Complete space-like submanifolds with constant scalar curvature in a de Sitter space

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Abstract

In this paper, we investigate n-dimensional complete space-like submanifolds M^n with constant normalized scalar curvature R in a de Sitter space $S_p^{n+p}(c)$. Suppose that the normalized mean curvature vector field is parallel. We prove that if the norm square $\|h\|^2$ of the second fundamental form of M^n satisfies $n\bar{R} \leq \|h\|^2 \leq \min\{\alpha(n,\bar{R}), \beta(n,\bar{R})\}$, then M^n is a totally umbilical submanifold; or $n = 3$ and M^3 is a hyperbolic cylinder $H^1(c-\lambda^2) \times S^2(c-\mu^2)$ in $S_1^4(c)$, where $\bar{R} = c - R \geq 0, \alpha(n,\bar{R})$ and $\beta(n,\bar{R})$ are constants only depend on n and $\bar{R}$.

Mathematics Subject Classification: 53C42,53A10.
Key words: space-like submanifolds, de Sitter space, totally umbilical manifolds, hyperbolic cylinder.

1. Introduction

A de Sitter space $S_p^{n+p}(c)$ is an $(n+p)$-dimensional connected complete pseudo-Riemannian manifold of index p with constant curvature $c > 0$. Goddard [7] conjectured that a complete space-like hypersurface in $S_1^{n+1}(c)$ with constant mean curvature H must be totally umbilical. Akutagawa [2] and Ramanathan [11] proved independently that the conjecture is true if $H^2 \leq c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Cheng [4] generalized this result to complete space-like submanifolds in $S_p^{n+p}(c)$ with parallel mean curvature vector. For the study of space-like hypersurfaces with constant scalar curvature in a de Sitter space, Zheng ([15], [16]) proved that the compact space-like hypersurface M^n in a de Sitter space $S_1^{n+1}(c)$ with constant scalar curvature is totally umbilical if $k(M) > 0$ and $R < c$, where $k(M)$ and R are the sectional curvature and the normalized scalar curvature of M^n. Later, Cheng and Ishikawa [5] showed that if the condition $K(M) > 0$ is deleted, then Zheng's result in [15], [16] is also true. Recently, Liu [8] proved the following theorem

Theorem 1. *Let M^n be an n-dimensional $(n \geq 3)$ complete space-like hypersurface with constant normalized scalar curvature R in an $(n+1)$-dimensional de Sitter space S_1^{n+1} and denote $\bar{R} = 1 - R$. If the norm square $\|h\|^2$ of the second fundamental*

Balkan Journal of Geometry and Its Applications, Vol.9, No.2, 2004, pp. 82-91.

form of M^n *satisfies* $n\bar{R} \leq \sup \|h\|^2 \leq D(n, \bar{R})$*, then either (i)* $\sup \|h\|^2 = n\bar{R}$ *and* M^n *is totally umbilical; or (ii)* $\sup \|h\|^2 = D(n, \bar{R})$ *and* M^n *is a hyperbolic cylinder* $H^1(1-coth^2 r) \times S^{n-1}(1-tanh^2 r)$*, where*

$$D(n, \bar{R}) = \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n].$$

On the other hand, it is natural and very important to study n-dimensional submanifolds with constant scalar curvature and higher codimension in a de Sitter space $S_p^{n+p}(c)$. But there are few results about it. In this paper, we shall prove the following

Theorem 2. *Let* M^n *be an* n*-dimensional* $(n \geq 3)$ *complete space-like submanifold with constant normalized scalar curvature* R *in an* $(n+p)$*-dimensional de Sitter space* $S_p^{n+p}(c)$*. Suppose that the normalized mean curvature vector field is parallel and* $\bar{R} = c - R \geq 0$*. If the norm square* $\|h\|^2$ *of the second fundamental form of* M^n *satisfies* $n\bar{R} \leq \|h\|^2 \leq \min\{\alpha(n, \bar{R}), \beta(n, \bar{R})\}$*, then* M^n *is a totally umbilical submanifold; or* $n = 3$ *and* M^3 *is a hyperbolic cylinder* $H^1(c - \lambda^2) \times S^2(c - \mu^2)$ *in* $S_1^4(c)$*, where*

$$\alpha(n, \bar{R}) = \frac{n}{n-2} \frac{(n-1)(n-2)^2\bar{R}^2 + [nc - (n-1)\bar{R}]^2}{(n-2)^2\bar{R} + 2[nc - (n-1)\bar{R}]}, \quad \beta(n, \bar{R}) = \frac{n}{n-2}[nc-(n-1)\bar{R}].$$

λ *and* μ *are the two distinct principal curvatures of* M^3 *such that one has the multiplicity 1 and the other the multiplicity 2.*

2 Preliminaries

Let $S_p^{n+p}(c)$ be an $(n+p)$-dimensional de Sitter space with index p. Let M^n be an n-dimensional connected space-like submanifold immersed in $S_p^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_{n+p}$ in $S_p^{n+p}(c)$ such that at each point of $M^n, e_1, \cdots, e_n$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices: $1 \leq A, B, C, \cdots \leq n+p; 1 \leq i, j, k, \cdots \leq n; n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p$. Let $\omega_1, \cdots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $S_p^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Then the structure equations of $S_p^{n+p}(c)$ are given by

$$(1) \qquad d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$(2) \qquad d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D,$$

$$(3) \qquad K_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

Restrict these form to M^n. Then we have

$$(4) \qquad \omega_\alpha = 0, \qquad n+1 \leq \alpha \leq n+p.$$

From Cartan's Lemma we have

$$\omega_{\alpha_i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{5}$$

The connection forms of M^n are characterized by the structure equations

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \omega_{ij} + \omega_{ji} = 0, \tag{6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{8}$$

where R_{ijkl} are the components of the curvature tensor of M^n.
Denote by h the second fundamental form of M^n. Then

$$h = \sum_{i,j,\alpha} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha. \tag{9}$$

Denote by ξ, H and $\|h\|^2$ the mean curvature vector field, the mean curvature and the norm square of the second fundamental form of M^n. Then they are defined by

$$\xi = \frac{1}{n}\sum_\alpha(\sum_i h^\alpha_{ii})e_\alpha, \quad H = \|\xi\| = \frac{1}{n}\sqrt{\sum_\alpha(\sum_i h^\alpha_{ii})^2}, \quad \|h\|^2 = \sum_{i,j,\alpha}(h^\alpha_{ij})^2. \tag{10}$$

Moreover, the normal curvature tensor $\{R_{\alpha\beta kl}\}$, the Ricci curvature tensor $\{R_{ik}\}$ and the scalar curvature $n(n-1)R$ are expressed as

$$R_{\alpha\beta kl} = \sum_m (h^\alpha_{km}h^\beta_{ml} - h^\alpha_{lm}h^\beta_{mk}), \tag{11}$$

$$R_{ik} = (n-1)c\delta_{ik} - n\sum_\alpha(\sum_l h^\alpha_{ll})h^\alpha_{ik} + \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk}, \tag{12}$$

$$n(n-1)(R-c) = \|h\|^2 - n^2H^2, \tag{13}$$

where R is the normalized scalar curvature.
Define the first and the second covariant derivatives of $\{h^\alpha_{ij}\}$, say $\{h^\alpha_{ijk}\}$ and $\{h^\alpha_{ijkl}\}$ by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{14}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_m h^\alpha_{mjk}\omega_{mi} + \sum_m h^\alpha_{imk}\omega_{mj} + \sum_m h^\alpha_{ijm}\omega_{mk} + \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

We obtain the Codazzi equation by straightforward computations

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{15}$$

It follows that the Ricci identities hold

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj} R_{mjkl} + \sum_m h^\alpha_{im} R_{mjkl} + \sum_\beta h^\beta_{ij} R_{\beta\alpha kl}. \tag{16}$$

The Laplacian of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (16) we obtain for any $\alpha, n+1 \le \alpha \le n+p$,

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} + \sum_{k,m} h^\alpha_{km} R_{mijk} + \sum_{k,m} h^\alpha_{im} R_{mkjk} + \sum_{k,\beta} h^\beta_{ik} R_{\beta\alpha jk}. \tag{17}$$

In the case of the mean curvature vector $\xi \neq 0$, we know that $e_{n+1} = \xi/H$ is a normal vector field defined globally on M^n. We define $\|\mu\|^2$ and $\|\tau\|^2$ by

$$\|\mu\|^2 = \sum_{i,j} (h^{n+1}_{ij} - H\delta_{ij})^2, \qquad \|\tau\|^2 = \sum_{\alpha > n+1} \sum_{i,j} (h^\alpha_{ij})^2, \tag{18}$$

respectively. Then $\|\mu\|^2$ and $\|\tau\|^2$ are functions defined on M^n globally, which do not depend on the choice of the orthonormal frame $\{e_1, \cdots, e_n\}$. And we have

$$\|h\|^2 = nH^2 + \|\mu\|^2 + \|\tau\|^2. \tag{19}$$

From the definition of the mean curvature vector ξ, we know $nH = \sum_i h^{n+1}_{ii}$ and $\sum_i h^\alpha_{ii} = 0$ for $n+2 \le \alpha \le n+p$ on M^n.
From (13),(18) and (19), we have

$$\Delta(n^2H^2) = \Delta\|h\|^2 = \Delta(\mathrm{tr} H^2_{n+1}) + \Delta\|\tau\|^2. \tag{20}$$

Hence, from (8),(11) and (17) and by a direct calculation we conclude

$$\begin{aligned} \tfrac{1}{2}\Delta(trH^2_{n+1}) &= \sum_{i,j,k} (h^{n+1}_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij} \Delta h^{n+1}_{ij} \\ &= \sum_{i,j,k} (h^{n+1}_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij} (nH)_{ij} + nc\mathrm{tr}H^2_{n+1} - n^2H^2c \\ &\quad -nH\mathrm{tr}(H^3_{n+1}) + [\mathrm{tr}(H^2_{n+1})]^2 + \sum_{\beta > n+1} [\mathrm{tr}(H_{n+1}H_\beta)]^2, \end{aligned} \tag{21}$$

$$\begin{aligned} \tfrac{1}{2}\Delta\|\tau\|^2 &= \sum_{i,j,k,\alpha>n+1} (h^\alpha_{ijk})^2 + \sum_{i,j,\alpha>n+1} h^\alpha_{ij}\Delta h^\alpha_{ij} \\ &= \sum_{i,j,k,\alpha>n+1} (h^\alpha_{ijk})^2 + nc\|\tau\|^2 - nH \sum_{\alpha>n+1} \mathrm{tr}(H^2_\alpha H_{n+1}) \\ &\quad + \sum_{\alpha>n+1} [\mathrm{tr}(H_{n+1}H_\alpha)]^2 + \sum_{\alpha,\beta>n+1} [\mathrm{tr}(H_\alpha H_\beta)]^2, \end{aligned} \tag{22}$$

where H_α denote the matrix (h^α_{ij}) for all α.
We need the following Lemmas.

Lemma 1([3]). *Let $\{\mu_i\}^n_{i=1}$ be a set of real numbers satisfying $\sum_i \mu_i = 0$ and $\sum_i \mu^2_i = \beta^2$, where $\beta \ge 0$. Then*

$$|\sum_i \mu^3_i| \le \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{23}$$

and the equalities hold if and only if at least $n-1$ of the μ_i's are equal with each other.

Lemma 2 ([10], [13])). *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below. If F is a C^2-function bounded from above on M^n, then for any $\varepsilon > 0$, there is a point $x \in M^n$ such that*

$$\sup F - \varepsilon < F(x), \|\nabla F\|(x) < \varepsilon, \Delta F(x) < \varepsilon. \tag{24}$$

Lemma 3([12]). *Let A, B be symmetric $n \times n$ matrices satisfying $AB = BA$ and $trA = trB = 0$. Then*

$$|\mathrm{tr}A^2B| \le \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{1/2}. \tag{25}$$

3 Proof of Theorem 2

For a C^2-function f defined on M^n, we defined its gradient and Hessian (f_{ij}) by the following formulas

$$df = \sum_i f_i\omega_i, \qquad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}. \tag{26}$$

Let $T = \sum_{i,j} T_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = nH\delta_{ij} - h_{ij}^{n+1}. \tag{27}$$

Following Cheng-Yau [6], we introduce an operator $\square$ associated to T acting on f by

$$\square f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})f_{ij}. \tag{28}$$

By a simple calculation and from (20), we obtain

$$\begin{aligned} \square(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} \\ &= \tfrac{1}{2}\Delta(n^2H^2) - \|\mathrm{grad}(nH)\|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &= \tfrac{1}{2}\Delta(\mathrm{tr}H_{n+1}^2) + \tfrac{1}{2}\Delta\|\tau\|^2 - \|\mathrm{grad}(nH)\|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}. \end{aligned} \tag{29}$$

We choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $h_{ij}^{n+1} = \lambda_i\delta_{ij}$. Since $\sum_i(\lambda_i - H) = 0$, then

$$\sum_i(\lambda_i - H)^2 = \sum_i \lambda_i^2 - nH^2 = \mathrm{tr}\, H_{n+1}^2 - nH^2 = \|\mu\|^2.$$

Then by Lemma 1

$$\begin{aligned} -nH\mathrm{tr}(H_{n+1}^3) &= -nH\sum_i \lambda_i^3 \\ &= -3nH^2\|\mu\|^2 - n^2H^4 - nH\sum_i(\lambda_i - H)^3 \\ &\ge -3nH^2\|\mu\|^2 - n^2H^4 - \tfrac{n(n-2)}{\sqrt{n(n-1)}}H\|\mu\|^3. \end{aligned} \tag{30}$$

From (21),(30) and $trH_{n+1}^2 = \|\mu\|^2 + nH^2$,we have

$$
\begin{aligned}
\tfrac{1}{2}\Delta(\mathrm{tr}H_{n+1}^2) \;&\geq\; \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\
&\quad + \|\mu\|^2\{\|\mu\|^2 - \tfrac{n(n-2)}{\sqrt{n(n-1)}}H\|\mu\| + nc - nH^2\} \\
&\geq\; \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\
&\quad + \|\mu\|^2\{nc - nH^2 - \tfrac{n(n-2)}{\sqrt{n(n-1)}}H\|\mu\|\}.
\end{aligned} \tag{31}
$$

Let M^n be a compele connected submanifold in $S_p^{n+p}(c)$ with nowhere zero mean curvature H. Suppose that the normalized mean curvature vector ξ/H is parallel in $T^\perp M^n$ and choose $e_{n+1} = \xi/H$. Then $\omega_{\alpha n+1} = 0$ for all α. Consequently $R_{\alpha n+1jk} = 0$. From (11) we have

$$\sum_i h_{ij}^\alpha h_{ik}^{n+1} = \sum_i h_{ik}^\alpha h_{ij}^{n+1}, \tag{32}$$

i.e.,

$$H_\alpha H_{n+1} = H_{n+1}H_\alpha. \tag{33}$$

If we set $B = H_{n+1} - HI$, then $\mathrm{tr}B = 0$. By means of (33) we get $H_\alpha B = BH_\alpha$ for $\alpha > n+1$.By virtue of Lemma 3

$$|\mathrm{tr}(H_\alpha^2 B)| \leq \frac{n-2}{\sqrt{n(n-1)}}\mathrm{tr}H_\alpha^2\sqrt{\mathrm{tr}B^2}, (\alpha > n+1). \tag{34}$$

Since

$$
\begin{aligned}
\mathrm{tr}(H_\alpha^2 B) &= \mathrm{tr}(H_\alpha^2 H_{n+1}) - H\mathrm{tr}H_\alpha^2, (\alpha > n+1), \\
\mathrm{tr}B^2 &= \mathrm{tr}H_{n+1}^2 - nH^2 = \|\mu\|^2,
\end{aligned} \tag{35}
$$

by (34),(35) we conclude

$$\mathrm{tr}(H_\alpha^2 H_{n+1}) \leq (H + \frac{n-2}{\sqrt{n(n-1)}}\|\mu\|)\mathrm{tr}H_\alpha^2, (\alpha > n+1). \tag{36}$$

From (22),(36) we get

$$\frac{1}{2}\Delta\|\tau\|^2 \geq \sum_{i,j,k,\alpha>n+1}(h_{ijk}^\alpha)^2 + \|\tau\|^2\{nc - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|\mu\|\}. \tag{37}$$

We need the following Lemma 4.

Lemma 4. *Let M^n be an n-dimensional space-like submanifold in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. Suppose that the normalized scalar curvature R is constant and $R \leq c$. Then*

$$\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 \geq \|\mathrm{grad}(nH)\|^2.$$

Proof. According to (13) and $R \leq c, \|h\|^2 \leq n^2H^2$ and

$$nH\nabla_k(nH) = \sum_{i,j,\alpha} h_{ij}^{\alpha} h_{ijk}^{\alpha}.$$

Therefore we get

$$n^2H^2\|\text{grad}(nH)\|^2 = \sum_k (\sum_{i,j,\alpha} h_{ij}^{\alpha} h_{ijk}^{\alpha})^2 \leq \|h\|^2 \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2.$$

Thus the Lemma 4 is true.
Since we have

$$\|\mu\|^2 \leq \|h\|^2 - nH^2, \tag{38}$$

from (29),(31),(37),(38) and Lemma 4 we have

$$\begin{aligned} \square(nH) &\geq (\|\mu\|^2 + \|\tau\|^2)\{nc - nH^2 - \tfrac{n(n-2)}{\sqrt{n(n-1)}} H\|\mu\|\} \\ &\geq (\|h\|^2 - nH^2)\{nc - nH^2 - \tfrac{n(n-2)}{\sqrt{n(n-1)}} H\sqrt{\|h\|^2 - nH^2}\}. \end{aligned} \tag{39}$$

Denote $\bar{R} = c - R$. By (13) we have

$$\|h\|^2 - nH^2 = \frac{n-1}{n}(\|h\|^2 - n\bar{R}). \tag{40}$$

By (39),(40) we have

$$\begin{aligned} \square(nH) \geq\ & \tfrac{n-1}{n}(\|h\|^2 - n\bar{R})\{nc - (n-1)\bar{R} - \tfrac{1}{n}\|h\|^2 \\ & - \tfrac{n-2}{n}\sqrt{(\|h\|^2 + n(n-1)\bar{R})(\|h\|^2 - n\bar{R})}\}. \end{aligned} \tag{41}$$

Since $n \geq 3$, then $\frac{1}{n} \leq \frac{n-2}{n}$. Hence we have

$$\square(nH) \geq \frac{n-1}{n}(\|h\|^2 - n\bar{R})P(\bar{R}, \|h\|^2), \tag{42}$$

where

$$\begin{aligned} P(\bar{R}, \|h\|^2) =\ & nc - (n-1)\bar{R} - \tfrac{n-2}{n}\|h\|^2 \\ & - \tfrac{n-2}{n}\sqrt{(\|h\|^2 + n(n-1)\bar{R})(\|h\|^2 - n\bar{R})}. \end{aligned} \tag{43}$$

(1). If $n\bar{R} \leq \|h\|^2 < \min\{\alpha(n, \bar{R}), \beta(n, \bar{R})\}$,then

$$n\bar{R} \leq \sup\|h\|^2 < \min\{\alpha(n, \bar{R}), \beta(n, \bar{R})\}. \tag{44}$$

It is directly checked that $\sup\|h\|^2 < \alpha(n, \bar{R})$ is equivalent to

$$\begin{aligned} & [nc - (n-1)\bar{R} - \tfrac{n-2}{n}\sup\|h\|^2]^2 \\ > & \tfrac{(n-2)^2}{n^2}[\sup\|h\|^2 + n(n-1)\bar{R}](\sup\|h\|^2 - n\bar{R}). \end{aligned} \tag{45}$$

But it is clear from (44) that (45) is equivalent to

$$\begin{aligned} & nc - (n-1)\bar{R} - \tfrac{n-2}{n}\sup\|h\|^2 \\ > & \tfrac{n-2}{n}\sqrt{[\sup\|h\|^2 + n(n-1)\bar{R}](\sup\|h\|^2 - n\bar{R})}. \end{aligned} \tag{46}$$

Hence we have

$$P(\bar{R}, \sup \|h\|^2) > 0. \tag{47}$$

On the other hand,

$$\begin{aligned} \Box(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} = \sum_i (nH - h_{ii}^{n+1})(nH)_{ii} \\ &= n\sum_i H(nH)_{ii} - \sum_i \lambda_i (nH)_{ii} \leq (n|H|_{\max} - C)\Delta(nH), \end{aligned} \tag{48}$$

where $|H|_{\max}$ is the maximum of the mean curvature H and C is the minimum of the principal curvatures $\{\lambda_i\}_{i=1}^n$ of M^n.

Now we consider the following smooth function on M^n defined by $F = -(f^2 + a)^{-1/2}$, where $a(> 0)$ is a real number and f is a non-negative C^2-function on M^n. From the hypothesis of the Theorem 2 and the Gauss equation which implies Ricci curvature $Ric \geq n - 1 - \frac{n^2H^2}{4}$, we know that the Ricci curvature is bounded below. Obviously, F is bounded, so we can apply Lemma 2 to F. For any $\varepsilon > 0$, there is a point $x \in M^n$, such that at which F satisfies the properties (24) in Lemma 2. By a simple and direct calculation, we have

$$F\triangle F = 3\|dF\|^2 - \frac{1}{2}F^4\Delta f^2. \tag{49}$$

From (24),(49)

$$\frac{1}{2}F^4(x)\triangle f^2(x) = 3\|dF\|^2(x) - F(x)\triangle F(x) < 3\varepsilon^2 - \varepsilon F(x). \tag{50}$$

Thus, for any convergent sequence $\{\varepsilon_m\}$ with $\varepsilon_m > 0$ and $\lim_{m\to\infty}\varepsilon_m = 0$, there exists a point sequence $\{x_m\}$ such that the sequence $\{F(x_m)\}$ converges to F_0 (we can take a subsequence if necessary) and satisfies (24), hence, $\lim_{m\to\infty}\varepsilon_m[3\varepsilon_m - F(x_m)] = 0$. From the definition of supremum and (24), we have $\lim_{m\to\infty} F(x_m) = F_0 = \sup F$ and hence the definition of F gives rise to $\lim_{m\to\infty} f(x_m) = f_0 = \sup f$.

Now we set $f = \sqrt{nH}$, so $\lim_{m\to\infty}(nH)(x_m) = \sup(nH)$, thus by (13) $\lim_{m\to\infty}\|h\|^2(x_m) = \sup\|h\|$. Under the hypothesis of the Theorem 2, by (42), (48) and (50) we have

$$\begin{aligned} 0 &\leq \tfrac{1}{2}F^4(x_m)\tfrac{n-1}{n}[\|h\|^2(x_m) - n\bar{R}]P(\bar{R}, \|h\|^2(x_m)) \leq \tfrac{1}{2}F^4(x_m)\Box[nH(x_m)] \\ &\leq (n|H|_{\max} - C)\tfrac{1}{2}F^4(x_m)\Delta(nH)(x_m) \\ &< (n|H|_{\max} - C)(3\varepsilon_m^2 - \varepsilon_m F(x_m)). \end{aligned} \tag{51}$$

Let $m \to \infty$ in (51). Then we have

$$[\sup\|h\|^2 - n\bar{R}]P(\bar{R}, \sup\|h\|^2) = 0. \tag{52}$$

By (47), we have $\sup\|h\|^2 = n\bar{R}$. From (40) and $\sup(\|h\|^2 - nH^2) = 0$ we get $\|h\|^2 = nH^2$, and so M^n is totally umbilical.

(2). If $\|h\|^2 = \min\{\alpha(n,\bar{R}), \beta(n,\bar{R})\}$, then we have

$$\|h\|^2 = \alpha(n,\bar{R}); \quad \text{or} \quad \|h\|^2 = \beta(n,\bar{R}).$$

(i). If $\|h\|^2 = \beta(n,\bar{R})$, then $\|h\|^2 \leq \alpha(n,\bar{R})$. This is equivalent to

$$(53)\quad [nc-(n-1)\bar{R}-\frac{n-2}{n}\|h\|^2]^2 \geq \frac{(n-2)^2}{n^2}[\|h\|^2+n(n-1)\bar{R}](\|h\|^2-n\bar{R}).$$

Hence, we have

$$0 \geq \frac{(n-2)^2}{n^2}[\|h\|^2+n(n-1)\bar{R}](\|h\|^2-n\bar{R}) \geq 0,$$

which means $\|h\|^2 = n\bar{R}$. By (40) $\|h\|^2 = nH^2$, i.e., M is totally umbilical.

(ii). If $\|h\|^2 = \alpha(n,\bar{R})$, then the equality in (53) holds. Since $\|h\|^2 \leq \beta(n,\bar{R})$, we have

$$nc-(n-1)\bar{R}-\frac{n-2}{n}\|h\|^2 = \frac{n-2}{n}\sqrt{[\|h\|^2+n(n-1)\bar{R}](\|h\|^2-n\bar{R})},$$

i.e., $P(\bar{R},\|h\|^2)=0$. Since $\|h\|^2=\alpha(n,\bar{R}) = const.$, from (13) we have $H = const.$. Therefore we know that $\Delta(nH)=0$. By (48) we have $\square(nH)\leq 0$. From (42) we get $\square(nH)=0$. Thus the equalities in (42),(41),(39),(38) and (23) in Lemma 1 hold. When the equalities in (42),(41) hold, we have $-\frac{1}{n}\|h\|^2 = \frac{n-2}{n}\|h\|^2$, i.e., $n=3$. When the equality in (38) holds, we have $\|\mu\|^2 = \|h\|^2 - nH^2$. Hence by (19), we have $\|\tau\|=0$. Since e_{n+1} is parallel on the normal bundle $T^{\perp}(M^n)$ of M^n, using the method of Yau [14], we know M^3 lies in a totally geodesic submanifold $S_1^4(c)$ of $S_p^{3+p}(c)$. When the equalities in (23) of Lemma1 hold, after renumberation if necessary, we can assume that $\lambda=\lambda_1\neq\lambda_2=\lambda_3=\mu$, i.e., M^3 has two distinct principal curvatures, one with the multiplicity 1 and the other with the multiplicity 2. Therefore by [9] or [1], M^3 is a hyperbolic cylinder $H^1(c-\lambda^2)\times S^2(c-\mu^2)$ in $S_1^4(c)$. This completes the proof of the Theorem 2.

Acknowledgements. The author is very grateful to the referee for a careful reading and very helpful suggestions on the earlier version of the manuscript.

This work is partially supported by the National Natural Science Foundation of China and Natural Science Foundation of Shaanxi province.

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