A convergence criterion for monotone global dynamical systems

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Abstract

Generalizing the notion of monotone dynamical system presented in [6], the new concept of monotone global dynamical system is defined in [5]. In this note is proved that the convergence criterion ([6]) for a monotone dynamical system also works for the monotone global dynamical systems in which the partial order relation on the vector bundle is fiberwise defined.

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1 Preliminaries

In [6], H. L. Smith has considered some particular topics concerning a structure (X, K, ϕ), where

- X is a Banach space;
- K is a convex, pointed, closed and with non-empty interior cone, which determines a partial order relation on X;
- ϕ is a semi-flow on X, i.e. a continuous map $\phi : X \times \mathbb{R}^+ \to X$ ($\mathbb{R}^+ = [0, \infty) \subset \mathbb{R}$); if $\phi(x, t)$ is denoted by $\phi_t(x)$, then for $(\forall)\ x \in X,\ (\forall)\ s, t \in \mathbb{R}^+$ the following relations hold true:

$$\phi_0 = id_X \quad , \quad \phi_{s+t} = \phi_s \circ \phi_t = \phi_t \circ \phi_s; \tag{1.1}$$

We say that ϕ is a *monotone* (and order preserving) semi-flow on X if for $(\forall)\ t \in \mathbb{R}^+$, $(\forall)\ x, y \in X$ we have

$$x \le y \Longrightarrow \phi_t(x) \le \phi_t(y)\ . \tag{1.2}$$

The structure (X, K, ϕ), named by Smith ([6]) a monotone dynamical system, was generalized by D.I.Papuc in [5] as a structure $((E, p, M); K, \phi)$, named a *monotone* (and

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order preserving) *global dynamical system*, where (E,p,M) is a regular vector bundle, i.e. M is a real, finite-dimensional, connected, paracompact, without boundary topological manifold and the type-fibre of (E,p,M) is $\mathbb{R}^m$. The vector bundle (E,p,M) endowed with a cone field K (i.e. a map $K : x \in M \mapsto K(x) \subset p^{-1}(x) = E_x \subset E$, where $K(x)$ is a convex, pointed, closed, with non-emtpy interior cone and the sets $\bigcup_{x\in M} \mathrm{int}K(x)$, $\bigcup_{x\in M}(E_x \setminus K(x))$ are open subsets of E) was studied in many notes (e.g. [3], [4]).

A partial order relation on $(E_x, K(x))$ is determined by the following relation:

$$X_x, Y_x \in E_x \setminus X_x \le Y_x \overset{def}{\Leftrightarrow} Y_x - X_x \in K(x);$$

then the pair $(E_x, \le)$ is a partially ordered topological vector space ([5]).

Relative to the structure $((E,p,M),K)$ we have a partial order relation on E, fiberwise induced:

$$X_x \le Y_y \Leftrightarrow x = y \qquad \text{and} \qquad Y_y - X_x \in K(x).$$

In a monotone global dynamical system $((E,p,M);K,\phi)$, ϕ is a semi-flow on E for which the following two supplementary conditions (3) and (4) hold:

$$(\forall)\ t \in \mathbb{R}^+;\ (\forall)\ x \in M \mid \phi_t(E_x) \subset E_{f_t(x)}, \tag{1.3}$$

where $f_t : x \in M \mapsto f_t(x) \in M$ $(\forall\ t \in \mathbb{R}^+)$ is the continuous map uniquely determined by the relation $p \circ \phi_t = f_t \circ p$. The map $f : (x,t) \in M \times \mathbb{R}^+ \mapsto f(x,t) = f_t(x) \in M$ is a semi-flow on M, called *the projection* of the semi-flow ϕ; we have:

$$(\forall)\ t \in \mathbb{R}^+;\ (\forall)\ X_x, Y_x \in E \mid X_x \le Y_x \Longrightarrow \phi_t(X_x) \le \phi_t(Y_x). \tag{1.4}$$

In the following, we introduce ([5]) some concepts concerning an arbitrary monotone global dynamical system:

- *nearly invariant* and *invariant sets*: a subset $B \subset E$ is *nearly invariant* if $\phi_t B \subset B$ for all $t \ge 0$ and it is *invariant* if $\phi_t B = B$ for all $t \ge 0$;

- *orbits*: the orbit of the vector $X_x \in E$, denoted by $\mathcal{O}(X_x)$, is defined as

$$\mathcal{O}(X_x) = \{\phi_t(X_x) : t \ge 0\};$$

  We note that any orbit is a neary invariant set.

- *periodic orbits*: $\mathcal{O}(X_x)$ is a *T-periodic orbit* if for some $T > 0$, we have $\phi_T(X_x) = X_x$. In this case $\phi_{t+T}(X_x) = \phi_t(X_x)$ for all $t \ge 0$ and hence

$$\mathcal{O}(X_x) = \{\phi_t(X_x) : 0 \le t \le T\};$$

- *equilibrium* (invariant) *vectors*: the vector X_x is said to be an *equilibrium vector* if $\mathcal{O}(X_x) = \{X_x\}$. We further denote by $\mathcal{E}$ the set of all equilibrium points for ϕ.

- the *convergent limit set* of a vector X_x, denoted by $\omega(X_x)$ is

$$\omega(X_x) \stackrel{\text{def}}{=} \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \phi_s(X_x)} \quad .$$

  When Φ_t are homeomorphisms for all $t \geq 0$ this set is closed and nearly invariant. If it is compact then it is conected ([1]).

- an *equilibrium vector* X_x is a vector for which the omega limit set $\omega(X_x)$ is an invariant set.

- a *convergent vector* X_x is a vector for which $\omega(X_x) = \{Y_y\}$ and Y_y is an invariant vector.

2 The convergence criterion for a monotone global dynamical system

We shall consider an arbitrary monotone global dynamical system $((E, p, M); K, \phi)$.

Lemma. If for a vector $X_x \in E_x$ the following conditions are satisfied:
1) $\overline{\mathcal{O}(X_x)}$ is a compact set;
2) there is a real number $T > 0$ such that $f_T(p(X_x)) = x$ and $\phi_T(X_x) \geq X_x$, then $\omega(X_x)$ is a T-periodic orbit.

Proof. The monotonicity of ϕ implies $\phi_{(n+1)T}(X_x) \geq \phi_{nT}(X_x)$, $n \in \mathbb{N}$ and, since $\overline{\mathcal{O}(X_x)}$ is compact, it follows that

$$\lim_{n \to \infty} \phi_{nT}(X_x) = \xi_y \quad .$$

Taking account of the continuity of ϕ, we have, for $(\forall)\ t > 0$, that:

$$\begin{aligned} \phi_{t+T}(\xi_y) &= \phi_{t+T}(\lim_{n \to \infty} \phi_{nT}(X_x)) = \\ &= \lim_{n \to \infty} \phi_{(n+1)T+t}(X_x) = \\ &= \lim_{n \to \infty} (\phi_t(\phi_{(n+1)T}(X_x)) = \\ &= \phi_t(\xi_y) \end{aligned}$$

It follows that $\mathcal{O}(\xi_y)$ is a T-periodic orbit. If $t_j \to \infty$ and $\phi_{t_j}(X_x) \longrightarrow \xi_q$, $j \to \infty$, we write $t_j = n_j T + r_j$ with $n_j \in \mathbb{N}$ and $0 \leq r_j < T$.

We can assume that $r_j \to r$, for $j \to \infty$ (passing to a subsequence if necessary). Since $n_j \to \infty$ as $j \to \infty$, we have

$$\phi_{t_j}(X_x) = \phi_{r_j}(\phi_{n_j T}(X_x)) \longrightarrow \phi_r(\xi_y) = \xi_q,$$

with $0 \leq r \leq T$. Therefore we conclude that $\omega(X_x) = \mathcal{O}(\xi_y)$. □

Theorem. (convergence criterion). Given a vector $X_x \in E_x$, and assuming that

1) $\overline{\mathcal{O}(X_x)}$ is a compact set;

2) $f_t(p(X_x)) = x$ and $\phi_t(X_x) \geq X_x$ for $t \in (a,b) \subset (0,\infty)$, $(a,b) \neq \emptyset$, then X_x is a convergent vector and $\xi_y = \lim_{t\to\infty} \phi_t(X_x)$ is an invariant vector.

Proof. Let $T > 0$ and $0 < \varepsilon < T$ such that $(T-\varepsilon, T+\varepsilon) \subset (a,b)$. By the previous Lemma we have that $\omega(X_x) = \mathcal{O}(\xi_y)$, where

$$\xi_y = \lim_{n\to\infty} \phi_{nT}(X_x)$$

and $\mathcal{O}(\xi_y)$ is a T-periodic orbit. Applying the same assertions for $\tau \in (T-\varepsilon, T+\varepsilon)$ and replacing T, we find that $\omega(X_x)$ is a τ-periodic orbit. But $\omega(X_x) = \mathcal{O}(\xi_y)$, and hence

$$\phi_{t+\tau}(\xi_y) = \phi_t(\xi_y), \text{ for all } t \geq 0.$$

It follows that $\phi_t(\xi_y)$ is τ-periodic for any $\tau \in (T-\varepsilon, T+\varepsilon)$.

Let G be the set of all periods of $\phi_t(\xi_y)$. Then G is closed with respect to addition and contains the interval $(T-\varepsilon, T+\varepsilon)$. If $0 \leq s < \varepsilon$ and $t \geq 0$ then

$$\phi_{t+s}(\xi_y) = \phi_t(\phi_s(\xi_y)) = \phi_t(\phi_{s+T}(\xi_y)) = \phi_t(\xi_y).$$

From this results that $[0,\varepsilon) \subset G$ and thus $G = \mathbb{R}^+$ and $\xi_y \in \mathcal{E}$. □

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