On a linear family of affine connections

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. The aim of this paper is to study some geometrical objects in the deformation algebra associated to a linear family of affine connections. It is pointed out the parallelism between certain algebraic and geometric properties.

Mathematics Subject Classification: 53B20, 53B21.
Key words: associative deformation algebras, conformal metrics.

1 Preliminaries

Let M be a n - dimensional ($n > 3$) differentiable manifold. One denotes by $\mathcal{F}(M)$ the ring of real valued functions, defined on M and by $\mathcal{T}_s^r(M)$ the $\mathcal{F}(M)$-module of tensor fields of type (r,s) on M. Particularly, one denotes $\mathcal{T}_0^1(M)$, respectively $\mathcal{T}_1^0(M)$, by $\mathcal{X}(\mathcal{M})$, respectively $\Lambda^1(M)$.

The differentiable manifolds, the differentiable mappings, the tensor fields and the linear connections are supposed to be of class C^∞.

Let $A \in \mathcal{T}_2^1(M)$. If one defines the product of two vector fields X and Y by

$$(*) \qquad X \circ Y = A(X,Y),$$

the $\mathcal{F}(M)$-module $\mathcal{X}(M)$ becomes an $\mathcal{F}(M)$-algebra. This algebra is called the algebra associated to A and it is denoted by $\mathcal{U}(M,A)$. If $A = \overline{\nabla} - \nabla$, where ∇ and $\overline{\nabla}$ are two affine connections on M, then $\mathcal{U}(M, \overline{\nabla} - \nabla)$ is called the deformation algebra associated to the pair of connections $(\nabla, \overline{\nabla})$.

Definition 1.1 *An element $X \in \mathcal{U}(M,A)$ is called an almost principal vector field if there exist $f \in \mathcal{F}(M)$ and a 1-form $\omega \in \Lambda^1(M)$ such that*

$$A(Z,X) = fZ + \omega(Z)X, \forall Z \in \mathcal{X}(M).$$

Remark 1.1 i) If $f = 0$, then X becomes a principal vector field;
ii) If $\omega = 0$, then X is an almost special vector field;
iii) If $f = 0$ and $\omega = 0$, then X is a special vector field;
iv) If $A(X,X) = 0$, then X is a 2-nilpotent vector field.

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 99-103.

2 Main result

Let $(M, \overset{\circ}{g})$ be a conex, n - dimensional ($n > 3$) Riemannian manifold. One denotes by $\overset{\circ}{\nabla}$, respectively $\overset{1}{\nabla}$, the Levi-Civita connection associated to $\overset{\circ}{g}$, respectively $\overset{1}{g}= e^{2u}\ \overset{\circ}{g}$, where $u \in \mathcal{F}(M)$. One gets the linear family of connections

$$(**) \qquad \{\overset{\circ}{\nabla} + \lambda(\overset{1}{\nabla} - \overset{\circ}{\nabla}) | \lambda \in \mathbf{R}\}.$$

Theorem 2.1 *Let* $\overset{\lambda}{\nabla}$ *be an affine connection from the linear family* $(**)$*. We denote by* $\overset{\lambda}{R}$*, respectively* $\overset{\circ}{R}$*, the curvature tensor field of the linear connection* $\overset{\lambda}{\nabla}$*, respectively* $\overset{\circ}{\nabla}$ *. Let* T_pM *be the tangent vector space in an arbitrary point* $p \in M$*. The following assertions are equivalent:*

(i) $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$;

(ii) $\overset{\lambda}{R} = \overset{\circ}{R}$, *if* $\overset{\circ}{R}_p : T_pM \times T_pM \times T_pM \mapsto T_pM$ is a surjective mapping, $\forall p \in M$;

(iii) $\overset{\lambda}{\nabla}\overset{\lambda}{R} = \overset{\lambda}{\nabla}\overset{\circ}{R}$, *if* $\overset{\circ}{R}_p : T_pM \times T_pM \times T_pM \mapsto T_pM$ *is a surjective mapping,* $\forall p \in M$*;*

(iv) *The deformation algebra* $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ *is associative;*

(v) $\overset{\lambda}{\nabla}$ *and* $\overset{\circ}{\nabla}$ *have the same geodesics;*

(vi) *All the elements of the algebra* $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ *are almost principal vector fields;*

(vii) *All the elements of the algebra* $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ *are almost special vector fields;*

(viii) *All the elements of the algebra* $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ *are principal vector fields;*

(ix) *All the elements of the algebra* $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ *are special vector fields;*

(x) *All the elements of the algebra* $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ *are 2-nilpotent vector fields.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (i) $\Rightarrow$ (iv), (i) $\Rightarrow$ (v), (i) $\Rightarrow$ (vi), (i) $\Rightarrow$ (vii), (i) $\Rightarrow$ (viii), (i) $\Rightarrow$ (ix), (i) $\Rightarrow$ (x) are obvious.

(iii) $\Rightarrow$ (i) From (iii) one gets

$$(\overset{\lambda}{\nabla}_X \overset{\lambda}{R})(Y, Z, V) = (\overset{\lambda}{\nabla}_X \overset{\circ}{R})(Y, Z, V), \forall X, Y, Z, V \in \mathcal{X}(M).$$

Moreover

$$(\overset{\lambda}{\nabla}_X \overset{\circ}{R})(Y, Z, V) = (\overset{\circ}{\nabla}_X \overset{\circ}{R})(Y, Z, V) + \lambda\{A(X, \overset{\circ}{R}(Y, Z)V) -$$

$$- \overset{\circ}{R}(A(X, Y), Z)V - \overset{\circ}{R}(Y, A(X, Z))V - \overset{\circ}{R}(Y, Z)A(X, V)\},$$

where $A = \overset{1}{\nabla} - \overset{\circ}{\nabla}$. The last two formulae imply

$$(2.1) \qquad \begin{aligned} (\overset{\lambda}{\nabla}_X \overset{\lambda}{R})(Y, Z, V) &= (\overset{\circ}{\nabla}_X \overset{\circ}{R})(Y, Z, V) + \lambda\{A(X, \overset{\circ}{R}(Y, Z)V) - \\ &- \overset{\circ}{R}(A(X, Y), Z)V - \overset{\circ}{R}(Y, A(X, Z))V - \overset{\circ}{R}(Y, Z)A(X, V)\}. \end{aligned}$$

Permuting circular X, Y, Z one gets another two analogous relations

$$(2.1') \qquad (\overset{\lambda}{\nabla}_Y \overset{\lambda}{R})(Z,X,V) = (\overset{\circ}{\nabla}_Y \overset{\circ}{R})(Z,X,V) + \lambda\{A(Y, \overset{\circ}{R}(Z,X)V) - \\ - \overset{\circ}{R}(A(Y,Z),X)V - \overset{\circ}{R}(Z,A(Y,X))V - \overset{\circ}{R}(Z,X)A(Y,V)\},$$

$$(2.1'') \qquad (\overset{\lambda}{\nabla}_Z \overset{\lambda}{R})(X,Y,V) = (\overset{\circ}{\nabla}_Z \overset{\circ}{R})(X,Y,V) + \lambda\{A(Z, \overset{\circ}{R}(X,Y)V) - \\ - \overset{\circ}{R}(A(Z,X),Y)V - \overset{\circ}{R}(X,A(Z,Y))V - \overset{\circ}{R}(X,Y)A(Z,V)\}.$$

The second Bianchi identities, the relations (2.1), (2.1') and (2.1") lead to

$$(2.2) \qquad \lambda\{A(X, \overset{\circ}{R}(Y,Z)V) + A(Y, \overset{\circ}{R}(Z,X)V) + A(Z, \overset{\circ}{R}(X,Y)V) - \\ - \overset{\circ}{R}(Y,Z)A(X,V) - \overset{\circ}{R}(Z,X)A(Y,V) - \overset{\circ}{R}(X,Y)A(Z,V)\} = 0.$$

From (2.2) we obtain $\lambda = 0$, so (i) or

$$(2.2') \qquad A(X, \overset{\circ}{R}(Y,Z)V) + A(Y, \overset{\circ}{R}(Z,X)V) + A(Z, \overset{\circ}{R}(X,Y)V) = \\ = \overset{\circ}{R}(Y,Z)A(X,V) - \overset{\circ}{R}(Z,X)A(Y,V) - \overset{\circ}{R}(X,Y)A(Z,V).$$

Let $\overset{\circ}{g}_{ij}, A^k_{ij}$, respectively $\overset{\circ}{R}{}^i_{jkl}$ be the components of $\overset{\circ}{g}, A$, respectively $\overset{\circ}{R}$, in a local system of coordinates $(x^1, x^2, \ldots, x^n)$. In local coordinates (2.2') becomes

$$(2.2'') \qquad A^s_{il} \overset{\circ}{R}{}^r_{sjk} + A^s_{jl} \overset{\circ}{R}{}^r_{ski} + A^s_{kl} \overset{\circ}{R}{}^r_{sij} = \\ A^r_{js} \overset{\circ}{R}{}^s_{lki} + A^r_{ks} \overset{\circ}{R}{}^s_{lij} + A^r_{is} \overset{\circ}{R}{}^s_{ljk} .$$

From $\overset{1}{g} = e^{2u} \overset{\circ}{g}$ and $A = \overset{1}{\nabla} - \overset{\circ}{\nabla}$ one has

$$(2.3) \qquad A^i_{jk} = \delta^i_j u_k + \delta^i_k u_j - \overset{\circ}{g}_{jk} u^i,$$

where $u_i = \frac{\partial u}{\partial x^i}, u^i = \overset{\circ}{g}{}^{ik} u_k, \overset{\circ}{g}{}^{ik} \overset{\circ}{g}_{ij} = \delta^k_j$. Relations (2.2') and (2.3) imply

$$(\delta^r_i \overset{\circ}{R}{}^s_{ljk} + \delta^r_j \overset{\circ}{R}{}^s_{lki} + \delta^r_k \overset{\circ}{R}{}^s_{lij})u_s + (\overset{\circ}{g}_{il} \overset{\circ}{R}{}^r_{sjk} + \overset{\circ}{g}_{jl} \overset{\circ}{R}{}^r_{ski} + \overset{\circ}{g}_{kl} \overset{\circ}{R}{}^r_{sij})u^s = 0.$$

Considering $r = j$ and summing, one gets

$$(2.4) \qquad (n-2) \overset{\circ}{R}{}^s_{lki} u_s + (\overset{\circ}{R}_{lski} + \overset{\circ}{g}_{il} \overset{\circ}{R}_{sk} - \overset{\circ}{g}_{kl} \overset{\circ}{R}_{is})u^s = 0,$$

where $\overset{\circ}{R}_{ijkl} = \overset{\circ}{g}_{is} \overset{\circ}{R}{}^s_{jkl}, \overset{\circ}{R}_{ij} = \overset{\circ}{R}{}^k_{ikj}$. Multiplying (2.4) by $\overset{\circ}{g}{}^{il}$ and summing, we obtain

$$(2.4') \qquad (n-2)R_{sk}u^s = 0.$$

From (2.4') and (2.4) one has

$$(2.4'') \qquad (n-3) \overset{\circ}{R}{}^s_{lki} u_s = 0.$$

Since $n > 3$, from (2.4") we get

$$(2.5) \qquad \omega(\overset{\circ}{R}(X,Y)Z) = 0, \forall X, Y, Z \in \mathcal{X}(M),$$

where ω is the 1-form having the components $u_1, u_2, \dots, u_n$. $\forall p \in M$, the relation (2.5) implies

$$\omega_p(\overset{\circ}{R}_p\,(X_p, Y_p)Z_p) = 0, \forall X_p, Y_p, Z_p \in T_pM. \tag{2.5'}$$

Since $\overset{\circ}{R}_p$: $T_pM \times T_pM \times T_pM \mapsto T_pM$ is a surjective mapping, $\forall p \in M$, from (2.5') one has $\omega_p(T_pM) = 0, \forall p \in M$, i.e. $\omega_p = 0, \forall p \in M$, so $\omega = 0$. Therefore $u_1 = u_2 = \dots = u_n = 0$ and u =constant. Hence $\overset{1}{\nabla} = \overset{\circ}{\nabla}$.

(iv) $\Rightarrow$ (i) Since the algebra $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ is abelian, then this algebra is associative if and only if

$$\lambda[A(X, A(Y, Z)) - A(Y, A(X, Z))] = 0, \forall X, Y, Z \in \mathcal{X}(M). \tag{2.6}$$

From (2.6) we get $\lambda = 0$, so (i) or

$$A(X, A(Y, Z)) = A(Y, A(X, Z)), \forall X, Y, Z \in \mathcal{X}(M). \tag{2.6'}$$

In local coordinates (2.6') becomes

$$A^i_{sk}A^s_{jl} = A^i_{sl}A^s_{jk}. \tag{2.6''}$$

Taking into account (2.6") and (2.3) one has

$$\delta^i_k u_l u_j - \delta^i_l u_k u_j - g_{il}u^i u_k + g_{jk}u^i u_l + (\delta^i_k g_{jl} - \delta^i_l g_{jk})u_s u^s = 0. \tag{2.6'''}$$

Considering $i = k$ and summing, one gets

$$n u_j u_l + (n-2)g_{jl}u_s u^s = 0. \tag{2.6^{iv}}$$

Multiplying the previous relation by g^{jl}, we have $u_s u^s = 0$ and also $u_j u_l = 0$. Therefore $u_1 = u_2 = \dots = u_n = 0$ and hence $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$.

(v) $\Rightarrow$ (i) The symmetric linear connections $\overset{\lambda}{\nabla}$ and $\overset{\circ}{\nabla}$ have the same geodesics if and only if there exists a 1-form $\overset{\lambda}{\omega} \in \Lambda^1(M)$ such that

$$\overset{\lambda}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\lambda}{\omega}(X)Y + \overset{\lambda}{\omega}(Y)X, \forall X, Y \in \mathcal{X}(M). \tag{2.7}$$

Since $\overset{1}{g} = e^{2u}\,\overset{\circ}{g}$, the deformation tensor $\overset{\lambda}{A} = \overset{\lambda}{\nabla} - \overset{\circ}{\nabla}$ is given by

$$\overset{\circ}{g}(\overset{\lambda}{A}(X, Y), Z) = \lambda\{X(u)\,\overset{\circ}{g}(Y, Z) + Y(u)\,\overset{\circ}{g}(X, Z) - Z(u)\,\overset{\circ}{g}(Y, X)\}. \tag{2.8}$$

The relations (2.7) and (2.8) lead to

$$\overset{\circ}{g}(Y, Z)[\overset{\lambda}{\omega}(X) - \lambda X(u)] + \overset{\circ}{g}(X, Z)[\overset{\lambda}{\omega}(Y) - \lambda Y(u)] - \overset{\circ}{g}(Y, X)Z(u) = 0. \tag{2.9}$$

For $Y = X$, from (2.9) one has

$$2\,\overset{\circ}{g}(X, Z)[\overset{\lambda}{\omega}(X) - \lambda X(u)] = Z(u)\,\overset{\circ}{g}(X, X), \forall X, Z \in \mathcal{X}(M). \tag{2.10}$$

From (2.10) we get

$$(2.10') \qquad 2 \overset{\circ}{g}_p (X_p, Z_p)[\overset{\lambda}{\omega}_p (X_p) - \lambda X_p(u)] = \\ = Z_p(u) \overset{\circ}{g}_p (X_p, X_p), \forall X_p, Z_p \in T_pM \setminus \{0\}.$$

Since $n > 3$, $\forall p \in M$ and $Z_p \in T_pM \setminus \{0\}$ there exists a vector $X_p \in T_pM \setminus \{0\}$ such that $\overset{\circ}{g}_p (X_p, Z_p) = 0$. From (2.10') one has $Z_p(u) = 0$, $\forall p \in M, \forall Z_p \in T_pM \setminus \{0\}$. Therefore $u =$ constant and from (2.8) we get $\overset{\circ}{g} (\overset{\lambda}{A} (X, Y), Z) = 0, \forall X, Y, Z \in \mathcal{X}(M)$. Hence $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$.

vi) $\Rightarrow$ v) All the elements of the deformation algebra $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ are almost special vector fields if and only if there exist two 1-forms ω and η on M such that

$$(2.11) \qquad \overset{\lambda}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \omega(X)Y + \eta(Y)X, \forall X, Y \in \mathcal{X}(M).$$

The linear connections $\overset{\lambda}{\nabla}$ and $\overset{\circ}{\nabla}$ are symmetric, so from (2.11) one has $\omega = \eta$, i.e. (v).

vii) $\Rightarrow$ i), viii) $\Rightarrow$ i), ix) $\Rightarrow$ i), x) $\Rightarrow$ i) (it is used the fact that $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$ is an abelian algebra).

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