* $\mathcal{F}$ *is a* 3×3*-grid.*

If $q > 2$*, then* $d_0 = d_1 = 4$ *and* $g = 3$.

d) The group $PGL_3(q^2)$ *acts flag-transitively on* $\mathcal{F}$.

Parts a), b), c) and d) are proved in 4.3, 4.6, 4.7 and 5.5, respectively. – The Frobenius planes can be used to construct a geometry Γ of rank 3 as follows: For, let $P = PG(2, q^2), q \equiv 2 \mod 3$.

- The points of Γ are the Baer subplanes of P.
- The lines of Γ are the sets $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$ for some Singer cycle σ and some Frobenius collineation ϱ such that $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$.
- The planes of Γ are the sets $\mathcal{E}_\varrho$, where ϱ is a Frobenius collineation.
- The incidence relation is defined by set-theoretical inclusion.

The geometry Γ is called a *Frobenius space of order* q. Our second result reads as follows:

Theorem 1.2. *Let* $q \equiv 2 \mod 3$, *and let* Γ *be the Frobenius space of order* q.

a) Γ *is a geometry of rank 3, whose planes are Frobenius planes of order* q.

b) The group $PGL_3(q^2)$ *acts flag-transitively on* Γ.

Part a) and b) are proved in 5.4 and 5.5, respectively. In Theorems 5.4 and 5.5 the combinatorial parameters (number of points, etc.) and the various flag stabilizers are stated.

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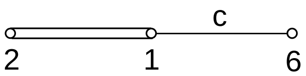
[1] The 0-diameter is the maximal distance from an element of type 0 in the incidence graph of $\mathcal{F}$. For details see Buekenhout, [3], Sec. 3.3.2

The "history" of this paper is as follows: In [17] I studied the dihedral groups generated by the Baer involutions τ_1 and τ_2 of two disjoint Baer subplanes B_1 and B_2 of $P = PG(2, q^2)$. It turned out that $\delta := \tau_1\tau_2$ is a projective collineation whose order is a divisor of $q^2 - q + 1$. If δ is of order $q^2 - q + 1$, then the point orbits of δ are complete $(q^2 - q + 1)$-arcs. Furthermore the orbit of Baer subplanes of δ containing B_0 and B_1 is a Singer Baer partition. In particular it turned out that any two disjoint Baer subplanes are contained in exactly one Singer Baer partition.

The last observation motivated me to define in [18] a geometry $\mathcal{B}_q$ (Baer geometry of order q) of rank 2 whose points are the Baer subplanes of P and whose lines are the Singer Baer partitions. The Baer geometry of order q admits $PGL_3(q^2)$ as a flag-transitive automorphism group. For $q = 2$ it turned out that the corresponding Baer geometry is the point-line-truncation of a geometry of rank 3 with diagram



still admitting $PGL_3(4)$ as flag-transitive automorphism group.

The present paper grew out of the attempt to find for all possible q a rank-3-geometry Γ_q such that the Baer geometry $\mathcal{B}_q$ is a point-line-truncation of Γ_q. It turned out that good candidates for the lines of such a geometry are not the whole Singer Baer partitions but parts of them consisting of exactly three Baer subplanes. In this way the Frobenius spaces have been found.

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The present paper is organized as follows: In Section 2 we shall study some elementary properties of Frobenius collineations and we shall introduce the sets $\mathcal{E}_\varrho$. Section 3 is devoted to the study of the possible intersections of a set $\mathcal{E}_\varrho$ and a Singer Baer partition. In Section 4 we shall define the Frobenius planes of order q and we shall prove Theorem 1.1. Finally, in Section 5, we shall give a proof of Theorem 1.2.

Acknowledgement. I wish to thank Albrecht Beutelspacher, Jorg Eisfeld, Udo Heim and Katja Valentin for many valuable discussions.

2 Frobenius Collineations

The present section is devoted to the study of the elementary properties of the Frobenius collineations (see 2.1) like the number of fixed points (2.4) or different matrix representations (2.5 and 2.6). In particular we shall show that in the case $q \equiv 1 \mod 3$ for any triangle $\{x, y, z\}$ of $PG(2, q)$ there exists exactly one Frobenius group admitting x, y, z as fixed points.

Given a Frobenius collineation ϱ of $PG(2, q^2)$ we shall denote by $\mathcal{E}_\varrho$ the set of Baer subplanes fixed by ϱ. In the second half of this section we shall study the following properties of $\mathcal{E}_\varrho$: possible intersections of two subplanes of $\mathcal{E}_\varrho$ (2.10, 2.12 and 2.13), computation of the groups fixing $\mathcal{E}_\varrho$ element- or setwise (2.14 and 2.15), computation of the number of the sets $\mathcal{E}_\varrho$ (2.17) and of the number of the sets $\mathcal{E}_\varrho$ containing a given Baer subplane (2.18).

Let V be the 3-dimensional vector space over $GF(q)$, that is, $V = GF(q)^3 \cong GF(q^3)$, and let $R : V \to V$ be defined by $R(x) := x^q$. Since R is a linear and bijective map from V onto V, it induces a projective collineation φ of the projective plane $P(V) = PG(2,q)$. (The map $R : GF(q^3) \to GF(q^3)$, $R : x \mapsto x^q$ is often considered as a collineation of the projective plane $PG(2,q^3)$ leaving a subplane $PG(2,q)$ pointwise invariant. Note that our approach is different.)

Definition 2.1. Let $P = PG(2,q)$, and let G be the group $PGL_3(q)$. A collineation ϱ of P is called a *Frobenius collineation* if ϱ is conjugate to φ by some element of G.

The group $< \varrho >$ is called a *Frobenius subgroup* of G.

Proposition 2.2. *Let $P = PG(2,q)$, and let ϱ be a Frobenius collineation of P.*

a) ϱ is of order 3.

b) ϱ has at least one fixed point.

c) ϱ is not a central collineation.

Proof. It is sufficient to prove the proposition for the collineation φ defined above.

a) Obviously, φ is of order 3.

b) The field $GF(q^3)$ has exactly one subfield F isomorphic to $GF(q)$. If we consider $GF(q^3)$ as a 3-dimensional subspace V over $GF(q)$, then F is a 1-dimensional subspace fixed by the map $R : V \to V$, $R : x \mapsto x^q$. Hence F is a fixed point of φ.

c) Again, consider $V = GF(q^3)$ as a 3-dimensional vector space over $GF(q)$. Then V admits a primitive normal base (see JUNGNICKEL [9], Result 3.1.13), that is, a base of the form $\{\omega, \omega^q, \omega^{q^2}\}$, where ω is a primitive element of $GF(q^3)$. So the elements ω, $\varphi(\omega)$ and $\varphi^2(\omega)$ define a triangle in the corresponding plane $PG(2,q)$. In particular φ cannot be a central collineation. ∎

Proposition 2.3. *Let $P = PG(2,q)$, and let α be a projective collineation of order 3. Then one of the following cases occurs.*

(i) α is a central collineation.

(ii) α has no fixed point.

(iii) α is a Frobenius collineation.

Proof. Suppose that $1 \neq \alpha$ is neither of type (i) nor of type (ii).

Step 1. There is a point z of P which is not incident with any fixed line of α. Assume that any point of P is incident with a fixed line of α. Then α admits at least $q+1$ fixed lines and therefore at least $q+1$ fixed points. Since α is not a central collineation, the fixed points form a k-arc with $k \geq q+1$. Since $1 \neq \alpha$, it follows $k \leq 3$, hence $q = 2$. A collineation of $PG(2,2)$ with at least three fixed points is either a central collineation or the identity, a contradiction.

Step 2. The points $z, \alpha(z)$ and $\alpha^2(z)$ form a triangle. Otherwise $z, \alpha(z)$ and $\alpha^2(z)$ would be collinear, and the line through $z, \alpha(z)$ and $\alpha^2(z)$ would be a fixed line contradicting the choice of z.

Step 3. Let p be a fixed point of α. Then the points $z, \alpha(z), \alpha^2(z)$ and p form a quadrangle. Assume for example that p, z and $\alpha(z)$ were collinear. Then the line l through p, z and $\alpha(z)$ is a fixed line, it follows that $\alpha^2(z)$ is incident with l, a contradiction.

Step 4. Let α and α' be two projective collineations of order 3 which are neither of type (i) nor of type (ii). Then α and α' are conjugate. By Steps 1 - 3 there exist fixed points p and p' and orbits $\{x,y,z\}$ and $\{x',y',z'\}$ forming triangles of α and α', respectively. Let β be a projective collineation of P with $\beta(x) = x'$, $\beta(y) = y'$, $\beta(z) = z'$ and $\beta(p) = p'$. Then we have $\beta\alpha\beta^{-1}(r') = \alpha'(r')$ for all $r' \in \{x',y',z',p'\}$. Hence α and $\alpha' = \beta\alpha\beta^{-1}$ are conjugate.

Step 5. Let α be a projective collineation of order 3 which is neither of type (i) nor of type (ii). Then α is a Frobenius collineation. Let ϱ be a Frobenius collineation of P. By Proposition 2.2, ϱ is a projective collineation of order 3 which is neither of type (i) nor of type (ii). By Step 4, α and ϱ are conjugate, hence α is a Frobenius collineation. ∎

Lemma 2.4. *Let $P = PG(2,q)$, and let ϱ be a Frobenius collineation.*
If $q \equiv 0 \mod 3$, then ϱ has exactly one fixed point.
If $q \equiv 1 \mod 3$, then ϱ has exactly three fixed points.
If $q \equiv 2 \mod 3$, then ϱ has exactly one fixed point.

Proof. We first observe that ϱ has at most three fixed points. (Otherwise ϱ would either have four fixed points forming a quadrangle which implies $\varrho = 1$ or ϱ would have at least three collinear fixed points which implies that ϱ is a central collineation. Both cases cannot occur.) So ϱ has either one, two or three fixed points.

If $q \equiv 0 \mod 3$, then $q^2 + q + 1 \equiv 1 \mod 3$. Since ϱ has point orbits of either one or three points, it follows that ϱ has exactly one fixed point. For the rest of the proof it suffices to observe that if $q \equiv 1 \mod 3$, then $q^2 + q + 1 \equiv 0 \mod 3$ and if $q \equiv 2 \mod 3$, then $q^2 + q + 1 \equiv 1 \mod 3$. ∎

Proposition 2.5. *Let $P = PG(2,q)$, and let ϱ be a Frobenius collineation of P. Then ϱ can be represented by the following matrix*

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. Let $V = GF(q^3)$ and consider V as a 3-dimensional vector space over $GF(q)$. It suffices to consider the map $R : V \to V$ defined by $R : x \mapsto x^q$. Let $B := \{\omega, \omega^q, \omega^{q^2}\}$ be a primitive normal base (see Jungnickel [9], Result 3.1.13) of V. With respect to B, R has the matrix representation described in the proposition. ∎

Proposition 2.6. *Let $P = PG(2,q)$, $q \equiv 1 \mod 3$, and let ϱ be a Frobenius collineation of P. Let $< \theta >$ be the (multiplicative) subgroup of order 3 of $GF(q)^*$. Then ϱ is induced by the linear map with matrix representation*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^2 \end{pmatrix}$$

with respect to the basis defined by the three fixed points of ϱ.

Proof. Since $q \equiv 1 \mod 3$, ϱ has exactly three fixed points, say p_1, p_2, p_3. If V is the 3-dimensional vector space over $GF(q)$, then there exist three vectors x, y, z of

V with $p_1 = < x >$, $p_2 = < y >$ and $p_3 = < z >$. With respect to the basis $\{x, y, z\}$ the collineation ϱ is induced by a linear map R with matrix representation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

for some elements $\mu, \nu \in GF(q^3)$. Because of $\varrho^3 = 1$ we have $R^3 = 1$, and it follows $\mu^3 = \nu^3 = 1$. Since ϱ is neither the identity nor a central collineation, it follows that the values $1, \mu$ and ν are pairwise distinct. Hence $\{1, \mu, \nu\} = < \theta >$.[2] ■

Corollary 2.7. *Let $P = PG(2, q)$, $q \equiv 1 \mod 3$. Then for any triangle $\{p_1, p_2, p_3\}$ of P there exist exactly two Frobenius collineations ϱ and ϱ' admitting p_1, p_2, p_3 as fixed points. Furthermore we have $\varrho' = \varrho^2 = \varrho^{-1}$.*

Corollary 2.8. *Let $P = PG(2, q)$, $q \equiv 1 \mod 3$. Then the number of Frobenius subgroups of P equals $\frac{1}{6}\,(q^2 + q + 1)\,(q^2 + q)\,q^2$.*

Proof. By 2.7, any triangle of P defines exactly one Frobenius subgroup and vice versa. It follows that the number of Frobenius subgroups equals the number of triangles. ■

The number of Frobenius groups of $PGL_3(q^2)$ with $q \equiv 2 \mod 3$ is determined in 2.19.

Definition 2.9. Let $P = PG(2, q^2)$, and let ϱ be a Frobenius collineation of P. Then we define $\mathcal{E}_\varrho$ to be the set of all Baer subplanes of P whose point and line set are fixed (setwise) by ϱ.

Proposition 2.10. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P. Let p_1, p_2, p_3 be the fixed points of ϱ.*

a) The points p_1, p_2, p_3 form a triangle.

b) For any plane $B \in \mathcal{E}_\varrho$ the collineation ϱ induces a Frobenius collineation in B. Furthermore we have $|B \cap \{p_1, p_2, p_3\}| = 1$.

c) Let x be a point which is not incident with any of the lines p_1p_2, p_2p_3, p_1p_3. Then the points $x, \varrho(x), \varrho^2(x)$ form a triangle.

d) Let x be a point not incident with any of the lines p_1p_2, p_2p_3, p_1p_3. Then for each $j \in \{1, 2, 3\}$ there is exactly one plane of $\mathcal{E}_\varrho$ containing x and p_j.

e) We have $|\mathcal{E}_\varrho| = 3\,(q^2 - 1)$.

f) Let l be the line p_2p_3, and let $\mathcal{E}_1$ be the set of planes of $\mathcal{E}_\varrho$ containing p_1.

(i) Let B and B' be two Baer subplanes of $\mathcal{E}_1$. Then either $B \cap B'$ does not contain any point of l or B and B' share $q + 1$ points of l.

(ii) The set $\{l \cap B \mid B \in \mathcal{E}_1\}$ is a partition of $l \setminus \{p_2, p_3\}$ into $q - 1$ pairwise disjoint Baer sublines of l.

(iii) If $B \in \mathcal{E}_1$, then there are $q+1$ elements B' of $\mathcal{E}_1$ with $B \cap l = B' \cap l$ (including B itself).

[2] Alternatively one also can consider the matrix representation of ϱ given in 2.5. Its characteristic polynomial is $x^3 - 1$. Because of $q \equiv 1 \mod 3$ the field $GF(q)$ has three third unit roots. As a consequence we get the matrix representation described above.

Proof. a) Since $q \equiv 2 \mod 3$, it follows that $q^2 \equiv 1 \mod 3$, by 2.4, ϱ has three fixed points. Since ϱ is not a central collineation, these three fixed points have to form a triangle.

b) Let $B \in \mathcal{E}_\varrho$. By definition $\varrho(B) = B$. Hence ϱ induces a projective collineation on B of order 3. Since $q \equiv 2 \mod 3$, it follows that $q^2+q+1 \equiv 1 \mod 3$. Therefore ϱ has at least one fixed point in B. Since ϱ is no central collineation in P, it cannot induce a central collineation in B. By 2.3, ϱ induces a Frobenius collineation in B admitting exactly one fixed point. In particular we have $|B \cap \{p_1, p_2, p_3\}| = 1$.

c) Assume that $x, \varrho(x)$ and $\varrho^2(x)$ were collinear. Then the line through x and $\varrho(x)$ is a fixed line different from p_1p_2, p_2p_3 and p_1p_3. Thus ϱ has at least four fixed lines and therefore four fixed points, a contradiction.

d) The points p_1, x, $\varrho(x)$ and $\varrho^2(x)$ form a quadrangle. Let B be the Baer subplane through these four points. Then B is fixed by ϱ, hence $B \in \mathcal{E}_\varrho$.

Let B' be a second Baer subplane through p_1 and x. Then it contains $p_1, x, \varrho(x)$ and $\varrho^2(x)$. Hence $B = B'$.

e) Let $\mathcal{E}_1$ be the set of Baer subplanes of $\mathcal{E}_\varrho$ containing p_1. By d), the planes of $\mathcal{E}_1$ cover the points of P outside the lines p_1p_2, p_2p_3, p_1p_3.

Let $B \in \mathcal{E}_1$. Since ϱ contains one fixed point in B, it also contains a fixed line l in B. Since $q+1 \equiv 0 \mod 3$, the line l cannot be incident with p_1. Hence $l = p_2p_3$. It follows that B contains $q^2 + q + 1 - (q + 2) = q^2 - 1$ points not incident with any of the lines p_1p_2, p_2p_3, p_1p_3. It follows that $|\mathcal{E}_1| = \frac{(q^2-1)^2}{q^2-1} = q^2 - 1$. Therefore $|\mathcal{E}| = 3(q^2 - 1)$.

f) (i) Let B, B' be two planes of $\mathcal{E}_1$ sharing a point $z \in l$. Then $z \neq p_2, p_3$. Hence $B \cap B'$ share the three collinear points $z, \varrho(z)$ and $\varrho^2(z)$. It follows that $B \cap B'$ contains $q + 1$ points on l.

(ii) and (iii) The set $\{B \cap l \mid B \in \mathcal{E}_1\}$ is by (i) a partial partion of $l \setminus \{p_2, p_3\}$ into at most $q - 1$ pairwise disjoint Baer sublines of l. Let $B \cap l$ be one of these Baer sublines. Through $B \cap l$ and p_1 there are exactly $q + 1$ Baer subplanes. It follows that $|\mathcal{E}_1| \leq (q - 1)(q + 1)$. In view of $|\mathcal{E}_1| = q^2 - 1$ it follows that all $q + 1$ Baer subplanes through $B \cap l$ and p_1 belong to $\mathcal{E}_1$ and that the set $\{B \cap l \mid B \in \mathcal{E}_1\}$ consists of $q - 1$ mutually disjoint Baer sublines of l. ∎

Corollary 2.11. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P with fixed points p_1, p_2, p_3. If l is the line p_2p_3, then there is exactly one partition $\{s_1, \ldots, s_{q-1}\}$ of $l \setminus \{p_2, p_3\}$ into $q - 1$ disjoint Baer sublines.*[3]

Then the planes of $\mathcal{E}_\varrho$ containing p_1 are exactly the Baer subplanes of P through p_1 intersecting l in one of the Baer sublines $s_1, \ldots, s_{q-1}$.

Proposition 2.12. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P. Let p_1, p_2, p_3 be the three fixed points of ϱ, and let B and B' be two planes of $\mathcal{E}_\varrho$ with $p_i \in B$, $p_j \in B'$ and $i \neq j$. Then $B \cap B' = \emptyset$ or there exists a point x not incident with any of the lines p_1p_2, p_2p_3, p_1p_3 such that $B \cap B' = \{x, \varrho(x), \varrho^2(x)\}$.*

Proof. W. l. o. g. let $i = 1$ and $j = 2$. Let Δ be the set of points incident with the lines p_1p_2, p_2p_3 or p_1p_3.

[3] The points and Baer sublines of l define a so-called miquelian inversive plane I. The partition $\{s_1, \ldots, s_{q-1}\}$ is the linear flock with carriers p_2 and p_3. For details see Thas [14], Sec. 10.

Step 1. We have $B \cap B' \cap \Delta = \emptyset$. By the proof of 2.10 e), $B \cap \Delta$ consists of the point p_1 and $q+1$ points on p_2p_3 different from p_2 and p_3. Similarly, $B' \cap \Delta$ consists of p_2 and $q+1$ points on p_1p_3 different from p_1 and p_3. Hence $B \cap B' \cap \Delta = \emptyset$.

Step 2. Let $x \in B \cap B'$. Then $x \notin \Delta$ and $B \cap B' = \{x, \varrho(x), \varrho^2(x)\}$. From $\varrho(B) = B$ and $\varrho(B') = B'$, it follows that $\{x, \varrho(x), \varrho^2(x)\} \subseteq B \cap B'$. Assume that there exists a further point $z \in B \cap B'$. Then $B \cap B'$ is a near pencil consisting of $q+2$ points. W. l. o. g. let z be incident with the line $x\varrho(x)$. Then $\varrho(z) \in B \cap B'$ and $\varrho(z)$ is incident with the line through $\varrho(x)$ and $\varrho^2(x)$, a contradiction. ∎

Proposition 2.13. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P. Let p_1, p_2, p_3 be the three fixed points of ϱ, and let $i, j \in \{1, 2, 3\}$ with $i \neq j$. If B is a plane of $\mathcal{E}_\varrho$ containing p_i, then there are exactly $\frac{2}{3}(q^2 - 1)$ planes B' of $\mathcal{E}_\varrho$ containing p_j with $B \cap B' = \emptyset$ and exactly $\frac{1}{3}(q^2 - 1)$ planes B' of $\mathcal{E}_\varrho$ containing p_j with $|B \cap B'| = 3$.*

Proof. Let Δ be the set of points incident with p_1p_2, p_2p_3 or p_1p_3. W. l. o. g. let $i = 1$ and $j = 2$. Let B' be a plane of $\mathcal{E}_\varrho$ containing p_2. By 2.12, we have $B \cap B' = \emptyset$ or $|B \cap B'| = 3$. In the latter case we have $B \cap B' = \{x, \varrho(x), \varrho^2(x)\}$ for some point x not contained in Δ.

Conversely, any set of the form $\{z, \varrho(z), \varrho^2(z)\}$ contained in B but not in Δ is contained in exactly one plane of $\mathcal{E}_\varrho$ containing p_2. Hence there are exactly $\frac{1}{3}(q^2+q+1-(q+2)) = \frac{1}{3}(q^2-1)$ planes B' of $\mathcal{E}_\varrho$ containing p_2 such that $|B \cap B'| = 3$.

The second part of the proposition follows from the fact that $\mathcal{E}_\varrho$ contains exactly $q^2 - 1$ planes through p_2. ∎

Proposition 2.14. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P. Let α be a projective collineation of P. Then $\alpha(B) = B$ for all $B \in \mathcal{E}_\varrho$ if and only if $\alpha \in <\varrho>$.*

Proof. By definition of $\mathcal{E}_\varrho$, we have $\alpha(B) = B$ for all $B \in \mathcal{E}_\varrho$ and all $\alpha \in <\varrho>$.

Conversely, let us suppose that α is a projective collineation such that $\alpha(B) = B$ for all $B \in \mathcal{E}_\varrho$. Let p_1, p_2, p_3 be the fixed points of ϱ. By 2.10 f), there are two planes B and B' of $\mathcal{E}_\varrho$ containing p_1 such that $\{p_1\} = B \cap B'$. It follows that $\alpha(p_1) = p_1$. Similarly, we have $\alpha(p_2) = p_2$ and $\alpha(p_3) = p_3$.

Let x be a point not incident with p_1p_2, p_2p_3 or p_1p_3, and let B_1 (and B_2) be the Baer suplanes through $\{x, \varrho(x), \varrho^2(x)\}$ and p_1 (and p_2, respectively). Then $B_1, B_2 \in \mathcal{E}_\varrho$ and $B \cap B' = \{x, \varrho(x), \varrho^2(x)\}$. It follows that $\alpha(x) \in \{x, \varrho(x), \varrho^2(x)\}$. If $\alpha(x) = x$, then $\alpha = 1$. If $\alpha(x) = \varrho(x)$, then $\alpha = \varrho$ and, finally, if $\alpha(x) = \varrho^2(x)$, then $\alpha = \varrho^2$. In particular $\alpha \in <\varrho>$. ∎

Proposition 2.15. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, let ϱ be a Frobenius collineation of P, and let $G = PGL_3(q^2)$. If $G_{\mathcal{E}} := \{\alpha \in G \mid \alpha(B) \in \mathcal{E}_\varrho$ for all $B \in \mathcal{E}_\varrho\}$, then we have $G_{\mathcal{E}} = N_G(<\varrho>)$.*

Proof. Firstly, let $\alpha \in G_{\mathcal{E}}$, and let $\varrho' := \alpha^{-1}\varrho\alpha$. Let $E \in \mathcal{E}_\varrho$. Then

$$\varrho'(E) = \alpha^{-1}\varrho(\alpha(E)) = \alpha^{-1}\alpha(E) = E.$$

By 2.14, it follows that $\varrho' \in <\varrho>$, hence $\alpha \in N_G(<\varrho>)$.

Conversely, let $\alpha \in N_G(<\varrho>)$. For all $E \in \mathcal{E}_\varrho$ we have

$$\varrho(\alpha(E)) = \alpha(\alpha^{-1}\varrho\alpha)(E) = \alpha(E).$$

It follows that $\alpha(E) \in \mathcal{E}_\varrho$, hence $\alpha \in G_{\mathcal{E}}$. ∎

Proposition 2.16. *Let $P = PG(2,q^2), q \equiv 2 \mod 3$, let ϱ and ϱ' be two Frobenius collineations of P. If $\mathcal{E}_\varrho = \mathcal{E}_{\varrho'}$, then $<\varrho> = <\varrho'>$.*

Proof. Because of $\mathcal{E}_\varrho = \mathcal{E}_{\varrho'}$ it follows that $\varrho'(E) = E$ for all $E \in \mathcal{E}_\varrho$. By 2.14, we have $\varrho' \in <\varrho>$, hence $<\varrho'> = <\varrho>$. ∎

Proposition 2.17. *Let $P = PG(2,q^2), q \equiv 2 \mod 3$. Then the number of sets $\mathcal{E}_\varrho$ in P equals $\frac{1}{6}(q^4+q^2+1)q^6(q^2+1)$.*

Proof. By 2.16, we have $\mathcal{E}_\varrho \neq \mathcal{E}_{\varrho'}$ for any two distinct Frobenius groups $<\varrho>$ and $<\varrho'>$ of P. Hence the number N of the sets $\mathcal{E}_\varrho$ equals the number of the Frobenius subgroups of $PGL_3(q^2)$. Since $q^2 \equiv 1 \mod 3$, it follows from 2.8 that $N = \frac{1}{6}(q^4+q^2+1)q^6(q^2+1)$. ∎

Proposition 2.18. *Let $P = PG(2,q^2), q \equiv 2 \mod 3$. Let B be a Baer subplane of P. Then there are exactly $\frac{1}{2}q^3(q^3-1)$ sets $\mathcal{E}_\varrho$ containing B.*

Proof. For a Baer subplane B let N_B be the number of sets $\mathcal{E}_\varrho$ containing B. Since the group $PGL_3(q^2)$ acts transitively on the set of Baer subplanes of P, it follows that $N := N_B$ is independent of the choice of B.

Let $\tilde{\mathcal{B}}$ be the set of Baer subplanes of P, and let $\tilde{\mathcal{E}}$ be the set of sets $\mathcal{E}_\varrho$. Consider the set $\{(B,\mathcal{E}_\varrho) \mid B \in \tilde{\mathcal{B}}, \mathcal{E}_\varrho \in \tilde{\mathcal{E}}, B \in \mathcal{E}_\varrho\}$. Computing its cardinality we obtain

$$|\tilde{\mathcal{B}}|N = |\tilde{\mathcal{E}}|3(q^2-1).$$

Because of $|\tilde{\mathcal{B}}| = q^3(q^3+1)(q^2+1)$ (see Hirschfeld [7], Cor. 3 of Lemma 4.3.1) and $|\tilde{\mathcal{E}}| = \frac{1}{6}q^6(q^4+q^2+1)(q^2+1)$ (2.17) it follows that $N = \frac{1}{2}q^3(q^3-1)$. ∎

Corollary 2.19. *Let $P = PG(2,q), q \equiv 2 \mod 3$. Then P admits exactly $\frac{1}{2}q^3(q^3-1)$ Frobenius groups.*

Proof. We embed P into the projective plane $P^* = PG(2,q^2)$. Any Frobenius group of P extends to a Frobenius group of P^*. Therefore the number N of Frobenius groups of P equals the number of Frobenius groups of P^* leaving P invariant. So, by 2.16, N equals the number of sets $\mathcal{E}_\varrho$ through P. By 2.18, it follows that $N = \frac{1}{2}q^3(q^3-1)$. ∎

3 Singer Cycles and Frobenius Collineations

A *Singer cycle* of the projective plane $P = PG(2, q)$ is a collineation of order q^2+q+1 permuting all points of P in a single cycle. Every finite desarguesian plane admits Singer cycles. If $P = PG(2, q^2)$ is a desarguesian projective plane of square order and if σ is a Singer cycle of P, then the point orbits of P under the action of $< \sigma^{q^2-q+1} >$ form a partition of P into $q^2 - q + 1$ disjoint Baer subplanes. Such a partition is called a *Singer Baer partition* and is denoted by $\mathcal{P}(\sigma)$ (see Singer [12] and Ueberberg [17]).

The main topic of this section are the possible intersections of a Singer Baer partition and a set $\mathcal{E}_\varrho$ introduced in Section 2.

If σ is a Singer cycle and if ϱ is a Frobenius collineation of $PG(2, q^2), q \equiv 2 \mod 3$, then we shall see that $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| \in \{0, 1, 3\}$ with $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$ if and only if $\varrho \in N_G(< \sigma >)$, where $G = PGL_3(q^2)$ (see 3.5). Furthermore if $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$ and if $< \gamma >$ is the subgroup of order 3 of $< \sigma >$, then γ and ϱ commute (3.3).

Proposition 3.1. *Let $P = PG(2, q^2)$, and let $G = PGL_3(q^2)$. Furthermore let $\mathcal{P} = \mathcal{P}(\sigma)$ be a Singer Baer partition for some Singer cycle σ, and let α be a projective collineation of P. Then we have $\alpha(\mathcal{P}) = \mathcal{P}$ if and only if $\alpha \in N_G(< \sigma >)$.*

Proof. See Ueberberg [18], Proposition 2.8 d). ■

Proposition 3.2. *Let $P = PG(2, q)$, and let $G = PGL_3(q)$. Let σ be a Singer cycle of P, and let $N := N_G(< \sigma >)$.*

a) $|N| = 3(q^2 + q + 1)$ and $N\setminus < \sigma >$ contains a Frobenius collineation.

b) If $q \equiv 2 \mod 3$, then any element of $N\setminus < \sigma >$ is a Frobenius collineation.

Proof. a) By Huppert [8], II, 7.3, we have $|N| = 3(q^2 + q + 1)$ and $N\setminus < \sigma >$ contains a Frobenius collineation ϱ.

b) Because of $q \equiv 2 \mod 3$ we have $q^2+q+1 \equiv 1 \mod 3$, that is, 3 and q^2+q+1 are relatively prime. Hence $< \varrho >$ is a 3-Sylow subgroup of N. By 2.4, ϱ has exactly one fixed point, say p.

Let $\alpha \in < \sigma > \cap N_N(< \varrho >)$. Then $\alpha^{-1}\varrho\alpha \in < \varrho >$, hence $\alpha^{-1}\varrho\alpha(p) = p$. It follows that $\varrho(\alpha(p)) = \alpha(p)$. Hence $\alpha(p)$ is a fixed point of ϱ implying that $\alpha(p) = p$. The only element of $< \sigma >$ admitting fixed points is the identity map. Hence $\alpha = 1$. It follows that $N_N(< \varrho >) \cap < \sigma > = < 1 >$. Hence $N_N(< \varrho >) = < \varrho >$. (Otherwise there would exist an element $\alpha\beta \in N_N(< \varrho >)$ with $1 \neq \alpha \in < \sigma >$ and $\beta \in < \varrho >$. It follows that $\alpha \in N_N(< \varrho >)$, a contradiction.)

The Theorem of Sylow implies

$$|Syl_3(N)| = |N : N_N(< \varrho >)| = 3(q^2 + q + 1)/3 = q^2 + q + 1.$$

It follows that N has exactly $q^2 + q + 1$ Frobenius subgroups. Since the elements of $< \sigma >$ are fixed point free, any element of $N\setminus < \sigma >$ has to be a Frobenius collineation. ■

Proposition 3.3. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let $G = PGL_3(q^2)$. Let σ be a Singer cycle of P, and let ϱ be a Frobenius collineation with $\varrho \in N_G(< \sigma >)$.*

a) We have $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$.

b) Let $<\gamma>$ *be the subgroup of order 3 in* $<\sigma>$. *Then*

$$<\gamma> = N_G(<\varrho>) \cap <\sigma> = C_G(<\varrho>) \cap <\sigma>,$$

in particular γ *and* ϱ *commute. Furthermore* $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$ *is an orbit under the action of* $<\gamma>$.

Proof. Let $\mathcal{P}(\sigma)$ be the Singer Baer partition defined by σ.

a) Since $\varrho \in N_G(<\sigma>)$, it follows from 3.1 that $\varrho(B) \in \mathcal{P}(\sigma)$ for all $B \in \mathcal{P}(\sigma)$. Let p_1, p_2, p_3 be the fixed points of ϱ. Then there exists an element $B_1 \in \mathcal{P}(\sigma)$ with $p_1 \in B_1$. It follows $\varrho(B_1) = B_1$, in other words, $B_1 \in \mathcal{E}_\varrho$. In particular we have $p_2, p_3 \notin B_1$. Similarly, we get two further planes $B_2, B_3 \in \mathcal{P}(\sigma)$ with $p_2 \in B_2$, $p_3 \in B_3$ and $B_2, B_3 \in \mathcal{E}_\varrho$. Hence $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$.

b) Since $q \equiv 2 \mod 3$, we have $q^4 + q^2 + 1 \equiv 0 \mod 3$. Since $<\sigma>$ is cyclic, it has exactly one subgroup $<\gamma>$ of order 3. Because of $\varrho \in N_G(<\sigma>)$ it follows that $<\gamma>^\varrho = <\gamma>$. Assume that $\varrho^{-1}\gamma\varrho = \gamma^{-1}$. Then $\gamma\varrho = \varrho\gamma^{-1}$ and $\gamma\varrho^{-1} = \varrho^{-1}\gamma^{-1}$. Therefore $<\gamma, \varrho> = \{\varrho^i\gamma^j \mid i, j = 0, 1, 2\}$ is a group of order 9. In particular it is abelian, contradicting the assumption $\varrho^{-1}\gamma\varrho = \gamma^{-1}$. So we have seen that

$$<\gamma> \leq C_G(<\varrho>) \cap <\sigma> \leq N_G(<\varrho>) \cap <\sigma> .$$

Let $1 \neq \alpha \in N_G(<\varrho>) \cap <\sigma>$. Then

$$\varrho(\alpha(p_1)) = \alpha(\alpha^{-1}\varrho\alpha)(p_1) = \alpha(p_1).$$

It follows that $\alpha(p_1)$ is a fixed point of ϱ. In the same way we see that $\alpha(\{p_1, p_2, p_3\}) = \{p_1, p_2, p_3\}$. Since $1 \neq \alpha \in <\sigma>$, α has no fixed points. So $\{p_1, p_2, p_3\}$ is an orbit of α which implies (again in view of $\alpha \in <\sigma>$) that α is of order 3. Hence $<\alpha> = <\gamma>$. Furthermore the three Baer subplanes of $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$ are exactly the three Baer subplanes B_1, B_2, B_3 of $\mathcal{P}(\sigma)$ containing p_1, p_2, p_3, respectively. In particular $\{B_1, B_2, B_3\}$ is an orbit under the action of $<\gamma>$. ∎

Definition 3.4. Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let σ be a Singer cycle of P. Then we denote by $<\gamma_\sigma>$ the subgroup of order 3 of $<\sigma>$.

Proposition 3.5. *Let* $P = PG(2, q^2), q \equiv 2 \mod 3$, *and let* $G = PGL_3(q^2)$. *Let* σ *be a Singer cycle of* P, *and let* ϱ *be a Frobenius collineation.*

a) We have $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| \in \{0, 1, 3\}$.

b) Let $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$. *Then* $\varrho \in N_G(<\sigma>)$.

c) Let $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho = \{B_1, B_2, B_3\}$. *If* τ_1, τ_2, τ_3 *are the Baer involutions of* B_1, B_2, B_3, *respectively, then we have*

$$<\tau_1\tau_2> = <\tau_1\tau_3> = <\tau_2\tau_3> = <\gamma_\sigma> .$$

Proof. a) and b) Let B and B' be two Baer subplanes contained in $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$. Then B and B' are disjoint therefore there exists a unique Singer Baer partition through B and B' (see Ueberberg [17], Th. 3.1), namely $\mathcal{P}(\sigma)$. Because of $\varrho(B) = B$ and $\varrho(B') = B'$ it follows that ϱ leaves $\mathcal{P}(\sigma)$ invariant. By 3.1, it follows that $\varrho \in N_G(<\sigma>)$. By 3.3, we have $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$.

c) By Ueberberg [17], Th. 1.1, we have $\tau_1\tau_2 \in < \sigma >$. Since $\varrho(B_1) = B_1$ and $\varrho(B_2) = B_2$, the Baer involutions τ_1 and τ_2 commute with ϱ (see [17], Prop. 2.1). Furthermore $\tau_1\tau_2$ is a projective collineation ([17], Prop. 2.2). If $G = PGL_3(q^2)$, then it follows

$$< \tau_1\tau_2 > \subseteq < \sigma > \cap C_G(< \varrho >) = < \gamma_\sigma >,$$

where the last equality follows from 3.3. ∎

4 The Frobenius Planes of Order q

In view of Propositions 3.3 and 3.5 we shall endow the sets $\mathcal{E}_\varrho$ (where ϱ is a Frobenius collineation) with an additional structure of points and lines.

Definition 4.1. Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P. Then we define a geometry $\mathcal{F} = \mathcal{F}(P, \varrho)$ of points and lines as follows:

- The points of $\mathcal{F}$ are the Baer subplanes of $\mathcal{E}_\varrho$.
- The lines of $\mathcal{F}$ are the sets $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$, where σ is a Singer cycle such that $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$.
- A point $B \in \mathcal{E}_\varrho$ and a line $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$ are incident if and only if $B \in \mathcal{P}(\sigma)$.

Then $\mathcal{F}$ is called a *Frobenius plane of order* q.

In the rest of this section we shall compute the parameters of the Frobenius planes (4.3), and we shall show that for any non-incident point-line-pair $(B, \mathcal{G})$ there exist either one or two lines through B intersecting $\mathcal{G}$ (4.6). As a corollary we shall determine the 0- and 1-diameters of the Frobenius planes (4.7).

Proposition 4.2. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of order 3. Let $\mathcal{F} = \mathcal{F}(P, \varrho)$ be the corresponding Frobenius plane. Two points B and B' of $\mathcal{F}$ are joined by a line if and only if B and B' are disjoint Baer subplanes of $\mathcal{E}_\varrho$.*

Proof. If B and B' are joined by a line, then B and B' are contained in some Singer Baer partition. In particular we have $B \cap B' = \emptyset$.

Conversely, let B and B' be two disjoint Baer subplanes of $\mathcal{E}_\varrho$. Then, by [17], Th. 1.1, there is a Singer Baer partition $\mathcal{P}(\sigma)$ containing B and B'. By 3.5, $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$ is a line of $\mathcal{F}$ through B and B'. ∎

Proposition 4.3. *Let $\mathcal{F}$ be a Frobenius plane of order q.*
a) $\mathcal{F}$ has $3(q^2 - 1)$ points.
b) $\mathcal{F}$ has $\frac{2}{3}(q^2 - 1)^2$ lines.
c) Any line of $\mathcal{F}$ is incident with exactly three points.
d) Any point is incident with $\frac{2}{3}(q^2 - 1)$ lines.

Proof. Let $P = PG(2, q^2)$, and let ϱ be a Frobenius collineation of P. Let p_1, p_2, p_3 be the three fixed points of ϱ.

a) By 2.10, $\mathcal{E}_\varrho$ contains $3(q^2-1)$ Baer subplanes, hence $\mathcal{F}$ has $3(q^2-1)$ points.

c) follows from the definition of $\mathcal{F}$.

d) Let B be an element of $\mathcal{E}_\varrho$ containing p_1. By 2.13, there are exactly $\frac{2}{3}(q^2-1)$ planes B' such that $p_2 \in B'$ and $B \cap B' = \emptyset$. It follows from 4.2 that there are exactly $\frac{2}{3}(q^2-1)$ lines in $\mathcal{F}$ through B.

b) Counting the pairs $(B, \mathcal{G})$, where B is a point and $\mathcal{G}$ is a line of $\mathcal{F}$ through B yields that $\mathcal{F}$ contains $\frac{2}{3}(q^2-1)^2$ lines. ■

Lemma 4.4. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P. Let p_1, p_2, p_3 be the three fixed points of ϱ, and let B and B' be two elements of $\mathcal{E}_\varrho$ such that $p_1 \in B$ and $p_2 \in B'$.*

If B and B' intersect in three points and if τ and τ' are the Baer involutions of B and B', then $\tau\tau'$ is a Frobenius collineation of P.

Proof. Let $\delta := \tau\tau'$. By [17], Proposition 2.2 a), δ is a projective collineation. Let $B \cap B' = \{x, y, z\}$. By 2.12, the points x, y, z form a triangle. By definition of δ, we have $\delta(x) = x, \delta(y) = y, \delta(z) = z$.

Since $B, B' \in \mathcal{E}_\varrho$, it follows that $\varrho(B) = B$ and $\varrho(B') = B'$. Hence $[\tau, \varrho] = 1$ and $[\tau', \varrho] = 1$. In particular we have $[\delta, \varrho] = 1$.

It follows that $\varrho(\delta(p_1)) = \delta(\varrho(p_1)) = \delta(p_1)$. So $\delta(p_1)$ is a fixed point of ϱ whence $\delta(p_1) \in \{p_1, p_2, p_3\}$. In the same way we get $\delta(\{p_1, p_2, p_3\}) = \{p_1, p_2, p_3\}$. It follows that $\{p_1, p_2, p_3\}$ is one orbit of δ (otherwise $\{p_1, p_2, p_3\}$ would contain a fixed point and, in view of $\delta(x) = x, \delta(y) = y, \delta(z) = z$, it follows that $\delta = 1$). So $\delta^3(p_1) = p_1$. Hence δ^3 has the four fixed points x, y, z, p_1 forming a quadrangle, that means, $\delta^3 = 1$. So δ is a projective collineation of order 3 admitting three fixed points and an orbit $\{p_1, p_2, p_3\}$ forming a triangle. It follows from 2.7 that δ is a Frobenius collineation. ■

Lemma 4.5. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let ϱ be a Frobenius collineation of P. Let α and α' be two further Frobenius collineations both commuting with ϱ. Then $\alpha\alpha'$ admits at least three fixed points.*

Proof. Let $P = P(V)$, where V is a 3-dimensional vector space over $GF(q^2)$. Let p_1, p_2, p_3 be the three fixed points of ϱ. By 2.5, there is a basis $\{v_1, v_2, v_3\}$ of V such that ϱ is induced by the linear map with matrix

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \bar{\theta} \end{pmatrix},$$

where $1, \theta, \bar{\theta}$ are the third unit roots in $GF(q^2)$. (Observe that $p_j = < v_j >$ for $j = 1, 2, 3$.) Since $\alpha\varrho = \varrho\alpha$ and $\alpha'\varrho = \varrho\alpha'$, there are matrices

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ and } A' = \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix}$$

inducing α and α', respectively such that $AR = \lambda RA$ and $A'R = \lambda' RA'$ for some elements $0 \neq \lambda, \lambda' \in GF(q^2)$. It follows that

$$\begin{aligned} AR &= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \bar{\theta} \end{pmatrix} = \begin{pmatrix} a & b\theta & c\bar{\theta} \\ d & e\theta & f\bar{\theta} \\ g & h\theta & i\bar{\theta} \end{pmatrix} \\ &= \lambda RA = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \bar{\theta} \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \lambda \begin{pmatrix} a & b & c \\ d\theta & e\theta & f\theta \\ g\bar{\theta} & h\bar{\theta} & i\bar{\theta} \end{pmatrix}. \end{aligned}$$

If $a \neq 0$, then $\lambda = 1$ and $b = c = d = f = g = h = 0$. Hence

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix}.$$

It follows that α fixes the points p_1, p_2, p_3. Since α is a Frobenius collineation, by 2.7, $< \alpha > = < \varrho >$.

Let $a = 0$. If $b = 0$, then $c \neq 0$ (otherwise $\det A = 0$). Hence $\lambda = \bar{\theta}$ and $e = f = g = i = 0$, thus

$$A = \begin{pmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{pmatrix}.$$

The characteristic polynomial of A equals $x^3 - cdh$. Since α is a Frobenius collineation, the matrix A can be diagonalized, hence $x^3 - cdh$ has three roots. In other words, cdh admits three third roots.

If $b \neq 0$, then $\lambda = \theta$ and $c = d = e = h = i = 0$, hence

$$A = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{pmatrix}.$$

As above, we see that bfg admits three third roots.

Similarly, A' is of the form

$$A' = \begin{pmatrix} a' & 0 & 0 \\ 0 & e' & 0 \\ 0 & 0 & i' \end{pmatrix} \text{ or } A' = \begin{pmatrix} 0 & 0 & c' \\ d' & 0 & 0 \\ 0 & h' & 0 \end{pmatrix} \text{ or } A' = \begin{pmatrix} 0 & b' & 0 \\ 0 & 0 & f' \\ g' & 0 & 0 \end{pmatrix},$$

where $c'd'h'$ and $b'f'g'$ both admit three third roots.

If $A = \begin{pmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{pmatrix}$ and $A' = \begin{pmatrix} 0 & 0 & c' \\ d' & 0 & 0 \\ 0 & h' & 0 \end{pmatrix}$, then $AA' = \begin{pmatrix} 0 & ch' & 0 \\ 0 & 0 & dc' \\ bd' & 0 & 0 \end{pmatrix}$.

The characteristic polynomial of AA' is $x^3 - ch'dc'bd' = x^3 - (bcd)(c'd'h')$. Since bcd and $c'd'h'$ both admit three third roots, $x^3 - ch'dc'bd'$ is reducible, hence AA' has three eigenvalues. It follows that $\alpha\alpha'$ has three fixed points.

If $A = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{pmatrix}$ and $A' = \begin{pmatrix} 0 & b' & 0 \\ 0 & 0 & f' \\ g' & 0 & 0 \end{pmatrix}$, then $AA' = \begin{pmatrix} 0 & 0 & bf' \\ fg' & 0 & 0 \\ 0 & gb' & 0 \end{pmatrix}$.

As above it follows that $\alpha\alpha'$ has three fixed points.

If $A = \begin{pmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{pmatrix}$ and $A' = \begin{pmatrix} 0 & b' & 0 \\ 0 & 0 & f' \\ g' & 0 & 0 \end{pmatrix}$, then $AA' = \begin{pmatrix} cg' & 0 & 0 \\ 0 & db' & 0 \\ 0 & 0 & hf' \end{pmatrix}$.

Obviously, $\alpha\alpha'$ has three fixed points.

If $A = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{pmatrix}$ and $A' = \begin{pmatrix} 0 & 0 & c' \\ d' & 0 & 0 \\ 0 & h' & 0 \end{pmatrix}$, then $AA' = \begin{pmatrix} bd' & 0 & 0 \\ 0 & fh' & 0 \\ 0 & 0 & gi' \end{pmatrix}$.

Again $\alpha\alpha'$ has three fixed points. ■

Theorem 4.6. *Let $q \equiv 2 \mod 3$, and let $\mathcal{F}$ be the Frobenius plane of order q. For a non-incident point line pair $(B, \mathcal{G})$ of $\mathcal{F}$, we denote by $\alpha(B, \mathcal{G})$ the number of lines through B intersecting $\mathcal{G}$. a) We have $\alpha(B, \mathcal{G}) \in \{1, 2\}$ for all non-incident point-line-pairs $(B, \mathcal{G})$ of $\mathcal{F}$.*

b) Given a line $\mathcal{G}$ of $\mathcal{F}$ there are exactly $2(q^2-1)$ points B with $\alpha(B, \mathcal{G}) = 1$ and $q^2 - 4$ points B' with $\alpha(B', \mathcal{G}) = 2$.

Proof. a) Let $(B, \mathcal{G})$ be a non-incident point-line-pair of $\mathcal{F}$.

Step 1. There exists a line through B intersecting $\mathcal{G}$.

For, assume that every line through B is disjoint to $\mathcal{G}$. Let B_1, B_2, B_3 be the points on $\mathcal{G}$. Translating the above situation to $P = PG(2, q^2)$ we obtain a Frobenius collineation ϱ such that the points of $\mathcal{F}$ are the Baer subplanes of $\mathcal{E}_\varrho$. Furthermore there is a Singer cycle σ such that $\{B_1, B_2, B_3\} = \mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$. Finally the property that there is no line through B intersecting $\mathcal{G}$ means that B has non-trivial intersection with any of the planes B_1, B_2, B_3.

Let p_1, p_2, p_3 be the three fixed points of ϱ. W. l. o. g. we can suppose that $p_1 \in B_1$, $p_2 \in B_2$, $p_3 \in B_3$ and, say, $p_2 \in B$.

By 2.12, there are two points x and y of P such that $\{x, \varrho(x), \varrho^2(x)\}$ and $\{y, \varrho(y), \varrho^2(y)\}$ form two triangles and such that $B_1 \cap B = \{x, \varrho(x), \varrho^2(x)\}$ and $B_3 \cap B = \{y, \varrho(y), \varrho^2(y)\}$. Let $\tau, \tau_1, \tau_2, \tau_3$ be the Baer involutions of B, B_1, B_2, B_3, respectively.

By Lemma 4.4, $\tau_1\tau$ and $\tau\tau_2$ are Frobenius collineations, both commuting with ϱ. By 4.5, $\tau_1\tau_2 = (\tau_1\tau)(\tau\tau_2)$ admits three fixed points. On the other hand, by 3.5, we have $< \tau_1\tau_2 > = < \gamma_\sigma >$ implying that $\tau_1\tau_2$ has no fixed points, a contradiction.

Step 2. We have $\alpha(B, \mathcal{G}) \in \{1, 2\}$. By Step 1, we have $\alpha(B, \mathcal{G}) \geq 1$. If B_1, B_2, B_3 are the points of $\mathcal{G}$ and if p_1, p_2, p_3 are the fixed points of ϱ, then we can assume w. l. o. g. that $p_i \in B_i$ for $i = 1, 2, 3$. The plane B does also contain a fixed point of ϱ, say p_2. It follows that B and B_2 are not joined by a line, hence $\alpha(B, \mathcal{G}) \leq 2$.

b) Let $\tilde{\mathcal{L}}$ be the set of lines intersecting $\mathcal{G}$ in a point. By a), $\tilde{\mathcal{L}}$ covers the point set of $\mathcal{F}$.

Let α_1 and α_2 be the number of points B with $\alpha(B, \mathcal{G}) = 1$ and the number of points B' with $\alpha(B', \mathcal{G}) = 2$, respectively.

Let $\tilde{\mathcal{B}}$ be the set of points not incident with $\mathcal{G}$, and let

$$\mathcal{S} := \{(B, \mathcal{L}) \mid B \in \tilde{\mathcal{B}}, \mathcal{L} \in \tilde{\mathcal{L}}, B \in \mathcal{L}\}.$$

Counting the elements of $\mathcal{S}$ we get

$$\alpha_1 + 2\alpha_2 = 2 \cdot 3\left(\frac{2}{3}(q^2-1) - 1\right) = 4(q^2-1) - 6.$$

Since $\alpha_1 + \alpha_2 = |\tilde{\mathcal{B}}| = 3(q^2-1) - 3$, it follows that $\alpha_1 = 2(q^2-1)$ and $\alpha_2 = q^2 - 4$. ■

Corollary 4.7. *Let $q \equiv 2 \mod 3$, and let $\mathcal{F}$ be a Frobenius plane of order q. Let d_0, d_1 and g be the 0-diameter, the 1-diameter and the gonality of $\mathcal{F}$, respectively.*

If $q = 2$, then $d_0 = d_1 = g = 4$. Actually, $\mathcal{F}$ is a 3×3-grid.

If $q > 2$, then $d_0 = d_1 = 4$ and $g = 3$.

Proof. Step 1. Let g be the gonality of $\mathcal{F}$. If $q = 2$, then $g = 4$. If $q > 2$, then $g = 3$.

It follows from 4.6 that $\mathcal{F}$ admits triangles if and only if $q > 2$. Hence $g = 3$ for $q > 2$. For $q = 2$, the assertion follows from Ueberberg [18], Th. 1.2.

Step 2. We have $d_0 = d_1 = 4$.

By 4.6, it follows that $d_0 \leq 4$. On the other hand, since there exist non-collinear points x, y of $\mathcal{F}$ we have $d_0 \geq \operatorname{dist}(x, y) = 4$, hence $d_0 = 4$. Similarly, it follows that $d_1 = 4$. ■

5 Frobenius Spaces

In this section we shall introduce a geometry Γ of rank 3 of points, lines and planes such that the planes of Γ are Frobenius planes. For this reason we shall call these geometries *Frobenius spaces.*

The main result of this section is the computation of the parameters of Γ (5.4) and the computation of the flag stabilizers of $\Gamma$for any type of flags (5.5). It turns out that the group $PGL_3(q^2)$ acts flag-transitively on Γ.

Definition 5.1. Let $P = PG(2, q^2), q \equiv 2 \mod 3$. Then we define a geometry Γ of rank 3 as follows:

- The points of Γ are the Baer subplanes of P.
- The lines of Γ are the sets $\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$, where σ and ϱ are a Singer cycle and a Frobenius collineation of P, such that $|\mathcal{P}(\sigma) \cap \mathcal{E}_\varrho| = 3$.
- The planes of Γ are the sets $\mathcal{E}_\varrho$, where ϱ is a Frobenius collineation of P.
- The incidence relation is induced by the set-theoretical inclusion.

Γ is called a *Frobenius space of order q.*

Lemma 5.2. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let Γ be the corresponding Frobenius space. Let $\mathcal{G}$ be a line of Γ with point set $\{B_1, B_2, B_3\}$.*

Then there is exactly one Singer cycle σ of P such that $\{B_1, B_2, B_3\} \subseteq \mathcal{P}(\sigma)$. If $< \gamma >$ is the subgroup of order 3 of σ, then $\{B_1, B_2, B_3\}$ is an orbit of $\mathcal{P}(\sigma)$ under the action of $< \gamma >$.

Proof. By definition, we have $\mathcal{G} = \mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$ for some Singer cycle σ and some Frobenius collineation ϱ. On the other hand, by Ueberberg [17] Th. 3.1 there is exactly one Singer Baer partition through B_1 and B_2. The rest of the lemma has been proved in 3.3. ■

Proposition 5.3. *Let $P = PG(2, q^2), q \equiv 2 \mod 3$, and let Γ be the corresponding Frobenius space. Let $\mathcal{G} := \{B_1, B_2, B_3\}$ be a set of three Baer subplanes of P.*

Then $\mathcal{G}$ is a line of Γ if and only if there exists a Singer cycle σ such that $\mathcal{G}$ is an orbit of the subgroup $< \gamma >$ of order 3 of $< \sigma >$ acting on $\mathcal{P}(\sigma)$.

Proof. We first suppose that $\mathcal{G}$ is a line of Γ. Then the assertion follows from 5.2.

Now suppose that σ is a Singer cycle of P and $< \gamma >$ is the subgroup of order 3 of $< \sigma >$. Let $\mathcal{G}$ be an orbit of $< \gamma >$ acting on $\mathcal{P}(\sigma)$. Let $G = PGL_3(q^2)$. By 3.2, there is a Frobenius collineation $\varrho \in N_G(< \sigma >)$. By 3.3, the set $\mathcal{G}' := \mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$ is a line of Γ and an orbit under the action of $< \gamma >$. If $\mathcal{G} = \mathcal{G}'$, then the proof is complete. Suppose that $\mathcal{G} \neq \mathcal{G}'$, and let $\mathcal{G}' := \{B_1', B_2', B_3'\}$. Since the group $< \sigma^{q^2+q+1} >$ permutes the Baer subplanes of $\mathcal{P}(\sigma)$ in a single cycle, there exists an element $\bar{\sigma} \in < \sigma >$ such that $\bar{\sigma}(B_1) = B_1'$. Since γ and $\bar{\sigma}$ are both contained in $< \sigma >$, they commute. It follows that $\bar{\sigma}(\mathcal{G}) = \mathcal{G}'$. Let $\bar{\varrho} := \bar{\sigma}^{-1}\varrho\bar{\sigma}$. Then $\bar{\varrho}$ is a Frobenius collineation, and one easily verifies that $\bar{\varrho}$ fixes the Baer subplanes of $\mathcal{G}$. Hence $\mathcal{G} = \mathcal{P}(\sigma) \cap \mathcal{E}_{\bar{\varrho}}$ is a line of Γ. ∎

Theorem 5.4. *Let $q \equiv 2 \mod 3$, and let Γ be the Frobenius space of order q.*

a) Γ has $q^3(q^3+1)(q^2+1)$ points, $\frac{1}{9}q^6(q^4-1)(q^2-1)(q^2-q+1)$ lines, and Γ has $\frac{1}{6}q^6(q^4+q^2+1)(q^2+1)$ planes.

b) The lines and planes of Γ are incident with 3 and $3(q^2-1)$ points, respectively.

c) The points and planes are incident with $\frac{1}{3}q^3(q^2-1)(q-1)$ and $\frac{2}{3}(q^2-1)^2$ lines, respectively.

d) The points and lines are incident with $\frac{1}{2}q^3(q^3-1)$ and q^2+q+1 planes, respectively.

e) Let $\mathcal{E}$ be a plane containing a point B. Then there are exactly $\frac{2}{3}(q^2-1)$ lines in $\mathcal{E}$ through B.

Proof. Let $P = PG(2, q^2)$, and let Γ be the corresponding Frobenius space.

a) By Hirschfeld [7] (Cor. 3 of Lemma 4.3.1), P contains $q^3(q^3+1)(q^2+1)$ Baer subplanes.

By 5.3, the number of lines of Γ equals the number of Singer Baer partitions times $\frac{1}{3}(q^2-q+1)$. Since P admits $\frac{1}{3}q^6(q^4-1)(q^2-1)$ Singer groups ([7], Cor. 3 of Th. 4.2.1), it follows that Γ has $\frac{1}{9}q^6(q^4-1)(q^2-1)(q^2-q+1)$ lines.

By 2.17, the number of planes of Γ equals $\frac{1}{6}q^6(q^4+q^2+1)(q^2+1)$.

b) These parameters have been computed in 4.3.

c) Let B be a point of Γ. Then the number of lines through B equals the number of Singer Baer partitions through B. This number equals $\frac{1}{3}q^3(q^2-1)(q-1)$ (see [18] Prop. 2.4 d)). The number of lines contained in a plane has been computed in 4.3.

d) Let B be a point of Γ. By 2.18, the number of planes of Γ through B equals $\frac{1}{2}q^3(q^3-1)$.

Let $G = PGL_3(q^2)$. By [18], Th. 1.1, the group G acts transitively on the Singer Baer partitions of P. Given a Singer Baer partition $\mathcal{P}(\sigma)$ the group $< \sigma^{q^2+q+1} >$ acts transitively on the lines of Γ contained in $\mathcal{P}(\sigma)$. It follows that G acts transitively on the lines of Γ. In particular any line of Γ is incident with the same number N of planes. Computing the pairs $(\mathcal{G}, \mathcal{E})$ of incident line-plane-pairs it follows from a) and c) that $N = q^2+q+1$.

e) This number has been computed in 4.3. ∎

Theorem 5.5. *Let* $P = PG(2, q^2), q \equiv 2 \mod 3$ *and let* Γ *be the corresponding Frobenius space, and let* $G = PGL_2(q^2)$.

Let $\{B, \mathcal{G}, \mathcal{E}\}$ *be a flag of* Γ, *where* B *is a Baer subplane of* P *admitting* τ *as a Baer involution,* $\mathcal{G} = \mathcal{P}(\sigma) \cap \mathcal{E}_\varrho$, *and* $\mathcal{E} = \mathcal{E}_\varrho$. *Furthermore let* $< \gamma >$ *be the subgroup of order 3 of* $< \sigma >$.

a) G *acts transitively on the chambers of* Γ.

b) Let $\bar{G}$ *be the stabilizer of* B *in* G. *Then* $\bar{G} \cong PGL_3(q)$. *Let* $\bar{\sigma} := \sigma^{q^2-q+1}$. *Then* $\bar{\sigma}$ *induces a Singer cycle on* B. *The stabilizers of the flags of* Γ *are as indicated in the following table:*

Flag	*Stabilizer*	*Order*
$\{B\}$	$\bar{G}$	$(q^2+q+1)(q^2+q)q^2(q-1)^2$
$\{\mathcal{G}\}$	$N_{\bar{G}}(< \bar{\sigma} >) \times < \gamma >$	$9(q^2+q+1)$
$\{\mathcal{E}\}$	$N_G(\varrho)$	$6\,(q^2-1)^2$
$\{B, \mathcal{G}\}$	$N_{\bar{G}}(< \bar{\sigma} >)$	$3(q^2+q+1)$
$\{B, \mathcal{E}\}$	$N_{\bar{G}}(\varrho)$	$2(q^2-1)$
$\{\mathcal{G}, \mathcal{E}\}$	$< \gamma > \times < \varrho >$	*9*
$\{B, \mathcal{G}, \mathcal{E}\}$	$< \varrho >$	*3*

Proof. We first compute the various stabilizers. By definition, $\bar{G}$ is the stabilizer of B.

Let H_1 be the stabilizer of $\mathcal{G}$. For each $\alpha \in H_1$ we have $\alpha(\mathcal{G}) = \mathcal{G}$ and hence $\alpha(\mathcal{P}(\sigma)) = \mathcal{P}(\sigma)$. It follows that H_1 is a subgroup of $N_G(< \sigma >)$. The subgroup of $< \sigma >$ leaving $\mathcal{G}$ invariant is the group $< \bar{\sigma} > \times < \gamma >$. Since ϱ fixes all elements of $\mathcal{G}$, we have $\varrho \in H_1$ and $\varrho \in N_{\bar{G}}(< \bar{\sigma} >)$. It follows that $H_1 =< \varrho, \bar{\sigma}, \gamma >$. Since ϱ and γ commute (3.3), we finally get $H_1 = N_{\bar{G}}(< \bar{\sigma} >) \times < \gamma >$.

By 2.15, the stabilzer H_2 of $\mathcal{E}$ equals the group $N_G(< \varrho >)$.

The stabilizer of $\{B, \mathcal{G}\}$ is the group

$$\bar{G} \cap H_1 = \bar{G} \cap N_{\bar{G}}(< \bar{\sigma} >) \times < \gamma >= N_{\bar{G}}(< \bar{\sigma} >).$$

The stabilizer of $\{B, \mathcal{E}\}$ is the group

$$\bar{G} \cap H_2 = \bar{G} \cap N_G(< \varrho >) = N_{\bar{G}}(< \varrho >).$$

The stabilizer of $\{\mathcal{G}, \mathcal{E}\}$ is the group $H_1 \cap H_2 = (N_{\bar{G}}(< \bar{\sigma} >) \times < \gamma >) \cap N_G(< \varrho >)$. Since $N_G(< \varrho >) \cap < \sigma > = < \gamma >$ (see 3.3), it follows that

$$H_1 \cap H_2 = (N_{\bar{G}}(< \bar{\sigma} >) \times < \gamma >) \cap N_G(< \varrho >) = < \varrho > \times < \gamma > .$$

Finally the stabilizer of $\{B, \mathcal{G}, \mathcal{E}\}$ is the group $\bar{G} \cap H_1 \cap H_2 = < \varrho >$.

The next step is to show that G acts transitively on the chambers of Γ. By [18], G acts transitively on the Singer Baer partitions of P. Given a Singer Baer partition $\mathcal{P}(\sigma)$, the group $< \sigma >$ acts transitively on the Baer subplanes of $\mathcal{P}(\sigma)$. Hence G acts transitively on the flags of type $\{0, 1\}$. Given a chamber $\{B, \mathcal{G}, \mathcal{E}\}$ we denote by S the stabilizer of $\{B, \mathcal{G}\}$ and by $\mathcal{O}$ the orbit of S containing $\mathcal{E}$ in the set of planes of Γ. If T is the stabilizer of the chamber $\{B, \mathcal{G}, \mathcal{E}\}$, then we get

$$|\mathcal{O}| = \frac{|S|}{|T|} = \frac{3(q^2+q+1)}{3} = q^2 + q + 1.$$

(For the second equality we have used 3.2.) Since there are exactly q^2+q+1 planes incident with $\{B, \mathcal{G}\}$, it follows that G acts transitively on the chambers of Γ.

The orders of the stabilizers are in the most cases easy to compute. If $\mathcal{F}$ is the set of flags of a given type and if F is one flag of this type with stabilizer G_F, then we have

$$|G_F| = \frac{|G|}{|\mathcal{F}|}.$$

Using this formula all orders can be computed. ■

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