

Introduction

In the two centuries following its creation (by Newton and Leibniz), the calculus was developed along two parallel paths: the *method of limits* and the *infinitesimal method*, both present in the work and the letters of the founding fathers. In the method of limits, the derivative of a function f appears as the limit of the differential ratio

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (1)$$

when Δx goes to zero; in the infinitesimal approach, the derivative is the value of (1) for an infinitely small increase Δx . Newton and Leibniz neither clearly define the concept of limit, nor that of infinitely small increase.

The approach of the immediate followers of Leibniz, Jacob and Johann Bernoulli and their pupil Euler, is essentially infinitesimal, and those mathematicians make important progress in mathematical analysis and its use in science and engineering. Faced with the difficulty in justifying the infinitesimal method, d'Alembert suggests to concentrate on the method of limits. This program is realized around 1820 by Cauchy, who clearly defines the concept of limit and put it as a firm basis to analysis. However, Cauchy refrains himself from banishing completely the infinitesimals, that he defines as variables having a zero limit.

In the second half of the XIXth century, Weierstrass and others give to Cauchy's approach its modern formulation, by realizing the program of *arithmetization of analysis* initiated by Bolzano. Based upon a preliminary construction of the real numbers, for which the non-existence of infinitely small or infinitely large numbers can be proved, the Weierstrassian presentation of mathematical analysis excludes the concept of infinitesimal. If a classics like the *Cours d'analyse infinitésimale* of Charles de La Vallée Poussin still maintains the description of the two approaches in the twelve editions of his book published between 1903 and 1959¹, the infinitesimal method and even the infinitesimal terminology is completely eradicated from the classical treatises of this century like those of Goursat, Tannery, Hardy, Valiron, Favard, Schwartz and Dieudonné.

But this arithmetization of analysis was also instrumental in the development of set theory and of new reflexions upon the foundations of mathematics and mathematical logic. If we except passionate discussions upon the axiom of choice, those

¹*La méthode des limites consiste à trouver des relations entre les quantités par passage à la limite. (...) La méthode infinitésimale est celle où l'on se sert de la considération des infiniment petits*

works do not modify substantially the approach of mathematical analysis until the fall of 1960, when Abraham Robinson shows that: *the concepts and methods of contemporary Mathematical Logic are capable of providing a suitable framework for the development of the Differential and Integral Calculus by means of infinitely small and infinitely large numbers. (...) The resulting subject was called by me Non-standard Analysis since it involves and was, in part, inspired by the so-called Non-standard models of Arithmetic whose existence was first pointed out by T. Skolem*².

Hence Robinson achieves, for the infinitesimal method, what Weierstrass had achieved, a century before, for the method of limits. The birth of non-standard analysis, and its early developments, is described in a very detailed way in J.W. Dauben's biography of Abraham Robinson³ analyzed in this issue of the *Bulletin*.

Since Robinson's book, a large number of expository books on non-standard analysis, whose approach may differ from Robinson's one, have been published, among which we may quote the following ones :

- M. Machover and J. Hirschfeld, Lectures on Non-standard Analysis, Springer, Berlin, 1969
- K.D. Stroyan and W.A. Luxemburg, Introduction to the Theory of Infinitesimals, Academic Press, New York, 1976
- H.J. Keisler, Elementary Calculus : An Approach Using Infinitesimals, Prindle, Weber and Schmidt, Boston, Mass., 1976
- H.J. Keisler, Foundations of Infinitesimal Calculus, Prindle, Weber and Schmidt, Boston, Mass., 1976
- M. Davis, Applied Non-Standard Analysis, Wiley, New York, 1977
- D. Laugwitz, Infinitesimalkalkül, Bibliographische Institut, Mannheim, 1978
- J.M. Henle and E.M. Kleinberg, Infinitesimal Calculus, MIT Press, Cambridge, Mass, 1979
- R. Lutz and M. Goze, Non-standard Analysis: A Practical Guide with Applications, Springer, Berlin, 1981
- M.M. Richter, Ideale Punkte, Monaden und Nichtstandard-Methoden, Vieweg, Wiesbaden, 1982
- A. Robert, Analyse non standard, Presses Polytechniques Romandes, Lausanne, 1985
- A.E. Hurd and P.A. Loeb, An Introduction to Non Standard Real Analysis, Academic Press, New York, 1985

²A. ROBINSON, *Non-Standard Analysis*, North-Holland, Amsterdam, 1966

³J.W. DAUBEN, *The Creation of Nonstandard Analysis. A Personal and Mathematical Odyssey*, Princeton University Press, Princeton, 1995

- S. Albeverio, J.E. Fenstad, R. Hoegh-Krohn and T. Lindstrom, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, New York, 1986
- D. Laugwitz, *Zahlen und Kontinuum*, Bibliographische Institut, Mannheim, 1986
- A. Deledicq et M. Diener, *Leçons de calcul infinitésimal*, Armand Colin, Paris, 1989
- F. Diener et G. Reeb, *Analyse Non Standard*, Hermann, Paris, 1989

Thirty-five years after its creation, non-standard analysis is to-day a vivid and fruitful method of mathematical analysis, with striking accomplishments in various areas like functional analysis, measure theory, differential equations, probability, stochastic processes and mathematical physics. This is exemplified by the recent books respectively edited by S.A. Albeverio, W.A. Luxemburg and M.P.H. Wolf⁴ and by F. and M. Diener⁵, analyzed in this *Bulletin*, and by the interesting papers which constitute this issue.

It still remains to see if nonstandard analysis will provide a real alternative for teaching calculus and mathematical analysis. Whatever the issue will be, there is no doubt that the reflexions that it has inspired and will suggest should contribute to a better understanding of the unavoidable difficulties of this fundamental branch of modern mathematics, and to a better methodology for its presentation.

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⁴*Advances in Analysis, Probability and Mathematical Physics*, Kluwer Academic Publishers, Dordrecht, 1995

⁵*Nonstandard analysis in Practice*, Springer-Verlag, Berlin, 1995