Compact manifolds of nonnegative isotropic curvature and pure curvature tensor

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Abstract

We show the vanishing of the Betti numbers $\beta_i(M)$, $2 \leq i \leq n-2$, of compact irreducible manifolds of nonnegative isotropic curvature and pure curvature tensor. We also study manifolds of constant isotropic curvature and show that this condition for dimensions $n \geq 5$ is equivalent to constant sectional curvature.

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1 Introduction

The concept of *isotropic curvature* for manifolds of dimension $n \geq 4$ was introduced by Micallef and Moore in [13]. In that paper, they proved that if M has positive isotropic curvature then the homotopy groups $\pi_i(M)$ vanish for $2 \leq i \leq [n/2]$. Therefore, if $\pi_1(M)$ is finite, the Betti numbers $\beta_i(M)$ are zero for $2 \leq i \leq n-2$. The question whether this conclusion remains true for manifolds with infinite fundamental group remains open, especially because the fundamental group of compact manifolds of positive isotropic curvature can be very large. We remark that a great contribution to the understanding of the fundamental group of a compact manifold of positive isotropic curvature was recently made by Frasier in [4]. She proves that if the fundamental group of compact manifolds of positive isotropic curvature and of dimension $n \geq 5$ does not contain a subgroup isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

We cannot expect the vanishing of the Betti numbers $\beta_i(M)$, $2 \leq i \leq n-2$, in the case of nonnegative isotropic curvature, since all symmetric spaces have nonnegative curvature operator and therefore, nonnegative isotropic curvature. However, under certain conditions, we have such a conclusion, namely, for hypersurfaces of Euclidean space, for conformally flat manifolds ([12]), and for manifolds of nonnegative Weitzenböck operator and whose Weyl tensor satisfies $\mathcal{W} \leq S/[(n-1)(n-2)]$, where S denotes the scalar curvature (see [6]). In addition, Micallef and Wang proved in [14] that the second Betti number of even dimensional compact manifolds nonnegative isotropic curvature is at most 1.

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Definition 1.1 *A Riemannian manifold is said to have* pure curvature tensor *if for every $x \in M$ there is an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space such that the 2-forms $e_i \wedge e_j$ are eigenvectors of the curvature operator $\mathcal{R}$.*

We recall that conformally flat manifolds and hypersurfaces of Euclidean space have pure curvature tensor. The main result of this paper is the following theorem, which generalizes the results in [12] .

Theorem 1.2 *Let $M^n, n \geq 4$ be an irreducible compact manifold of nonnegative isotropic curvature. If the curvatute tensor of M is pure, then the Betti numbers $\beta_i(M)$ are zero, for $2 \leq i \leq n-2$.*

We point out that submanifolds of Space Forms with flat normal bundle also have pure curvature tensor and thus Theorem 1.2 can be applied to study such compact submanifolds in the presence of nonnegative isotropic curvature.

The key points in the proof of Theorem 1.2 are Lemma 2.2 in Section 2 and the study of the holonomy algebra of locally irreducible compact manifolds of nonnegative isotropic curvature and pure of curvature tensor in Section 3. For the latter, we show that the holonomy algebra is the whole orthogonal algebra (see Proposition 3.1).

Since manifolds of constant sectional curvatures have pure curvature tensor, we investigate next if constant isotropic curvature implies the purity of the tensor. Surprisingly, we obtain that for dimensions greater than 4, constant isotropic curvature implies constant sectional curvature.

Theorem 1.3 *Let $M^n, n \geq 4$, be a Riemannian manifold. Then:*
(1) If $n = 4$, the isotropic curvature is constant if and only if M^n is conformally flat with constant scalar curvature.
(2) If $n \geq 5$, M^n has constant isotropic curvature if and only if M^n has constant sectional curvature.

This paper is organized as follows. In Section 2 we prove the main lemma, in Section 3 we study the holonomy algebra, and Theorem 1.2 is proved in Section 4, after we derive the Weitzenböck for manifolds with pure curvature tensor. Theorem 1.3 is proved in Section 5.

2 The Main Lemma

In what follows M will always denote an n-dimensional Riemannian manifold with $n \geq 4$ and R its curvature tensor. For x in M we consider the complexified tangent space $T_xM \otimes \mathbf{C}$ and we extend the Riemannian metric $\langle \ , \ \rangle$ to a complex bilinear form $(,)$. An element Z in $T_xM \otimes \mathbf{C}$ is said to be *isotropic* if $(Z, Z) = 0$. A two plane $\sigma \subset T_xM \otimes \mathbf{C}$ is *totally isotropic* if $(Z, Z) = 0$ for any $Z \in T_xM \otimes \mathbf{C}$. If σ is a totally isotropic two-plane then there exists a basis $\{Z, W\}$ of σ such that

$$Z = e_i + \sqrt{-1}e_j \quad \text{and} \quad W = e_k + \sqrt{-1}e_m$$

where e_i, e_j, e_k, e_m are orthonormal vectors of T_xM. Conversely, any two such vectors span a totally isotropic two plane. Let $\widetilde{\mathcal{R}}$ denote the complex linear extension of the curvature operator $\mathcal{R} : \Lambda^2(T_xM) \to \Lambda^2(T_xM)$ given by

$$\langle \mathcal{R}(X \wedge Y), (U \wedge V)\rangle = \langle R(X,Y)V, U\rangle.$$

Definition 2.1 *A Riemannian manifold is said to have* nonnegative isotropic curvature *if* $\langle \widetilde{\mathcal{R}}(Z \wedge W), (Z \wedge W)\rangle \geq 0$ *whenever* $\{Z, W\}$ *is a totally isotropic two plane.*

It follows from this definition that for Z and W as above

$$\langle \widetilde{\mathcal{R}}(Z \wedge W), (Z \wedge W)\rangle = K_{ik} + K_{im} + K_{jk} + K_{jm} - 2\langle R(e_i, e_j)e_k, e_m\rangle \geq 0$$

where K_{ik} denotes the sectional curvature of the plane spanned by e_i, e_k and R is the curvature tensor of M (see [13] pg.203). If we consider $\bar{Z} = e_i - \sqrt{-1}e_j$, then $\{\bar{Z}, W\}$ is a totally isotropic two plane. Performing the same computation for $\langle \widetilde{\mathcal{R}}(\bar{Z} \wedge W), (\bar{Z} \wedge W)\rangle$ and adding with $\langle \widetilde{\mathcal{R}}(Z \wedge W), (Z \wedge W)\rangle$ we get that the nonnegativity of the isotropic curvature implies

$$K_{ik} + K_{im} + K_{jk} + K_{jm} \geq 0, \tag{2.1}$$

for all sets of orthonormal vectors e_i, e_j, e_m, e_k in T_xM.

Lemma 2.2 (Main Lemma) *Let M be a manifold with nonnegative isotropic curvature and $\{e_1, \ldots, e_n\}$ an orthonormal basis of T_xM. Then, for $2 \leq p \leq n-2$, we have*

$$\sum_{i=1,\ k=p+1}^{p,n} K_{ik} \geq 0. \tag{2.2}$$

Proof We divide the proof in the following cases:

Case 1: n and p are both even. Using (2.1) we get that

$$K_{1,p+1} + K_{2,p+1} + K_{1,p+2} + K_{2,p+2} + \ldots + K_{p-1,p+1} + K_{p,p+1} + K_{p-1,p+2} + K_{p,p+2} + \ldots$$
$$\ldots + K_{p-1,n-1} + K_{p,n-1} + K_{p-1,n} + K_{p,n} \geq 0.$$

Case 2: n is even, but p is odd. Therefore so is $n-p$.

In this case, we first show that there exists $k \in \{p+1, \ldots, n\}$, for which we find $i, j \in \{1, \ldots, p\}$ such that

$$K_{ik} + K_{jk} \geq 0. \tag{2.3}$$

In fact, suppose that $K_{ik} + K_{jk} < 0$ for all $k \geq p+1$ and for all $i, j \in \{1, \ldots, p\}$. Then for $k, m \in \{p+1, \ldots, n\}$ we would have $K_{ik} + K_{im} + K_{jk} + K_{jm} < 0$, contradicting equation (2.1). Therefore, for simplicity, we suppose that

$$K_{p-1,n} + K_{p,n} \geq 0. \tag{2.4}$$

Now we claim that either

$$K_{p-1,p+1} + K_{p-1,p+2} + \ldots + K_{p-1,n} \geq 0$$

or

$$K_{p,p+1} + K_{p,p+2} + \ldots + K_{p,n} \geq 0.$$

If not,

$$K_{p-1,p+1} + K_{p-1,p+2} + K_{p,p+1} + K_{p,p+2} + \ldots$$
$$\ldots + K_{p-1,n-2} + K_{p,n-2} + K_{p-1,n-1} + K_{p,n-1} +$$
$$+ K_{p-1,n} + K_{p,n} < 0,$$

which contradicts (2.1) and (2.4). Let us then suppose that $K_{p,p+1} + K_{p,p+2} + \ldots + K_{p,n} \geq 0$.

Now we claim that there exists $k > p$ such that $K_{1,k} + K_{2,k} + \ldots + K_{p-1,k} \geq 0$. If not,

$$K_{1,k} + K_{2,k} + \ldots + K_{p-1,k} + K_{1,l} + K_{2,l} + \ldots + K_{p-1,l} < 0,$$

which again contradicts (2.1). Let us then suppose that $K_{1,n} + K_{2,n} + \ldots + K_{p-1,n} \geq 0$

Now, to obtain the desired result, just observe that

$$\sum_{i=1,\ k=p+1}^{p,n} K_{ik} = K_{1,p+1} + K_{2,p+1} + K_{1,p+2} + K_{2,p+2} + \ldots$$
$$\ldots + K_{p-2,p+1} + K_{p-1,p+1} + K_{p-2,p+2} + K_{p-1,p+2} + \ldots$$
$$+ K_{p,p+1} + K_{p,p+2} + \ldots + K_{p,n} + K_{1,n} + K_{2,n} + \ldots + K_{p-1,n} \geq 0.$$

Case 3: n is odd. If p is odd, $n-p$ is even. It is proved in a similar manner. □

3 The holonomy algebra of manifolds with pure curvature tensor

Notice that the definition of pure curvature tensor is equivalent to saying that there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for which $\langle R(e_i, e_j)e_k, e_m\rangle = 0$ whenever the set $\{i, j, k, m\}$ contains more than two elements. We will call this basis an $\mathcal{R}$-*basis*.

The aim of this section is to prove the following result.

Proposition 3.1 *Let M be a compact locally irreducible manifold of nonnegative isotropic curvature. If the curvature tensor is pure then the holonomy algebra h of M is the whole orthogonal algebra.*

Before we prove Proposition 3.1 we recall two well known facts about the orthogonal algebra $o(U)$, where U is a vector space. The reader is refered to [3] for their proofs.

Lemma 3.2 *Let v be a non-zero element of U. Then*

$$vU = \{v \wedge u \mid u \in U\}$$

generates $o(U)$.

Lemma 3.3 *Let $U = V + W$ with $V = W^\perp$ and not both have dimension two. Then $o(V) + o(W)$ is a maximal proper subalgebra of $o(U)$.*

Proof of Proposition 3.1

First, we recall a result in [5], which states that *a locally irreducible locally symmetric space with pure curvature tensor has constant sectional curvature* (see also [15]). Therefore, the result is proved for locally symmetric spaces, since the only possible case is that M has constant positive curvature.

Next we show that if the sectional curvature of M is nonnegative then $h = o(n)$. In fact, since the curvature tensor is pure, the sectional curvatures are eigenvalues of the curvature operator, which in turn, implies that the curvature operator is nonnegative. It follows then from Gallot-Meyer Theorem ([7]) that the only possible cases for the restricted holonomy groups are the orthogonal group $O(n)$ or the unitary group $U(n/2)$. In the latter case, a result of Micallef-Wang in [14], implies that M is a simply connected Kähler manifold, that having nonnegative curvature operator, would be biholomorphic to the Complex Projective Space $\mathbf{CP}^{n/2}$. But this is a contradiction, since it is well known that manifolds of pure curvature tensor have zero Pontrjagin forms (see [2] p. 439 or [9]). Therefore the only possible case for the restricted holonomy group and hence for the holonomy group is $O(n)$, which implies that $h = o(n)$

Now we show that if $h \neq o(n)$ and $n > 4$, then M has nonnegative sectional curvature, which by the above paragraph is a contradiction. For that, let $r(x)$ denote the Lie algebra generated by $Im\mathcal{R} \subset \Lambda_x(M)$, where $Im\mathcal{R}$ denote the image of $\mathcal{R}$. It is well known that $r(x)$ is a subalgebra of h for all $x \in M$ (see [1] for instance).

Let us consider an $\mathcal{R}$-*basis* $\{e_1, \ldots, e_n\}$ and suppose that $K_{12} \neq 0$. We then consider

$$\begin{aligned} V &= \{e_1\} \cup \{e_i \mid K_{1i} \neq 0\} \\ W &= V^{\perp} \end{aligned}$$

and denote $T_xM = U = V + W$. We have that $dimV \geq 2$. Notice that $dimW \geq 1$, otherwise, $e_1 \wedge e_i \in Im\mathcal{R}$, for all i and hence $r(x)$ would contain e_1U. But from Lemma 3.2 we get that e_1U generates $o(n)$, contradicting that $h \neq o(n)$. Moreover, if $dimW = 1$, Lemma 3.2 implies $r(x) = o(n-1)$ and from Lemma 3.3 we obtain that $h = o(1) + o(n-1)$, which contradicts that M is locally irreducible. Thus, henceforth, we assume $dimW \geq 2$. Since we are also supposing that $n > 4$ and $dimV \geq 2$, not both have dimension two.

We claim now that there exists $e_j \in W$ such that $K_{2j} = 0$. If not, we would have that $e_2 \wedge e_j \in Im\mathcal{R}$, for all $e_j \in W$. Since $e_1 \wedge e_2 \in Im\mathcal{R}$ and

$$[e_1 \wedge e_2, e_2 \wedge e_j] = -e_1 \wedge e_j,$$

we would conclude that $e_1 \wedge e_i \in r(x)$, for all $i = 2, \ldots, n$. But e_1U generates $o(n)$, contradicting again that $h \neq o(n)$. Now we show that there exists $e_l \in W$ such that $K_{jl} = 0$. If not, $e_j \wedge e_l \in Im\mathcal{R}$ for all $e_l \in W$ and hence $r(x)$ must contain $o(W)$, the Lie algebra generated by e_jW. Since $r(x)$ also contains $o(V)$, the Lie algebra generated by e_1V, we conclude that by Lemma 3.3 that $h = o(V) + o(W)$. But this contradicts that M is locally irreducible.

Therefore we have obtained that for the totally isotropic two-plane spanned by $Z = e_1 + \sqrt{-1}\, e_j$ and $W = e_2 + \sqrt{-1}\, e_l$ we have

$$K_{12} + K_{1l} + K_{2j} + K_{jl} = K_{12} \geq 0.$$

If $n = 4$ and $h \neq o(4)$, $o(s) \times o(4-s)$, the only possible case is $h = u(2)$. In this case, the results of [14] and [16] combined imply that M is biholomorphic to the Complex Projective Space $\mathbf{CP}^2$, which is again a contradiction. □

4 The Weitzenböck Formula and the Proof of Theorem 1.2

Conditions on the isotropic curvature fit well the classical Bochner technique and this will be the basic tool to prove Theorem 1.2.

Recall that for a p-form ω, the Weitzenböck formula is given by

$$(\Delta\omega, \omega) = \sum_{i=1}^{n} (\nabla_{X_i}\omega, \nabla_{X_i}\omega) + (Q_p\omega, \omega),$$

where

$$(Q_p\omega, \omega) = \int_M \langle Q_p\omega(x), \omega(x)\rangle dM.$$

We want to estimate $(Q_p\omega, \omega)$ in the case that the manifold has pure curvature tensor and nonnegative isotropic curvature. For that we use the formulae obtained by Maillot in [10] (see also [11]).

Let $\omega \in \Lambda^2_x M$ and $X \in T_x M$. Consider the operator $\theta_\omega : \Lambda^p_x M \to \Lambda^p_x M$ given by a linear extension of

$$\theta_\omega(\widetilde{X}_{i_1} \wedge \ldots \wedge \widetilde{X}_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} (i(X_{i_k})\omega) \wedge \widetilde{X}_{i_1} \wedge \ldots \wedge \widehat{\widetilde{X}_{i_k}} \wedge \ldots \wedge \widetilde{X}_{i_p},$$

where $\{X_1, \ldots X_n\}$ denotes an orthonormal basis of $T_x M$ and $i(X)\omega$ is a 1-form given by $(i(X)\omega)Y = \omega(X, Y)$. Maillot proved that θ_ω is an skew-symmetric operator, that is,

$$\langle \theta_\omega(\widetilde{X}_1 \wedge \ldots. \wedge \widetilde{X}_p), \widetilde{Y}_1 \wedge \ldots. \wedge \widetilde{Y}_p\rangle = -\langle \widetilde{X}_1 \wedge \ldots. \wedge \widetilde{X}_p, \theta_\omega(\widetilde{Y}_1 \wedge \ldots. \wedge \widetilde{Y}_p)\rangle. \tag{4.5}$$

He also proved that if $\{\omega_s\}$, $s = 1, \ldots, n!/[2!(n-2)!])$ is an orthonormal basis of eigenvectors of the curvature operator $\mathcal{R}$ with corresponding egenvalues λ_s, then

$$\langle Q_p\, \omega, \tau\rangle = \sum_{s=1}^{n!/[2!(n-2)!]} \lambda_s \langle \theta_{\omega_s}\omega, \theta_{\omega_s}\tau\rangle. \tag{4.6}$$

Lemma 4.1 *Let M be a Riemannian manifold with pure curvature tensor and $\{e_1, \ldots, e_n\}$ an $\mathcal{R}$-basis, with corresponding eigenvalues K_{ij}, the sectional curvature of the plane spanned by e_i, e_j. Then*

$$Q_p(e_{i_1} \wedge \ldots \wedge e_{i_p}) = (\sum_{h=1,k=p+1}^{p,n} K_{i_h i_k}) e_{i_1} \wedge \ldots \wedge e_{i_p}.$$

Proof From (4.5) and (4.6) we get that

$$\langle Q_p\omega, \tau\rangle = \sum_{s<t} K_{st}\langle\theta_{e_s\wedge e_t}\omega, \theta_{e_s\wedge e_t}\tau\rangle = -\sum_{s<t} K_{st}\langle\theta_{e_s\wedge e_t}\circ\theta_{e_s\wedge e_t}\omega, \tau\rangle, \quad \forall\, \tau,$$

implying that

$$Q_p\omega = -\sum_{s<t} K_{st}\theta_{e_s\wedge e_t}\circ\theta_{e_s\wedge e_t}\omega.$$

Therefore, all we need is to find $\theta_{e_s\wedge e_t}\circ\theta_{e_s\wedge e_t}(e_{i_1}\wedge\ldots\wedge e_{i_p})$. For that we consider the following cases:

(i) $\{s,t\}\subset\{i_1,...,i_p\}$ or $\{s,t\}\cap\{i_1,...,i_p\}=\emptyset$.

In this case the definition of θ implies

$$\theta_{e_s\wedge e_t}(e_{i_1}\wedge\ldots\wedge e_{i_p}) = 0.$$

(ii) If $s\in\{i_1,...,i_p\}$ and $t\notin\{i_1,...,i_p\}$, then

$$\theta_{e_s\wedge e_t}\circ\theta_{e_s\wedge e_t}(e_{i_1}\wedge...\wedge e_{i_p}) = -e_{i_1}\wedge...\wedge e_{i_p}.$$

It follows that

$$Q_p(e_{i_1}\wedge...\wedge e_{i_p}) = (\sum_{h=1,k=p+1}^{p,n} K_{i_h i_k})e_{i_1}\wedge...\wedge e_{i_p}.$$

□

Proof of Theorem 1.2

It follows from Lemma 2.2 and 4.1 that the operator Q_p is nonnegative for all $2\le p\le n-2$. This in the Weitzenböck formula implies that a p-form is harmonic if and only if it is parallel. Since the only possibility for the holonomy group G of M is $SO(n)$, if $\beta_p(M)>0$ for $2\le p\le[n/2]$ there would exist a parallel p-form ϕ, which would be left invariant by $SO(n)$. But, by the holonomy principle, the existence of ϕ would give rise to a parallel and hence harmonic p-form on the sphere S^n, which is a contradiction. □

5 Manifolds with constant isotropic curvature

We begin by recalling that in [8], Kulkarni proved that a manifold of dimension $n\ge 4$ is conformally flat if and only if the sectional curvatures satisfy the relation

$$K_{12}+K_{34} = K_{13}+K_{24}, \tag{5.7}$$

for all orthonormal vectors e_1, e_2, e_3, e_4.

Proof of Theorem 1.3

First we will show that constant isotropic curvature implies the following two facts:

(a) For e_i, e_j, e_k, e_l orthonormal vectors in T_xM, we have $\langle\mathcal{R}(e_i\wedge e_j), e_k\wedge e_l\rangle = 0$.

(b) For $n\ge 4$, the sectional curvatures satisfy the Kulkarni equation (5.7), and hence M is conformally flat.

In fact, for (a), we just consider the isotropic vectors

$$Z = e_i + \sqrt{-1}e_j,\ W = e_k + \sqrt{-1}e_l,\ \bar{W} = e_k - \sqrt{-1}e_l.$$

Then since $K_I(Z,W) = K_I(Z,\bar{W})$, where $K_I(Z,W)$ denotes the isotropic curvature of the plane spanned by $\{Z,W\}$, we have

$$K_{ik} + K_{il} + K_{jk} + K_{jl} - 2\langle \mathcal{R}(e_i \wedge e_j), e_k \wedge e_l\rangle =$$

$$K_{ik} + K_{il} + K_{jk} + K_{jl} + 2\langle \mathcal{R}(e_i \wedge e_j), e_k \wedge e_l\rangle,$$

which gives (a).

For (b), we again consider the isotropic vectors $Z = e_i + \sqrt{-1}\, e_k$, $W = e_j + \sqrt{-1}\, e_l$, $Z' = e_i + \sqrt{-1}\, e_j$, and $W' = e_l + \sqrt{-1}\, e_k$. Since $K_I(Z,W) = K_I(Z',W')$, using (a) we obtain

$$K_{ij} + K_{il} + K_{kj} + K_{kl} = K_{il} + K_{ik} + K_{jl} + K_{jk}.$$

This gives

$$K_{ij} + K_{kl} = K_{ik} + K_{jl}, \tag{5.8}$$

which implies (b).

Now suppose $n \geq 5$ and consider the isotropic vectors $Z = e_i + \sqrt{-1}\, e_j$, $W = e_m + \sqrt{-1}\, e_l$, $W' = e_m + \sqrt{-1}\, e_k$. Since we have $K_I(Z,W) = K_I(Z,W')$, we get

$$K_{il} + K_{jl} = K_{ik} + K_{jk},$$

(5.9)

for all orthonormal vectors e_i, e_j, e_k, e_l.

Now, from (5.7) we have

$$K_{il} + K_{jk} = K_{ik} + K_{jl}. \tag{5.10}$$

and subtracting (5.9) from (5.10), we have $K_{jk} = K_{jl}$. This implies

$$Ric(e_k) = K_{kl} + \sum_{j\neq l} K_{kj} = K_{kl} + \sum_{j\neq k} K_{lj} = Ric(e_l).$$

Therefore M is Einstein and since it is also conformally flat, M has constant curvature. The converse is obvious.

Now we consider the case $n = 4$. Recall that in dimension 4, the isotropic curvature is given by the Weitzenböck operator on 2-forms, which in turn is given by

$$Q_2^+ = W^+ + \frac{S}{3} \qquad Q_2^- = W^- + \frac{S}{3}, \tag{5.11}$$

where $\pm$ indicates the selfdual and anti-selfdual components. Since M^4 is conformally flat, $W^+ = W^- = 0$ and since Q_2 has constant eigenvalues, we conclude that S is constant. The converse follows immediately from (5.11). $\square$

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