Contact CR-warped product submanifolds in generalized Sasakian Space Forms

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Abstract. In [4] B. Y. Chen studied warped product CR-submanifolds in Kaehler manifolds. Afterward, I. Hasegawa and I. Mihai [5] obtained a sharp inequality for the squared norm of the second fundamental form for contact CR-warped products in Sasakian space form. Recently Alegre, Blair and Carriago [1] introduced generalized Sasakian space form. The aim of present paper is to study contact CR-warped product submanifolds in generalized Sasakian space form.

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1 Preliminaries

An odd-dimensional Riemannian manifold $(\overline{M}, g)$ is said to be an *almost contact metric manifold* if there exist on $\overline{M}$ a $(1,1)$-tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1, \ \phi^2 X = -X + \eta(X)\xi \quad \text{and} \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector field X, Y on $\overline{M}$.

In particular, in an almost contact metric manifold we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$. Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X,Y) = g(X, \phi Y)$ is called the *fundamental* 2-*form* of $\overline{M}$.

On the other hand, the almost contact metric structure of $\overline{M}$ is said to be normal if $[\phi, \phi](X,Y) = -2d\eta(X,Y)\xi$, for any X, Y, where $[\phi, \phi]$ denotes by the Nijenhuis torsion of ϕ, given by

$$[\phi, \phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

An almost contact metric manifold is called *Sasakian manifold* if

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$$(\overline{\nabla}_X\phi)Y = -g(X,Y)\xi + \eta(Y)X, \quad \overline{\nabla}_X\xi = \phi X \tag{1.1}$$

for any X, Y where $\overline{\nabla}$ denotes the Riemannian connection of g.

In 1985, J. A. Oubina introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold $\overline{M}$ is a *trans-Sasakian manifold* if there exist two smooth functions α and β on $\overline{M}$ such that

$$(\overline{\nabla}_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X), \tag{1.2}$$

for any X, Y on $\overline{M}$ and we say that trans-Sasakian structure is of type (α,β). In particular, from (1.2), it is easy to see that the following equations hold for a trans-Sasakian manifold

$$\overline{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{1.3}$$

$$d\eta = \alpha\Phi. \tag{1.4}$$

In particular, if $\beta = 0$, $\overline{M}$ is said to be an *α-Sasakian manifold*. Sasakian manifolds appear as examples of α-Sasakian manifolds, with $\alpha = 1$. Another important kind of trans-Sasakian manifold is that of *cosymplectic manifolds*, obtained for $\alpha = \beta = 0$. If $\alpha = 0$, $\overline{M}$ is said to be a *β-Kenmotsu manifold*. Kenmotsu manifolds are particular examples with $\beta = 1$.

Recently, Alegre, Blair and Carriazo [1] introduced the notion of a generalized Sasakian space form. Given an almost contact metric manifold $(\overline{M}, \phi, \xi, \eta, g)$ we say that $\overline{M}$ is a *generalized Sasakian space form* denoted by $\overline{M}(f_1, f_2, f_3)$ if there exist three functions f_1, f_2 and f_3 on $\overline{M}$ such that [1].

$$\begin{aligned} \overline{R}(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ &- g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.5}$$

for any vector fields X, Y, Z on $\overline{M}$, where $\overline{R}$ denotes the curvature tensor of $\overline{M}$.

This kind of a manifold appears as a natural generalization of the well known Sasakian space form, which can be obtained as a particular case of generalized Sasakian space forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$. Moreover, we can also find some other examples.

Example 1.1 A *Kenmotsu space form* i.e a Kenmotsu manifold with constant ϕ-sectional curvature c is a generalized Sasakian space form with $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$.

Example 1.2 A *cosymplectic space form* $\overline{M}(c)$ i.e a cosymplectic manifold with constant ϕ-sectional curvature c, is a generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

Example 1.3 An almost contact metric manifold is said to be an *almost $C(\alpha)$-manifold* if its Riemannian curvature tensor satisfies

$$
\begin{aligned}
\overline{R}(X,Y,Z,W) &= \overline{R}(X,Y,\phi Z,\phi W)+\alpha\{g(X,W)g(Y,Z)\\
&- g(X,Z)g(Y,W)+g(X,\phi Z)g(Y,\phi W)\\
&- g(X,\phi W)g(Y,\phi Z)\},
\end{aligned} \tag{1.6}
$$

for any vector fields X,Y,Z,W on $\overline{M}$, where α is a real number. Moreover, if such a manifold has constant ϕ-sectional curvature equal to c, then its curvature tensor is given by

$$
\begin{aligned}
\overline{R}(X,Y)Z &= \frac{c+3\alpha^2}{4}\{g(Y,Z)X-g(X,Z)Y\}\\
&+ \frac{c-\alpha^2}{4}\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\}\\
&+ \frac{c-\alpha^2}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\
&+ g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\},
\end{aligned} \tag{1.7}
$$

and so, it is a generalized Sasakian space form with $f_1=\frac{c+3\alpha^2}{4}$, $f_2=f_3=\frac{c-\alpha^2}{4}$.

Let M be an n-dimensional submanifold immersed in a generalized Sasakian space form $\overline{M}(f_1,f_2,f_3)$. Let $\overline{\nabla}$ and ∇ be the Riemannian connection and the induced Levi-Civita connection of $\overline{M}(f_1,f_2,f_3)$ and M respectively. Then the Gauss and Weingarten formulas are given respectively by

$$
\overline{\nabla}_X Y=\nabla_X Y+h(X,Y),\quad \overline{\nabla}_X N=-A_N X+\nabla^{\perp}_X N, \tag{1.8}
$$

for vector fields X,Y tangent to M and a vector field N normal to M, where h denotes the second fundamental form, $\nabla^{\perp}$ the normal connection and A_N the shape operator in the direction of N. The second fundamental form and the shape operator are related by

$$
g(h(X,Y),N)=g(A_N X,Y). \tag{1.9}
$$

Let R be the Riemannian curvature tensor of M, then the equation of Gauss is given by [5]

$$
\begin{aligned}
\overline{R}(X,Y,Z,W) &= R(X,Y,Z,W)+g(h(X,W),h(Y,Z))\\
&- g(h(X,Z),h(Y,W)),
\end{aligned} \tag{1.10}
$$

for any vectors X,Y,Z and W tangent to M.

Let $p\in M$ and $\{e_1,\dots,e_n,\dots,e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\overline{M}(f_1,f_2,f_3)$ such that $e_1,\dots,e_n$ are tangent to M at p.

We denote by H the mean curvature vector that is

$$
H(p)=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i). \tag{1.11}
$$

We put

(1.12) $$h^r_{ij} = g(h(e_i, e_j), e_r),\ i, j = \{1, \ldots, n\},\ r \in \{n+1, \ldots, n+m\},$$

and

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

Let M a Riemannian manifold of dimension k and a a smooth function on M, we recall

(i) $\bigtriangledown_a$, the gradient of a is defined by

$$\langle \bigtriangledown_a, X\rangle = X(a),$$

for all vector field X on M.

(ii) $\triangle_a$, the Laplacian of a is defined by

$$\triangle_a = \sum_{j=1}^{k} \{(\bigtriangledown_{e_j} e_j)a - e_j e_j(a)\} = -\mathrm{div}\bigtriangledown_a,$$

where $\bigtriangledown$ is the Levi-Civita connection on M and $\{e_1, ..., e_k\}$ is an orthonormal frame on M.

As a consequence, we have

$$||\bigtriangledown_a||^2 = \sum_{j=1}^{k} (e_j(a))^2.$$

There are different classes of submanifold. For submanifolds tangent to the structure vector field ξ. We mention the following three cases

(i) A submanifold M tangent to ξ is called an *invariant submanifold* if ϕ-preserves any tangent space of M, that is $\phi(T_pM) \subset T_pM$, for every $p \in M$.

(ii) A submanifold M tangent to ξ is called *anti-invariant submanifold* if ϕ maps a tangent space of M into the normal space, that is, $\phi(T_pM) \subset T_p^{\perp}M$ for all $p \in M$ where $T_p^{\perp}M$ denotes the normal space at $p \in M$.

(iii) A submanifold M tangent to ξ is called a *contact CR-submanifold* if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is, $T_pM = D_p \oplus D_p^{\perp}$, with $\phi(D_p) \subset D_p$ and $\phi(D_p^{\perp}) \subset T_p^{\perp}M$, for every $p \in M$.

2 Warped product submanifolds

Let (M_1, g_1) and (M_2, g_2) be the Riemannian manifolds and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold $M_1 \times_f M_2 = (M_1 \times M_2, g)$, where $g = g_1 + f^2 g_2$. On a warped product one has [5]

$$\nabla_U V = \nabla_V U = (U \ln f)V, \tag{2.1}$$

for any vector fields U tangent to M_1 and V tangent to M_2.

B. Y. Chen [4] established a sharp relationship between the warping function f of a warped product CR-submanifold $M_1 \times_f M_2$ of a Kaehler manifold $\overline{M}$ and the squared norm of the second fundamental form $||h||^2$. In [5] Hasegawa and Mihai proved a similar inequality for contact CR-warped product submanifold in a Sasakian manifold.

In this section, we investigate warped products $M = M_1 \times_f M_2$ which are contact CR-submanifolds of a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Such submanifolds are tangent to the structure vector field ξ. We distinguish two cases

(a) ξ is tangent to M_1,

(b) ξ is tangent to M_2.

In case (a), one has two subcases :

(1) M_1 is an anti-invariant submanifold and M_2 is an invariant submanifold of $\overline{M}$.

(2) M_1 is an invariant submanifold and M_2 is an anti-invariant submanifold of $\overline{M}$.

We start with the subcase (1):

Theorem 2.1 *Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional generalized Sasakian space form. Then there do not exist warped product submanifolds $M = M_1 \times_f M_2$ such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of $\overline{M}$.*

Proof. Assume $M = M_1 \times_f M_2$ is a warped product submanifold of a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of $\overline{M}$. From equation (2.1) we have

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X, \tag{2.2}$$

for any vector fields Z and X tangent to M_1 and M_2 respectively.

If in particular, we take $Z = \xi$, we get $\xi f = 0$. Using (1.1) and (2.2), we have

$$0 = \overline{\nabla}_X \xi = \nabla_X \xi = (\xi \ln f)X.$$

Thus M_2 cannot exist. □

Now for the subcase(2), we have

Theorem 2.2 *Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional generalized Sasakian space form and $M = M_1 \times_f M_2$ an n-dimensional warped product submanifold such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional totally real submanifold of $\overline{M}(f_1, f_2, f_3)$. Then* *(i) the squared norm of the second fundamental form of M satisfies*

$$||h||^2 \geq 2\beta[||\nabla(\ln f)||^2 - \Delta(\ln f) + 1] + 4\alpha\beta(f_2 + 1), \tag{2.3}$$

where Δ denotes the Laplace operator on M_1.

(ii) the equality sign of (2.3) holds if M_1 is a totally geodesic submanifold of $\overline{M}(f_1, f_2, f_3)$. Hence M_1 is a generalized Sasakian space form of constant ϕ-sectional curvature $(f_1 + 3f_2)$.

Proof. Let $M = M_1 \times_f M_2$ be a contact CR-warped product submanifold in a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ such that $\dim M_1 = 2\alpha + 1$ and $\dim M_2 = \beta$. Let $\{X_0 = \xi, X_1, X_2, \ldots, X_\alpha, X_{\alpha+1} = \phi X_1, \ldots, X_{2\alpha} = \phi X_\alpha, Z_1, \ldots, Z_\beta\}$ be a local orthonormal frame on M such that $X_0, \ldots, X_{2\alpha}$ are tangent to M_1 and $Z_1, \ldots, Z_\beta$ are tangent to M_2. For any unit vector field X tangent to M_1 and Z, W tangent to M_2 respectively, we have

$$\begin{aligned} g(h(\phi X, Z), \phi Z) &= g(\overline{\nabla}_Z \phi X, \phi Z) = g(\phi \overline{\nabla}_Z X, \phi Z) \\ &= g(\overline{\nabla}_Z X, Z) = g(\nabla_Z X, Z) = X \ln f. \end{aligned} \tag{2.4}$$

On the other hand since Z is a vector field tangent to a totally real submanifold M_2, we have

$$h(\xi, Z) = \phi Z. \tag{2.5}$$

We denote by $h_{\phi D^\perp}(X, Z)$ the component of $h(X, Z)$ in $\phi D^\perp$. Therefore from (2.4) and (2.5) we have

$$\begin{aligned} g(h(\phi X, Z), \phi W) &= g(A_{\phi W} Z, \phi X) = g(\overline{\nabla}_Z \phi W, \phi X) \\ &= g(\overline{\nabla}_Z W, X) = (X \ln f) g(Z, W). \end{aligned} \tag{2.6}$$

Putting $X = \phi X, W = \phi W$ in (2.6) we get

$$g(h(X, Z), W) = \phi X(\ln f) g(Z, \phi W) = -\phi X(\ln f) g(\phi Z, W),$$

from which we obtain

$$h(X, Z) = -\phi X(\ln f) \phi Z.$$

Therefore for $X \in TM_1$, $Z \in TM_2$

$$\begin{aligned} ||h(X, Z)||^2 &= (\phi X(\ln f))^2 g(\phi Z, \phi Z) = (\phi X(\ln f))^2 g(Z, Z) \\ &= (\phi X(\ln f))^2. \end{aligned} \tag{2.7}$$

Let ν be the normal subbandle orthogonal to $\phi D^\perp$. Obviously, we have

$$T^\perp M = \phi D^\perp \oplus \nu, \quad \phi\nu = \nu.$$

Let $\{e_i\}_{i=0,\ldots,2\alpha}$ and $\{Z_t\}_{t=1,\ldots,\beta}$ are (local) orthonormal frame on M_1 and M_2 respectively. On M_1, we consider a ϕ-adapted orthonormal frame namely $\{e_i, \phi e_i, \xi\}_{i=1,\ldots,\alpha}$. We evaluate $||h(X,Z)||^2$ for $X \in D$ and $Z \in D^\perp$. We know that

$$h(X,Z) = h_{\phi D^\perp}(X,Z) + h_\nu(X,Z),$$

where $h_{\phi D^\perp}(X,Z) \in \phi D^\perp$ and $h_\nu(X,Z) \in \nu$.

For $X \in TM_1$, $Z \in TM_2$, we have

$$||h(X,Z)||^2 = \sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta}\{||h(e_i,Z_t)||^2 + ||h(\phi e_i, Z_t)||^2\} + \sum_{t=1}^{\beta}||h_{\phi D^\perp}(\xi, Z_t)||^2.$$

Now from (2.7), we have

$$||h_{\phi D^\perp}(e_i,Z_t)||^2 = (\phi e_i(\ln f))^2$$

$$||h_{\phi D^\perp}(\phi e_i,Z_t)||^2 = (\phi^2 e_i(\ln f))^2 = (e_i(\ln f))^2.$$

Since

$$||\triangledown a||^2 = \sum_{i=1}^{2\alpha}(e_i(a))^2.$$

Then we get

$$\begin{aligned} ||\triangledown(\ln f)||^2 &= \sum_{i=1}^{2\alpha}(e_i(\ln f))^2 + \sum_{i=1}^{2\alpha}(\phi e_i(\ln f))^2 \\ &= \sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta}(||h_{\phi D^\perp}(\phi e_i,Z_t)||^2 + ||h_{\phi D^\perp}(e_i,Z_t)||^2). \end{aligned} \tag{2.8}$$

Therefore from (2.5) and (2.8), we have

$$\begin{aligned} \sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta}||h_{\phi D^\perp}(X_i,Z_t)||^2 &= \sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta}(||h_{\phi D^\perp}(e_i,Z_t)||^2 + ||h_{\phi D^\perp}(\phi e_i,Z_t)||^2) \\ &+ \sum_{t=1}^{\beta}||h_{\phi D^\perp}(\xi,Z_t)||^2 \\ &= \sum_{t=1}^{\beta}(||\triangledown(\ln f)||^2 + ||\phi Z_t||^2). \end{aligned}$$

Since $||\phi Z_t||^2 = 1$, thus we get

$$\begin{aligned} \sum_{i=0}^{2\alpha}\sum_{t=0}^{\beta}||h_{\phi D^\perp}(X_i,Z_t)||^2 &= \sum_{t=0}^{\beta}||\triangledown(\ln f)||^2 + \sum_{t=0}^{\beta}||\phi Z_t||^2 \\ &= \beta(||\triangledown(\ln f)||^2 + 1) \end{aligned} \tag{2.9}$$

Next, for any unit vector field X tangent to M_1 and orthogonal to ξ and Z tangent to M_2 orthogonal to ξ, equation (1.5) gives

$$\begin{aligned}
\overline{R}(X,\phi X,Z,\phi Z) &= f_1\{g(\phi X,Z)g(X,\phi Z)-g(X,Z)g(\phi X,\phi Z)\}\\
&+ f_2\{g(X,\phi Z)g(\phi^2 X,\phi Z)-g(\phi X,\phi Z)g(\phi X,\phi Z)\\
&+ 2g(X,\phi^2 X)g(\phi Z,\phi Z)\}+f_3\{\eta(X)\eta(Z)g(\phi X,\phi Z)\\
&- \eta(\phi X)\eta(Z)g(X,\phi Z)+g(X,Z)\eta(\phi X)\eta(\phi Z)\\
&- g(\phi X,Z)\eta(X)\eta(\phi Z)\}\\
&= 2f_2\{g(X,\phi^2 X)g(\phi Z,\phi Z)\}\\
&= -2f_2. \qquad (2.10)
\end{aligned}$$

On the other hand, by Codazzi equation, we have

$$\begin{aligned}
\overline{R}(X,\phi X,Z,\phi Z) &= -g(\triangledown^{\perp}_X h(\phi X,Z)-h(\triangledown_X \phi X,Z)\\
&- h(\phi X,\triangledown_X Z),\phi Z)+g(\triangledown^{\perp}_{\phi X} h(X,Z)-h(\triangledown_{\phi X} X,Z)\\
&- h(X,\triangledown_{\phi X} Z),\phi Z) \qquad (2.11)
\end{aligned}$$

By using equation (2.1) and structure equation of a generalized Sasakian manifold, we get

$$\begin{aligned}
g(\triangledown^{\perp}_X h(\phi X,Z),\phi Z) &= Xg(h(\phi X,Z),\phi Z)-g(h(\phi X,Z),\overline{\triangledown}_X \phi Z)\\
&= Xg(\triangledown_Z X,Z)-g(h(\phi X,Z),\phi\overline{\triangledown}_X Z)\\
&= X(X\ln f)g(Z,Z)-(X\ln f)g(h(\phi X,Z),\phi Z)\\
&- g(h(\phi X,Z),\phi h_\nu(X,Z))\\
&= (X^2\ln f)g(Z,Z)+(X\ln f)^2 g(Z,Z)-||h_\nu(X,Z)||^2,
\end{aligned}$$

where we denote by $h_\nu(X,Z)$ the ν-component of $h(X,Z)$. Also, we have

$$\begin{aligned}
g(h(\triangledown_X \phi X,Z),\phi Z) &= g(\overline{\triangledown}_Z \triangledown_X \phi X,\phi Z)\\
&= g(\overline{\triangledown}_Z \overline{\triangledown}_X \phi X,\phi Z)-g(\overline{\triangledown}_Z h(X,\phi X),\phi Z)\\
&= -g(X,X)g(Z,Z)+((\triangledown_X X)\ln f)g(Z,Z).
\end{aligned}$$

$$g(h(\phi X,\triangledown_X Z),\phi Z)=(X\ln f)g(h(\phi X,Z),\phi Z)=(X\ln f)^2 g(Z,Z).$$

Substituting the above relations in (2.11) we find

$$\begin{aligned}
\overline{R}(X,\phi X,Z,\phi Z) &= 2||h_\nu(X,Z)||^2-(X^2\ln f)g(Z,Z)\\
&+ ((\triangledown_X X)\ln f)g(Z,Z)-2g(X,X)g(Z,Z)\\
&- ((\phi X)^2\ln f)g(Z,Z)\\
&+ ((\triangledown_{\phi X}\phi X)\ln f)g(Z,Z). \qquad (2.12)
\end{aligned}$$

By Summing the equation (2.12) using equation (2.10), we get

$$\sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta}||h_\nu(X,Z)||^2 = 2\alpha\beta(f_2+1) - \beta\Delta(\ln f) \tag{2.13}$$

Combining (2.9) and (2.13), we obtain the inequality (2.3). □

Denote by $h^{''}$ the second fundamental form of M_2 in M, then we get

$$g(h^{''}(Z,W),X) = g(\triangledown_Z W, X) = -X(\ln f)g(Z,W),$$

or equivalently

$$h^{''}(Z,W) = -g(Z,W)\triangledown(\ln f). \tag{2.14}$$

If the equality sign of (2.3) holds identically then we obtain

$$h(D,D) = 0, \quad h(D^\perp, D^\perp) = 0. \tag{2.15}$$

The first condition (2.15) implies that M_1 is totally geodesic in M, on the other hand, one has

$$g(h(X,\phi Y),\phi Z) = g(\overline{\triangledown}_X \phi Y, \phi Z) = g(\triangledown_X Y, Z) = 0.$$

Thus M_1 is totally geodesic in $\overline{M}(f_1,f_2,f_3)$ and hence is a generalized Sasakian space form with constant ϕ-sectional curvature (f_1+3f_2). The second condition (2.15) and (2.14) imply that M_2 is totally umbilical in $\overline{M}(f_1,f_2,f_3)$. Moreover, by (2.15), it follows that M is a minimal submanifold of $\overline{M}(f_1,f_2,f_3)$.

Corollary 2.1 *We have the following table :*

Manifold	$M_1 \times_f M_2, \quad \xi \in T_pM_1$
$\overline{M}(f_1,f_2,f_3)$	$\lvert\lvert h\rvert\rvert^2 \geq 2\beta[\lvert\lvert\triangledown(\ln f)\rvert\rvert^2 - \Delta(\ln f)+1] + 4\alpha\beta(f_2+1)$
$\overline{M}_{Sas}(c)$	$\lvert\lvert h\rvert\rvert^2 \geq 2\beta[\lvert\lvert\triangledown \ln f\rvert\rvert^2 - \Delta(\ln f)+1] + \alpha\beta(c+3)$
$\overline{M}_{cosy}(c)$	$\lvert\lvert h\rvert\rvert^2 \geq 2\beta[\triangledown \ln f\rvert\rvert^2 - \Delta \ln f+1] + \alpha\beta(c+4)$
$\overline{M}_{Ken}(c)$	$\lvert\lvert h\rvert\rvert^2 \geq 2\beta[\lvert\lvert\triangledown \ln f\rvert\rvert^2 - \Delta \ln f+1] + 2\beta(c+5)$
$\overline{M}_{C(\alpha)}(c)$	$\lvert\lvert h\rvert\rvert^2 \geq \beta[\lvert\lvert\triangledown \ln f\rvert\rvert^2 - \Delta \ln f+1] + \alpha\beta(c-\alpha^2+4)$

where $\overline{M}_{Sas}(c)$, $\overline{M}_{cosy}(c)$, $\overline{M}_{Ken}(c)$, $\overline{M}_{C(\alpha)}(c)$ *denote Sasakian space form, cosymplectic space form, Kenmotsu space form and* $C(\alpha)$*-space form respectively.*

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