Variational problems and crystallographic groups

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. The aim of this article is to consider crystallographic groups acting as gauge groups and associate this actions with some variational problems. The natural action on the euclidian spaces of such groups has been extended to some Sobolev spaces of functions and on such spaces are considered actions associated to Lagrangians with potential which is invariant with respect to a crystallographic group $\mathcal{C}$. A result of $\mathcal{C}$-invariance of the action and of the first variational corresponding problem is obtained. The $\mathcal{C}$- invariance of the minimizing sequences of the action is obtaind as a direct consequence. Extending a result from [4] related to periodic potentials, this article presents a case when from $\mathcal{C}$-invariance of the potential follows the existence of absolute minima (and hence of critical points) for the respective actions.

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The results presented in this article are inspired by an idea of Professor Constantin Udrişte to generalise conditions of periodicity related in many ways to different variational problems an to their associated Hamilton equations (see [2], [9], [7], [8]) and by an idea of Professor Kostake Teleman to generalise conditions of periodicity by invariance with respect to the action of a crystallographic group and the compact sets in $\mathbb{R}^n$ of the form $\Pi_{i=1}^{n}[0, T_i]$ by fundamental domains of such crystallographic groups (see [6]). Some important theorms and definitions from [1] have been also used.

The mathematical description of crystals is very important for many scientifical fields such as geology, geography, chemistry, physics to medicine and computer science and also to the most revollutionary technological fields. The symmetry of crystals is described by the crystallographic groups in case $n = 3$. These groups allow the classification of the crystals from both mathematical and physical point of view. Crystals are considered as subsets of the Euclidian space E^3. We recall

The fundamental theorem of Euclidian geometry. *The isometries of the Euclidian space E^n, $n \in \mathbf{N}^*$, are functions $f : E^n \to E^n$, $f(x) = Ax + b$, for any $x \in E^n$, where $A \in O(n)$ and $b \in E^n$.*

Definition 1. A *(generalised) crystallographic group* is a discrete group of isometries of E^n: $\mathcal{C} = \{f = [x \to Ax + b]; A \in G, b \in \Gamma\}$, where $G \subset O(n)$ is a finite group

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and Γ contains an abelian free group generated by n linearly independent translations of E^n.

Definition 2. A group of transformations of a topological space is called *a discrete group of transformations* when all its orbits are discrete.

We consider on E^n the topology associated with the Euclidian metric.

Example. $G = \{Id_{\mathbb{R}^n}\}$ and Γ is generated by $T_1 e_1, \ldots, T_n e_n$, where $\{e_1, ..., e_n\}$ is the canonical basis of E^n and $T_i > 0$, $i = 1, ..., n$. In this case $\mathbb{R}^n/\mathcal{C} \simeq T^n$, the n-dimensional torus and the fundamental domain is $\Pi_{i=1}^n [0, T_i]$. This corresponds to the periodicity cases.

Let $T \in (0, \infty)$ and $W^2_{1,T}$ be the Hilbert reflexive space of the functions in $L^2_1([0,T], \mathbb{R}^n)$ endowed with the inner product

$$((u,v)) = \int_o^T [< u(t), v(t) > + < \dot{u}(t), \dot{v}(t) >] dt$$

and the corresponding norm $\|\|$.

Definition 3. If $\mathcal{C}$ is a crystallographic group acting on $\mathbb{R}^n$, there is a naturally induced action of $\mathcal{C}$ in $W^2_{1,T}$, defined by $fu = f \circ u$, for any $f \in \mathcal{C}$ and $u \in W^2_{1,T}$.

Theorem 1. *Let $\mathcal{C}$ be a crystallographic group acting on $\mathbb{R}^n$ and $F : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that:*
1) F is measurable with respect to t for every $x \in \mathbb{R}^n$,
2) F is $\mathcal{C}^1$ with respect to $x \in \mathbb{R}^n$ for every $t \in [0,T]$,
3) F is $\mathcal{C}$-invariant with respect to $x \in \mathbb{R}^n$ for every $t \in [0,T]$.
Let $L(t,x,y) = F(t,x) + \frac{1}{2}\|y\|^2$ be the Lagrangian whose potential is F.
Let $\varphi : W^2_{1,T} \longrightarrow \mathbb{R}$ be the associated functional:

$$\varphi(u) = \int_o^T L(t, u(t), \dot{u}(t)) dt. \tag{0.1}$$

Then:
1) The action φ is $\mathcal{C}$-invariant and
2) The Euler-Lagrange associated equation

$$\varphi'(u) = 0 \tag{0.2}$$

is $\mathcal{C}$-invariant: [$u \in W^2_{1,T}$, $\varphi'(u) = 0$ and $f \in \mathcal{C}$] $\Rightarrow \varphi'(fu) = 0$.

Proof. 1) Let $f \in \mathcal{C}$ be defined by $f(x) = Ax + b$, for any $x \in E^n$, where $A \in O(n)$ and $b \in E^n$ and let $u \in W^2_{1,T}$. Then $\dot{\overbrace{fu}} = \dot{\overbrace{Au + b}} = A\dot{u}$ and $\|A\dot{u}(t)\|^2 = \|\dot{u}(t)\|^2$ for every $t \in [0,T]$. It follows that

$$\varphi(fu) = \int_o^T L(t, fu(t), \dot{\overbrace{fu}}(t)) dt = \int_o^T [F(t, fu(t)) + \frac{1}{2}\|A\dot{u}(t)\|^2] dt$$

$$= \int_o^T [F(t, u(t)) + \frac{1}{2}\|\dot{u}(t)\|^2] dt = \varphi(u).$$

2) Let $f \in \mathcal{C}$ and let $u \in W^2_{1,T}$ such that $\varphi'(u) = 0$. Then, for any $v \in W^2_{1,T}$:
$\varphi'(fu)(v) = \lim_{s\to 0}\{\frac{1}{s}[\varphi(fu + sv) - \varphi(fu)]\} = \lim_{s\to 0}\{\frac{1}{s}[\varphi(f(u + sf^{-1}v)) - \varphi(fu)]\} = \lim_{s\to 0}\{\frac{1}{s}[\varphi(u + sf^{-1}v) - \varphi(u)]\} = \varphi'(u)(f^{-1}v) = 0.$ □

Corollary 1. *In conditions of Theorem 1, if $\{u_k\}$ is a minimizing sequence for φ and if $f \in \mathcal{C}$, then $\{fu_k\}$ is a minimizing sequence for φ.*

Proof. If $\lim_{k\to\infty} \varphi(u_k) = inf\varphi$ and $f \in \mathcal{C}$, then $\lim_{k\to\infty} \varphi(fu_k) = \lim_{k\to\infty} \varphi(u_k) = inf\varphi$. □

Corollary 2. *In conditions of Theorem 1, if u is a critical point of φ and if $f \in \mathcal{C}$, then fu is a critical point of φ.*

Proof. It follows from Theorem 1, 2). □

Corollary 3 *In conditions of Theorem 1 if u is an absolute minima of φ and $f \in \mathcal{C}$, then fu is an absolute minima of φ.*

Proof. From $\varphi(u) = \varphi(fu)$ and $\varphi(u) = \inf\varphi$ it follows that $\varphi(fu) = inf\varphi$. □

Theorem 2. *Let F and φ be like in Theorem 1 and assume that there exist $a \in \mathcal{C}^0(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0,T], \mathbb{R}^+)$ such that*
4) $|F(t,x)| \le a(\|x\|)b(t)$ and
5) $|\nabla F(t,x)| \le a(\|x\|)b(t)$, for every $t \in [0,T]$ and $x \in \mathbb{R}^n$.

Then φ has absolute minima.

Proof. We use Theorem 1, Corollary 1 and the ideas from Theorem 1.6 in [1] to show that φ has a bounded minimizing sequence. In our case, the existence of a function $h \in L^1([0,T])$ such that $F(t,x) \ge h(t)$, for every $x \in \mathbb{R}$ and for almost every $t \in [0,T]$ follows from the fact that the fundamental domain of the crystallographic group is compact, the restriction of $F(t,)$ to this domain is surjective and from the regularity of F. □

Corollary. *In conditions of Theorem 2 the Euler-Lagrange equation (0.2) has solutions.*

Proof. Any absolute minima of φ is also a critical point of φ and hence a solution of the Euler-Lagrange equation (0.2). □

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