Beppo Levi's Theorem for the Vector-Valued McShane Integral and Applications

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Abstract

Using only elementary properties of the McShane integral for vector-valued functions, we establish a convergence theorem which for the scalar case of the integral yields the classical Beppo Levi (monotone) convergence theorem as an immediate corollary. As an application, the convergence theorem is used to prove that the space of McShane integrable functions, although not usually complete, is ultrabornological.

1 Introduction

The gauge-type integral of E.J. McShane has been extended to vector-valued functions and the resulting space of integrable functions has been studied by several authors ([F1-3], [FM], [G2]). For Banach space valued functions the space of McShane integrable functions is contained in the space of Pettis integrable functions and under certain conditions coincides with the space of Pettis integrable functions ([F1-3], [FM], [G2]). It is well-known that the space of Pettis integrable functions is usually not complete ([P], [T2]) so, in general, the space of Banach valued McShane integrable functions is also not a complete space. However, it has recently been shown that the space of Pettis integrable functions is a barrelled space ([DFP1], [DFP2], [Sw3]); indeed, in [DFFP] it has been shown that the space of McShane (Pettis) integrable functions defined on a bounded interval is ultrabornological, a property which implies barrelledness. The results in [DFFP] employ some sophisticated results of Fremlin and Mendoza relating the McShane and Pettis integrals and

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general convergence theorems. In this note we use only elementary properties of the McShane integral for vector-valued functions to establish an interesting convergence theorem for the vector-valued McShane integral which for the scalar case of the integral yields the Beppo Levi (monotone) convergence theorem as an immediate corollary. As an application of the convergence theorem, we show that the space of McShane integrable functions has several properties which imply that the space, while usually not complete, is barrelled and ultrabornolgical.

2 The integral

We now give a description of the McShane integral in $\mathbb{R}$. Let $\mathbb{R}^*$ be the extended reals (with $\pm\infty$ adjoined to $\mathbb{R}$). Let X be a (real) Banach space. Any function $f : R \to X$ is assumed to be extended to $\mathbb{R}^*$ by setting $f(\pm\infty) = 0$. A gauge on $\mathbb{R}^*$ is a function γ which associates to each $t \in \mathbb{R}^*$ a neighborhood $\gamma(t)$ of t. A partition of $\mathbb{R}$ is a finite collection of left-closed intervals I_i, $i = 1, \dots, n$, such that $\mathbb{R} = \bigcup_{i=1}^{n} I_i$ (here we agree that $(-\infty, a)$ is left-closed). A tagged partition of $\mathbb{R}$ is a finite collection of pairs $\{(I_i, t_i) : 1 \le i \le n\}$ such that $\{I_i\}$ is a partition of $\mathbb{R}$ and $t_i \in \mathbb{R}^*$; t_i is called the tag associated with I_i. Note that it is not required that the tag t_i belong to I_i; this requirement is what distinguishes the McShane integral from the Henstock-Kurzweil integral ([Mc], [G1], [LPY], [DeS], [M]). If γ is a gauge on $\mathbb{R}^*$, a tagged partition $\{(I_i, t_i) : 1 \le i \le n\}$ is said to be γ-fine if $\bar{I}_i \subset \gamma(t_i)$ for every $i = 1, \dots, n$. If J is an interval in $\mathbb{R}^*$, we write $m(J)$ for its length and make the usual agreement that $0 \cdot \infty = 0$. If $\mathcal{D} = \{(I_i, t_i) : 1 \le i \le n\}$ is a tagged partition and $f : \mathbb{R} \to X$, we write $S(f, \mathcal{D}) = \sum_{i=1}^{n} f(t_i) m(I_i)$ for the Riemann sum of f with respect to $\mathcal{D}$.

Definition 1. A function $f : \mathbb{R} \to X$ is (McShane) integrable over $\mathbb{R}$ if there exists $v \in X$ such that for every $\varepsilon > 0$ there exists a gauge γ on $\mathbb{R}^*$ such that $\|S(f, \mathcal{D}) - v\| < \varepsilon$ for every γ-fine tagged partition $\mathcal{D}$ of $\mathbb{R}$.

The vector v is called the (McShane) integral of f over $\mathbb{R}$ and is denoted by $\int_{\mathbb{R}} f$. We refer the reader to [Mc], [G1] or [G2] for basic properties of the McShane integral. We emphasize that we use only basic properties of the McShane integral.

For later use we record one important result for the McShane integral usually referred to as Henstock's Lemma. If (I_i, t_i), $i = 1, \dots, n$, is any pairwise disjoint collection of half-closed subintervals of $\mathbb{R}$ and $t_i \in \mathbb{R}^*$, the collection $\{(I_i, t_i) : i = 1, \dots, n\}$ is called a partial tagged partition of $\mathbb{R}$ (it is not required that $\bigcup_{j=1}^{n} I = \mathbb{R}$), and such a collection is called γ-fine if $\bar{I}_j \subset \gamma(t_j)$ for $1 \le j \le n$.

Lemma 2. (Henstock) Let $f : \mathbb{R} \to X$ be McShane integrable and let $\varepsilon > 0$. Suppose the gauge γ on $\mathbb{R}^*$ is such that $\left\| \sum_{i=1}^{n} f(t_i) m(I_i) - \int_{\mathbb{R}} f \right\| < \varepsilon$ for every γ-fine tagged partition $\{(I_i, t_i) : 1 \le i \le n\}$ of $\mathbb{R}$. If $\{(I_i, t_i) : 1 \le i \le n\}$ is any γ-fine partial tagged partition of $\mathbb{R}$, then $\left\| \sum_{i=1}^{n} \left\{ f(t_i)m(I_i) - \int_{I_i} f \right\} \right\| \le \varepsilon$.

See [G2] for Lemma 2.

Let $M(\mathbb{R}, X)$ be the space of X-valued McShane integrable functions; if $X = \mathbb{R}$, we abbreviate $M(\mathbb{R},\mathbb{R}) = M(\mathbb{R})$. The space $M(\mathbb{R})$ is complete under the semi-norm $\|f\|_1 = \int_{\mathbb{R}} |f|$ ([Mc] VI.4.3; the proof of this fact does not use any measure theory except the definition of a null set). Let $f \in M(\mathbb{R}, X)$. Then for each $x' \in X'$, $x'f$ belongs to $M(\mathbb{R})$ and $< x', \int_{\mathbb{R}} f > = \int_{\mathbb{R}} x'f$ ([G2]) so we can define a linear map $F : X' \to M(\mathbb{R})$ by $Fx' = x'f$. It is easily checked that F has a closed graph and is, therefore, continuous by the Closed Graph Theorem (use VI.4.3 p. 455 of [Mc]). Hence, we may define a semi-norm on $M(\mathbb{R},X)$ by $\|f\|_1 = \sup \left\{ \int_{\mathbb{R}} |x'f| : \|x'\| \le 1 \right\}$; this quantity is finite by the continuity of the map F. It is convenient in some of the computations which follow to use a semi-norm which is equivalent to $\| \ \|_1$.

In what follows let $\mathcal{A}$ be the algebra of subsets of $\mathbb{R}$ generated by the half-closed intervals $[a, b)$ of $\mathbb{R}$; thus, the elements of $\mathcal{A}$ are finite pairwise disjoint unions of half-closed subintervals of $\mathbb{R}$ ([Sw1] 2.1.11). We extend the length function m to $\mathcal{A}$ in the natural way to obtain an additive function, still denoted by m, on $\mathcal{A}$ ([Sw1] p.25). If $f \in M(\mathbb{R},X)$, then f is integrable over every $A \in \mathcal{A}$, and we set $\|f\|_1' = \sup \left\{ \left\| \int_A f \right\| : A \in \mathcal{A} \right\}$. We show that $\| \ \|_1$ and $\| \ \|_1'$ are equivalent. For this we require a lemma for scalar-valued integrable functions.

Lemma 3. Let $\varphi \in M(\mathbb{R})$ and let $\Phi : \mathcal{A} \to \mathbb{R}$ be the indefinite integral of φ defined by $\Phi(A) = \int_A \varphi$. Then $v(\Phi)(\mathbb{R}) = \int_{\mathbb{R}} |\varphi|$. [Here, $v(\Phi)$ denotes the variation of Φ computed with respect to the algebra $\mathcal{A}$([Sw1] p. 30).]

Proof: If $\{A_i : 1 \le i \le n\}$ is any partition of $\mathbb{R}$ with $A_i \in \mathcal{A}$, then $\sum_{i=1}^{n} |\Phi(A_i)| \le \sum_{i=1}^{n} \int_{A_i} |\varphi| = \int_{\mathbb{R}} |\varphi|$ so $v(\Phi)(\mathbb{R}) \le \int_{\mathbb{R}} |\varphi|$.

For the reverse inequality, first assume that φ is a step function $\varphi = \sum_{k=1}^{n} t_k C_{A_k}$, where $A_k \in \mathcal{A}$ and $\{A_k : 1 \le k \le n\}$ is a partition of $\mathbb{R}$ and C_A denotes the characteristic function of A. Then $\sum_{k=1}^{n} |\Phi(A_k)| = \sum_{k=1}^{n} |t_k| \, m(A_k) = \int_{\mathbb{R}} |\varphi| \le v(\Phi)(\mathbb{R})$ so $v(\Phi)(\mathbb{R}) = \int_{\mathbb{R}} |\varphi|$ for step functions.

Now assume $\varphi \in M(\mathbb{R})$. Pick a sequence of step functions $\{\varphi_k\}$ such that $\int_{\mathbb{R}} |\varphi_k - \varphi| \to 0$ ([Mc] Th. 13.1, p. 155). Note $\int_{\mathbb{R}} |\varphi_k| \to \int_{\mathbb{R}} |\varphi|$. Set $\Phi_k(A) = \int_A \varphi_k$ for $A \in \mathcal{A}$. By the inequality in the first part of the proof, we have $|v(\Phi_k)(\mathbb{R}) - v(\Phi)(\mathbb{R})| \le v(\Phi_k - \Phi)(\mathbb{R}) \le \int_{\mathbb{R}} |\varphi_k - \varphi|$ so by the equality established above for step functions, we obtain $\lim v(\Phi_k)(I) = \lim \int_{\mathbb{R}} |\varphi_k| = v(\Phi)(\mathbb{R}) = \int_{\mathbb{R}} |\varphi|$.

Remark 4. The equality in Lemma 3 is "well-known" for the Lebesgue integral, at least when the variation is computed with respect to the σ-algebra of Lebesgue measurable sets. Since for scalar-valued functions $M(\mathbb{R})$ is just the space of Lebesgue integrable functions, the equality follows fairly easily from the equality for Lebesgue integrable functions. However, we were not able to find a statement and proof of the equality which used only basic properties of the McShane integral.

We now establish the equivalence of $\| \ \|_1$ and $\| \ \|_1'$.

Proposition 5. $\| \ \|_1' \le \| \ \|_1 \le 2 \| \ \|_1'$.

Proof: Let $f \in M(\mathbb{R}, X)$. Then

$$\| f\|_1' = \sup\left\{\left|\int_A x'f\right| : A \in \mathcal{A},\ \|x'\| \le 1\right\} \le \sup\left\{\int_{\mathbb{R}} |x'f| : \|x'\| \le 1\right\} = \| f\|_1 .$$

For $x' \in X', \|x'\| \le 1$, using Lemma 3 and Theorem 2.2.1.7, p. 30, of [Sw1], we obtain $\| f\|_1 = \sup\left\{\int_{\mathbb{R}} |x'f| : \|x'\| \le 1\right\} = \sup\left\{v\left(\int x'f\right)(\mathbb{R}) : \|x'\| \le 1\right\} \le 2\sup\left\{\left|\int_A x'f\right| : A \in \mathcal{A},\ \|x'\| \le 1\right\} = 2\| f\|_1'$.

Remark 6. The two semi-norms above are "well-known" to be equivalent for the space of Pettis integrable functions ([P]) and since any McShane integrable function is Pettis integrable ([FM]), the semi-norms are equivalent on $M(\mathbb{R}, X)$. However, the proof above uses only basic properties of the McShane integral and no properties of the Pettis integral.

We now establish a convergence theorem for the McShane integral which for the scalar case of the integral implies the Beppo Levi (monotone) convergence Theorem. Since we are integrating functions defined on $\mathbb{R}$ we will use a technique employed by McLeod ([M]); for this we require a lemma.

Lemma 7. There exists a positive McShane integrable function $\varphi : \mathbb{R} \to (0, \infty)$ and a gauge $\gamma (= \gamma_\varphi)$ such that $0 \le S(\varphi, \mathcal{D}) \le 1$ for every γ-fine partial tagged partition $\mathcal{D}$.

Proof: Pick any positive McShane integrable function φ such that $\int_{\mathbb{R}} \varphi = 1/2$. There exists a gauge γ such that $|S(\varphi, \mathcal{D}) - 1/2| < 1/2$ whenever $\mathcal{D}$ is γ-fine. Suppose $\mathcal{D} = \{(I_i, t_i) : 1 \le i \le n\}$ is a γ-fine partial tagged partition and set $I = \bigcup_{i=1}^{n} I_i$. By Henstock's Lemma $\left|S(\varphi, \mathcal{D}) - \int_I \varphi\right| \le 1/2$ so $S(\varphi, \mathcal{D}) \le 1$ as required because $\int_I \varphi \le 1/2$ since φ is positive.

Theorem 8. *For each k let g_k belong to $M(\mathbb{R}, X)$ and suppose $g = \sum_{k=1}^{\infty} g_k$ pointwise on $\mathbb{R}$ with $\sum_{k=1}^{\infty} \|g_k\|_1 < \infty$. Then $g \in M(\mathbb{R}, X)$, $\int_{\mathbb{R}} g = \sum_{k=1}^{\infty} \int_{\mathbb{R}} g_k$ and $\left\|\sum_{k=1}^{n} g_k - g\right\|_1 \to 0$ as $n \to \infty$.*

Proof: Let $\varepsilon > 0$ and set $G_n = \sum_{k=1}^{n} g_k$. Observe that since $\sum_{k=1}^{\infty} \left\| \int_A g_k \right\| \leq \sum_{k=1}^{\infty} \|g_k\|_1' < \infty$ (Proposition 5), the series $\sum_{k=1}^{\infty} \int_A g_k$ is (absolutely) convergent by the completeness of X. For convenience, set $v = \sum_{k=1}^{\infty} \int_{\mathbb{R}} g_k$. For each k let γ_k be a gauge for G_k such that $\left\| S(G_k, \mathcal{D}) - \int_{\mathbb{R}} G_k \right\| < \varepsilon/2^k$ when $\mathcal{D}$ is γ_k-fine. Pick n_0 such that $\sum_{k=n_0}^{\infty} \|g_k\|_1 < \varepsilon$. For every $t \in \mathbb{R}$ there exists $n(t) \geq n_0$ such that $k \geq n(t)$ implies $|G_k(t) - g(t)| < \varepsilon\varphi(t)$, where φ is the function in Lemma 7. Define a gauge γ by setting $\gamma(t) = \gamma_{n(t)}(t) \cap \gamma_\varphi(t)$ for $t \in \mathbb{R}$ and $\gamma(\pm\infty) = \mathbb{R}^*$. Suppose $D = \{(I_i, t_i) : 1 \leq i \leq n\}$ is a γ-fine partition of $\mathbb{R}^*$. Then

(1) $$|S(g, \mathcal{D}) - v| = \left\| \sum_{i=1}^{n} \left\{ \sum_{k=1}^{\infty} g_k(t_i) m(I_i) - \sum_{k=1}^{\infty} \int_{I_i} g_k \right\} \right\|$$

$$\begin{aligned} &\leq \left\| \sum_{i=1}^{n} \sum_{k=n(t_i)+1}^{\infty} g_k(t_i) m(I_i) \right\| + \left\| \sum_{i=1}^{n} \left\{ \sum_{k=1}^{n(t_i)} g_k(t_i) m(I_i) - \sum_{k=1}^{n(t_i)} \int_{I_i} g_k \right\} \right\| \\ &\quad + \left\| \sum_{i=1}^{n} \sum_{k=n(t_i)+1}^{\infty} \int_{I_i} g_k \right\| = T_1 + T_2 + T_3, \end{aligned}$$

with obvious definitions for the T_i. We estimate each T_i. First,

$$T_1 \leq \sum_{i=1}^{n} \left\| \sum_{k=n(t_i)+1}^{\infty} g_k(t_i) \right\| m(I_i) \leq \sum_{i=1}^{n} \varepsilon\varphi(t_i) m(I_i) = \varepsilon S(\varphi, \mathcal{D}) \leq \varepsilon$$

by Lemma 7. Next for estimating T_2 let $s = \max\{n(t_i), \ldots, n(t_n)\}$. Then, by Henstock's Lemma,

$$\begin{aligned} T_2 &= \left\| \sum_{i=1}^{n} \left\{ G_{n(t_i)}(t_i) m(I_i) - \int_{I_i} G_{n(t_i)} \right\} \right\| \\ &= \left\| \sum_{k=1}^{s} \sum_{\substack{i \\ n(t_i)=k}} \left\{ G_{n(t_i)}(t_i) m(I_i) - \int_{I_i} G_{n(t_i)} \right\} \right\| \\ &\leq \sum_{k=1}^{s} \left\| \sum_{\substack{i \\ n(t_i)=k}} \left\{ G_{n(t_i)}(t_i) m(I_i) - \int_{I_i} G_{n(t_i)} \right\} \right\| \leq \sum_{k=1}^{s} \varepsilon/2^k < \varepsilon. \end{aligned}$$

For T_3, note that the series in T_3 converges (absolutely) by the observation above. Then

$$T_3 = \sup_{\|x'\| \leq 1} \left| \sum_{i=1}^{n} \sum_{k=n(t_i)+1}^{\infty} \int_{I_i} x' g_k \right| \leq \sup_{\|x'\| \leq 1} \sum_{i=1}^{n} \sum_{k=n_0+1}^{\infty} \int_{I_i} |x' g_k|$$

$$\leq \sup_{\|x'\|\leq 1} \sum_{k=n_0+1}^{\infty} \int_{\mathbb{R}} |x'g_k| \leq \sum_{k=n_0+1}^{\infty} \|g_k\|_1 < \varepsilon.$$

From (1), $\|S(g,\mathcal{D})-v\| < 3\varepsilon$ and g is McShane integrable with integral equal to v.

For the last statement, by the first part of the theorem, if $A \in \mathcal{A}$, then

$$\left\|\int_A (g-G_n)\right\| = \left\|\int_A \sum_{k=n+1}^{\infty} g_k\right\| = \left\|\sum_{k=n+1}^{\infty} \int_A g_k\right\| \leq \sum_{k=n+1}^{\infty} \left\|\int_A g_k\right\| \leq \sum_{k=n+1}^{\infty} \|g_k\|_1'$$

which implies that $\|g-G_n\|_1' \to 0$ as $n \to \infty$.

Fremlin and Mendoza ([F1-3],[FM]) give general convergence theorems for the McShane integral. Their proofs use properties of vector-valued measures and Pettis integrals whereas the proof of Theorem 8 uses only basic properties of the McShane integral.

Theorem 8 can be viewed as an analogue of the Monotone Convergence Theorem for vector-valued McShane integrable functions. Indeed, it is easy to see that the (scalar) Monotone Convergence Theorem follows immediately from Theorem 8. For example, suppose that $f_k \in M(\mathbb{R})$ satisfies $0 \leq f_1 \leq f_2 \leq \ldots$, $f = \lim f_k$ and $\sup \int_{\mathbb{R}} f_k < \infty$. Set $g_k = f_k - f_{k-1}$, where $f_0 = 0$. Then the conditions of Theorem 8 are satisfied so $\sum_{k=1}^{\infty} g_k = f$ is integrable and $\sum_{k=1}^{n} \int_{\mathbb{R}} g_k = \int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f$.

Theorem 8 also has the following corollary which is useful in establishing the completeness of $M(\mathbb{R})$.

Corollary 9. Let $f_k \in M(\mathbb{R}, X)$ and suppose $\lim f_k = f$ pointwise on $\mathbb{R}$. If $\{f_k\}$ is $\|\ \|_1$-Cauchy, then f is integrable and $\|f_k - f\|_1 \to 0$.

Proof: Pick a subsequence $\{n_k\}$ satisfying $\left\|f_{n_{k+1}} - f_{n_k}\right\| < 1/2^k$ and set $g_k = f_{n_{k+1}} - f_{n_k}$. Then $\sum_{j=1}^{k} g_k = f_{n_{k+1}} - f_{n_1} \to f - f_{n_1}$ pointwise and $\sum_{k=1}^{\infty} \|g_k\|_1 < \infty$ so Theorem 8 implies that $f - f_{n_1}$ is integrable and $\left\|\sum_{j=1}^{k} g_k - (f - f_{n_1})\right\|_1 = \left\|f_{n_{k+1}} - f\right\|_1 \to 0$. Since the same argument can be applied to any subsequence of $\{f_k\}$, it follows that $\|f_k - f\|_1 \to 0$.

Upon the introduction of null sets, Corollary 9 can be applied directly to establish the completeness of $M(\mathbb{R})$ as in [Mc] VI.4.3.

We now prove two absolute continuity properties of the integral which are interesting in their own right but which will also be used later to establish topological properties of the space $M(\mathbb{R}, X)$.

Theorem 10. *Let $f \in M(\mathbb{R}, X)$ and let $F : \mathcal{A} \to X$ be the indefinite integral of f, i.e., $F(A) = \int_A f$. Then $\lim_{m(A)\to 0} \|F(A)\| = 0$, i.e., F is m-continuous, and $\lim_{m(A)\to 0} \|C_A f\|_1 = 0$.*

Proof: Let $\varepsilon > 0$. There exists a gauge γ on $\mathbb{R}^*$ such that

$$\left\| \sum_{i=1}^{n} f(t_i)m(I_i) - \int_{\mathbb{R}} f \right\| < \varepsilon/2 \text{ when } \mathcal{D}_0 = \{(I_i, t_i) : i = 1, \ldots, n\}$$

is a γ-fine tagged partition of $\mathbb{R}$. Fix such a $\mathcal{D}_0$ and set

$$M = \max\{\|f(t_i)\| : i = 1, \ldots, n\}.$$

Set $\delta = \varepsilon/2M$. Suppose $A \in \mathcal{A}$ and $m(A) < \delta$. Then A is a pairwise disjoint union of half-closed intervals $\{K_j : j = 1, \ldots, m\}$, where we may assume, by subdividing each K_j if necessary, that each K_j is contained in some I_i. Let $\pi_i = \{j : K_j \subset I_i\}$. Then $\{(K_j, t_i) : j \in \pi_i,\ i = 1, \ldots, n\}$ is a γ-fine partial tagged partition of $\mathbb{R}$ so Henstock's Lemma implies that $\left\| \sum_{i=1}^{n} \sum_{j \in \pi_i} \left\{ f(t_i)m(K_j) - \int_{K_j} f \right\} \right\| \le \varepsilon/2$. Hence,

$$\left\| \int_A f \right\| = \left\| \sum_{j=1}^{m} \int_{K_j} f \right\| \le \varepsilon/2 + \left\| \sum_{i=1}^{n} \sum_{j \in \pi_i} f(t_i)m(K_j) \right\| \le \varepsilon/2 + Mm(A) < \varepsilon.$$

The last statement follows from the first part and Proposition 5.

Theorem 11. Let $f \in M(\mathbb{R}, X)$. Then $\lim_{b\to\infty} \left\| C_{[b,\infty)} f \right\|_1 = 0$. In particular, $\lim_{b\to\infty} \left\| \int_b^\infty f \right\| = 0$.

Proof: Let $\varepsilon > 0$. There exists a gauge γ with $\gamma(z)$ bounded for every $z \in \mathbb{R}$ such that $\left\| S(f, \mathcal{D}) - \int_{\mathbb{R}} f \right\| < \varepsilon$ whenever $\mathcal{D}$ is a γ-fine tagged partition. Fix such a partition $\mathcal{D} = \{(I_i, t_i) : 1 \le i \le n\}$ and assume $I_1 = [b, \infty)$, $t_1 = \infty$. If $a > b$, let $K \in \mathcal{A}$, $K \subset [a, \infty)$ with $K = \bigcup_{i=1}^{k} J_i$, $\{J_i\}$ pairwise disjoint half-closed intervals. Then $\mathcal{K} = \{(J_i, \infty) : 1 \le i \le k\}$ is γ-fine so Henstock's Lemma implies that $\left\| \int_K f - S(f, \mathcal{K}) \right\| = \left\| \int_K f \right\| \le \varepsilon$ so $\left\| C_{[a,\infty)} f \right\|_1 \le \varepsilon$ when $a > b$.

3 Topological Properties of $M(\mathbb{R}, X)$

We now show that the convergence result in Theorem 8 and Theorems 10 and 11 can be used to show that the space $M(\mathbb{R}, X)$ is barrelled and ultrabornological. For this we use results of [Sw2], [Sw3] and [DFFP]; these papers contain general results based on continuous versions of the gliding hump property which imply that spaces are either Banach-Mackey spaces ([Sw1], [Sw2]) or ultrabornological ([DFFP]). Of course, an ultrabornological space is barrelled, but a comparison of the two methods and the corresponding properties of $M(\mathbb{R}, X)$ may be of some interest.

We begin by discussing the gliding hump property employed in [Sw2], [Sw3]. For $A \in \mathcal{A}$ let P_A be the projection operator defined on $M(\mathbb{R}, X)$ by $P_A f = C_A f$, $f \in M(\mathbb{R}, X)$. The family of projections $\{P_A : A \in \mathcal{A}\} = \mathcal{P}$ is said to have the strong gliding hump property (SGHP) if for every pairwise disjoint sequence $\{A_j\}$

from $\mathcal{A}$ and every null sequence $\{f_j\}$ from $M(\mathbb{R}, X)$, there is a subsequence $\{n_j\}$ such that the series $\sum_{j=1}^{\infty} P_{A_{n_j}} f_{n_j}$ converges in $M(\mathbb{R}, X)$ ([Sw2], [Sw3]; the description of SGHP in these papers is given in a more abstract setting). We begin by noting that Theorem 8 can be used to show that $M(\mathbb{R}, X)$ has SGHP.

Theorem 12. *$M(\mathbb{R}, X)$ has SGHP.*

Proof: Let $f_j \to 0$ in $M(\mathbb{R}, X)$ and let $\{A_j\} \subset \mathcal{A}$ be pairwise disjoint. Pick a subsequence such that $\left\|f_{n_j}\right\|_1 \leq 1/2^j$ and let f be the pointwise sum of the series $\sum_{j=1}^{\infty} P_{A_{n_j}} f_{n_j}$. By Theorem 8 f is McShane integrable and the series converges to f in $\| \ \|_1$. Hence, SGHP is satisfied.

We next consider a decomposition property for the family $\mathcal{P}$ that is necessary to apply the results of [Sw2], [Sw3].

(D) For every $y' \in M(\mathbb{R}, X)'$, $f \in M(\mathbb{R}, X)$ and $\varepsilon > 0$ there is a partition $\{B_1, \ldots, B_k\}$ of $\mathbb{R}$ with $B_i \in \mathcal{A}$ such that $v(y'P_{B_i}f) < \varepsilon$ for $i = 1, \ldots, k$, where $v(y'P_\bullet f)$ denotes the variation of the finitely additive set function $A \to < y', P_A f >$. [Rao and Rao refer to this property as "strongly continuous" ([RR] 5.1.4); again there is a more abstract definition of property (D) in [Sw2], [Sw3].]

Theorem 13. $\mathcal{P}$ has property (D).

Proof: Let $\varepsilon > 0$, $f \in M(\mathbb{R}, X)$, $y' \in M(\mathbb{R}, X)'$. It follows from Theorems 10 and 11 that $\lim_{m(A)\to 0} y'P_A f = 0$ and $\lim_{b\to\infty} y'P_{[b,\infty)} f = 0$.

Hence, from 2.2.1.7 of [Sw1], $\lim_{m(A)\to 0} v(y'P_A f) = 0$ and $\lim_{b\to\infty} v\left(y'P_{[b,\infty)} f\right) = 0$. Pick $a < b$ such that $v(y'P_{[b,\infty)} f) < \varepsilon$ and $v\left(y'P_{(-\infty,a)} f\right) = 0$. Now partition $[a, b]$ into intervals $\{I_i : i = 1, \cdots, n\}$ such that $v\left(y'P_{I_i} f\right) < \epsilon$. Then $\{(-\infty, a), [b, \infty), I_i\}$ is a partition of $\mathbb{R}$ satisfying the condition in (D).

If a space has a family of projections satisfying properties SGHP and (D), it is shown in [Sw2], [Sw3] that the space is a Banach-Mackey space, and since $M(\mathbb{R}, X)$ is a normed space, from Theorems 12 and 13, we have

Corollary 14. $M(\mathbb{R}, X)$ is barrelled.

It is shown in [DFFP] that the space of vector-valued McShane integrable functions defined on a compact interval is an ultrabornological space, a condition which implies that the space is barrelled. This result is also obtained from an abstract gliding hump theorem. We will use their results along with Theorem 8 to show that $M(\mathbb{R}, X)$ is also ultrabornological.

Let $\sum$ be the power set of $\mathbb{N}$ and μ the counting measure on $\sum$. For $f \in M(\mathbb{R}, X)$, set $Q_1 f = f\, C_{[-1,1)}$ and $Q_n f = f C_{[-n,-n+1)\cup[n-1,n)}$ for $n \geq 2$. For $A \in \sum$ and $f \in M(\mathbb{R}, X)$ define $Q_A f = \sum_{n\in A} Q_n f$ (pointwise sum). Then $Q = \{Q_A : A \in \sum\}$ is an equicontinuous family of projections on $M(\mathbb{R}, X)$, and from Theorem 3 of [DFFP], it follows that $Q_n(M(\mathbb{R}, X))$ is ultrabornological for every $n \in \mathbb{N}$. To show that $M(\mathbb{R}, X)$ is ultrabornological we use Corollary 1 of [DFFP]; in order to apply this result we need to check the following gliding hump property:

(*) if $\{\Omega_n\}$ is a decreasing sequence in Σ with $\mu\left(\bigcap_{n=1}^{\infty}\Omega_n\right) = 0$, $\{f_n\}$ is a bounded sequence such that $Q_{\Omega_n} f_n = f_n$, and $\{\alpha_n\} \in \ell^1$, then the series $\sum_{n=1}^{\infty} \alpha_n f_n$ converges in $M(\mathbb{R}, X)$.

(Again, there is an abstract definition of property (*) given in [DFFP].)

Theorem 14. *$M(\mathbb{R}, X)$ has property (*).*

Proof: Using the notation in (*), it follows from Lemma 1 of [DFFP] that the series $\sum_{n=1}^{\infty} \alpha_n f_n$ converges pointwise to a function f. It then follows immediately from Theorem 8 that the series converges to f in $\| \|_1$ so (*) is satisfied.

Thus, from Corollary 1 of [DFFP], we have

Corollary 15. $M(\mathbb{R}, X)$ is ultrabornological.

There are further examples of ultrabornological function spaces given in [Gi].

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