Invariant submanifolds of Sasakian manifolds

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Abstract. In this paper, the geometry of invariant submanifolds of a Sasakian manifold are studied. Necessary and sufficient conditions are given on an submanifold of a Sasakian manifold to be invariant submanifold and the invariant case is considered. In this case, we investigate further properties of invariant submanifolds of a Sasakian manifold.

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1 Introduction

The geometry of invariant submanifolds of a Riemannian manifold was studied by many geometers (see [1], [4], [5], [6]). In general, the geometry an invariant submanifold inherits almost all properties of the ambient manifold.

In this paper, we give necessary and sufficient conditions for a submanifold of Sasakian manifold to be an invariant submanifold and we consider the invariant case. Also, necessary and sufficient conditions are given for an invariant submanifold of Sasakian manifold to have an almost complex and Sasakian structure.

2 Preliminaries

A $(2m+1)$-dimensional Riemannian manifold (M, g) is said to be a Sasakian manifold if it admits an endomorphism ϕ of its tangent bundle TM, a vector field ξ and a 1-form η, satisfying;

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta o \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

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$$(\nabla_X \phi)Y = -g(X,Y)\xi + \eta(Y)X, \quad \nabla_X \xi = \phi X, \tag{2.3}$$

for any vector fields X, Y on M, where ∇ denotes the Riemannian connection on M.

A plane section π in T_xM of a Sasakian manifold M is called a ϕ-section if it is spanned by a unit vector X orthogonal to ξ and ϕX. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature.

A Sasakian manifold M is called *a Sasakian space form* and is denoted by $M(c)$ if it has constant ϕ-sectional curvature c. The curvature tensor R of a Sasakian space form $M(c)$ is given by

$$\begin{aligned} R(X,Y)Z &= \frac{1}{4}(c+3)\{g(Y,Z)X - g(X,Z)Y\} - \frac{1}{4}(c-1)\{\eta(Y)\eta(Z)X \\ &- \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &- g(\phi Y,Z)\phi X + g(\phi X,Z)\phi Y + 2g(\phi X,Y)\phi Z\}, \end{aligned} \tag{2.4}$$

for any vector fields X, Y, Z on M.

3 Submanifolds of a Sasakian Manifold

Let $\bar{M}$ be an m-dimensional submanifold of a Sasakian manifold M. We denote by $\bar{\nabla}$, h and $\bar{R}$ the Riemannian connection, the second fundamental form and Riemannian curvature tensor of $\bar{M}$, respectively.

Then the Gauss equation is given by

$$\begin{aligned} R(X,Y)Z &= \bar{R}(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y \\ &+ (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \end{aligned} \tag{3.1}$$

for any vector fields X, Y, Z on $\bar{M}$, where A denotes the shape operator of $\bar{M}$.

Let $\bar{M}$ be an $(n+1)$-dimensional immersed submanifold of a Sasakian manifold M, let $i : \bar{M} \longrightarrow M$ be an immersion; we denote by B the differential of i. The induced Riemannian metric $\bar{g}$ of $\bar{M}$ is given by $\bar{g} = i^*g$.

We denote by $T_x\bar{M}$ the tangent space of $\bar{M}$ at $x \in \bar{M}$, $T_x(\bar{M})^\perp$ the normal space of $\bar{M}$ in M, respectively. Furthermore, we denote by $\{N_1, N_2, ..., N_k\}$, $\quad k = 2m - n$, an orthonormal basis of the normal space $T_x(\bar{M})^\perp$. Then the vector field BX can be written in the following way:

$$\phi BX = B\varphi X + \sum_{i=1}^{k} \upsilon_i(X)N_i, \tag{3.2}$$

for any $X \in T_x\bar{M}$, where φ and υ_i are respectively induced $(1,1)$ tensor and 1-forms on $\bar{M}$. Similarly, ϕN_i can be written in the following way:

$$\phi N_i = BU_i + \sum_{j=1}^{k} \lambda_{ij} N_j, \tag{3.3}$$

where, U_i are vector fields on $\bar{M}$ and λ_{ij} are functions on M. Moreover, the vector field ξ can be expressed as follows:

$$\xi = BV + \sum_{i=1}^{k} \alpha_i N_i, \tag{3.4}$$

where V a vector field on $\bar{M}$, α_i are functions on M and the indices i, j, l, s, p run over the range $\{1, 2, ..., k\}$. Thus we have

$$\begin{aligned} \bar{g}(\varphi X, Y) &= g(B\varphi X, BY) = g(\phi BX, BY) = -g(BX, \phi BY) \\ &= -g(BX, B\varphi Y) = -\bar{g}(X, \varphi Y), \end{aligned} \tag{3.5}$$

for any $X, Y \in \Gamma(T\bar{M})$. Moreover, from $g(\phi BX, N_i) = -g(BX, \phi N_i)$ and $g(\phi N_i, N_j) = -g(N_i, \phi N_j)$, we get the equations

$$\upsilon_j(X) = -\bar{g}(X, U_j), \quad \lambda_{is} = -\lambda_{si}.$$

So λ_{is} is skew-symmetric. Moreover,

Lemma 3.1. *Let $\bar{M}$ be an immersed submanifold of a Sasakian manifold M. Then the following equations hold good:*

$$\varphi^2 = -I + \bar{\eta} \otimes V - \sum_{i=1} \upsilon_i \otimes U_i, \tag{3.6}$$

and

$$\upsilon_p(\varphi X) + \sum_{i=1}^{k} \upsilon_i(X)\lambda_{ip} - \alpha_p \bar{\eta}(X) = 0, \tag{3.7}$$

or

$$\varphi U_p + \sum_{i=1}^{k} \lambda_{ip} U_i + \alpha_p V = 0,$$

where $\bar{\eta}$ is an induced 1-form on $\bar{M}$ and $\bar{\eta}(X) = \bar{g}(X, V)$.

Proof. From (2.1), (3.2) and (3.3), we have

$$\begin{aligned}\phi^2 BX &= B\varphi^2 X + \sum_{i=1}^{k} \upsilon_i(\varphi X)N_i + \sum_{l=1}^{k} \upsilon_l(X)\{BU_l + \sum_{j=1}^{k} \lambda_{lj}N_j\} \\ &= B\varphi^2 X + \sum_{i=1}^{k} \upsilon_i(\varphi X)N_i + \sum_{l=1}^{k} \upsilon_l(X)BU_l \\ &+ \sum_{l=1}^{k} \upsilon_l(X) \sum_{j=1}^{k} \lambda_{lj}N_j, \qquad (3.8)\end{aligned}$$

for any $X \in \Gamma(T\bar{M})$. Moreover, from (2.1) and (3.4), we can write

$$\begin{aligned}\phi^2 BX &= -BX + \eta(BX)\xi \\ &= -BX + \eta(BX)BV + \eta(BX)\sum_{t=1}^{k} \alpha_t N_t. \qquad (3.9)\end{aligned}$$

From the equations (3.8) and (3.9), we get

$$B\varphi^2 X + \sum_{l=1} \upsilon_l(X)BU_l = -BX + \eta(BX)BV,$$

that is,

$$\varphi^2 X = -X + \bar{\eta}(X)V - \sum_{l=1} \upsilon_l(X)U_l,$$

or

$$\varphi^2 = -I + \bar{\eta} \otimes V - \sum_{l=1} \upsilon_l \otimes U_l.$$

Furthermore,

$$\sum_{i=1} \upsilon_i(\varphi X)N_i + \sum_{l=1} \upsilon_l(X) \sum_{j=1} \lambda_{lj}N_j = \eta(BX)\sum_{t=1} \alpha_t N_t.$$

Moreover,

$$\eta(BX) = g(BX, \xi) = g(BX, BV) = \bar{g}(X, V) = \bar{\eta}(X).$$

Thus we have

$$\upsilon_p(\varphi X) + \sum_{l=1} \upsilon_l(X)\lambda_{lp} - \bar{\eta}(X)\alpha_p = 0,$$

or,

$$\bar{g}(U_p, \varphi X) - \sum_{l=1} \lambda_{lp}\bar{g}(U_l, X) - \alpha_p \bar{g}(X, V) = 0,$$

which implies that

$$\varphi U_p + \sum_{l=1} \lambda_{lp}U_l + \alpha_p V = 0.$$

□

Lemma 3.2. *Let $\bar{M}$ be an immersed submanifold of a Sasakian manifold M. Then the following equations hold good:*

$$\varphi V + \sum_{i=1} \alpha_i U_i = 0, \quad \upsilon_t(V) + \sum_{i=1}^{k} \alpha_i \lambda_{it} = 0, \tag{3.10}$$

and

$$\upsilon(V) = 1 - \sum_{i=1}^{k} \alpha_i^2. \tag{3.11}$$

Proof. Making use of $\phi\xi = 0$, we have

$$\phi BV + \sum_{i=1}^{k} \alpha_i \phi N_i = 0$$

$$B\varphi V + \sum_{j=1}^{k} \upsilon_j(V) N_j + \sum_{i=1}^{k} \alpha_i \{BU_i + \sum_{s=1}^{k} \lambda_{is} N_s\} = 0$$

$$B\varphi V + \sum_{j=1}^{k} \upsilon_j(V) N_j + \sum_{i=1}^{k} \alpha_i BU_i + \sum_{i=1}^{k} \alpha_i \sum_{s=1}^{k} \lambda_{is} N_s = 0. \tag{3.12}$$

Thus from the tangential and normal components of (3.12), we get

$$\varphi V + \sum_{i=1}^{k} \alpha_i U_i = 0, \quad \upsilon_t(V) + \sum_{i=1}^{k} \alpha_i \lambda_{it} = 0.$$

Also, by means of $\eta(\xi) = 1$, we conclude that

$$\upsilon(V) = 1 - \sum_{i=1}^{k} \alpha_i^2.$$

□

4 Invariant Submanifolds of a Sasakian Manifold

Let $\bar{M}$ be an immersed submanifold of a Sasakian manifold M. If $\phi(B(T_x\bar{M})) \subset T_x\bar{M}$, for any point $x \in \bar{M}$, then $\bar{M}$ is said to be an invariant submanifold of M. In this case, we have

$$\phi BX = B\varphi X, \tag{4.1}$$

$$\phi N_i = \sum_{j=1} \lambda_{ij} N_j, \tag{4.2}$$

$$\xi = BV + \sum_{i=1} \alpha_i N_i. \tag{4.3}$$

Lemma 4.1. *Let $\bar{M}$ be an immersed submanifold of a Sasakian manifold M. If the vector field ξ is normal to $\bar{M}$, then $\bar{M}$ is an anti-invariant submanifold of M, that is, $\phi(B(T_x\bar{M})) \subset T_x(\bar{M})^{\perp}$ for any point $x \in \bar{M}$.*

From Lemma 3.1 and Lemma 3.2, we have

Lemma 4.2. *Let $\bar{M}$ be an invariant submanifold of a Sasakian manifold M. Then the following equations hold good:*

$$\begin{aligned} \varphi^2 &= -I + \bar{\eta} \otimes V, \quad \alpha_i \bar{\eta} = 0, \quad i, p = 1, 2, ..., k. \\ \varphi V &= 0, \ \sum_{i=1}^{k} \alpha_i \lambda_{ip} = 0, \end{aligned} \tag{4.4}$$

Proof. For any $X \in \Gamma(T\bar{M})$, we have

$$\begin{aligned} B\varphi^2 X &= \phi^2 BX \\ &= -BX + \eta(BX)\xi \\ &= -BX + \eta(BX)BV + \eta(BX) \sum_{i=1}^{k} \alpha_i N_i. \end{aligned}$$

Thus we get

$$\varphi^2 X = -X + \bar{\eta}(X)V, \quad \sum_{i=1}^{k} \bar{\eta}(X) \alpha_i N_i = 0, \tag{4.5}$$

or

$$\varphi^2 = -I + \bar{\eta} \otimes V, \quad \alpha_p \bar{\eta} = 0.$$

Furthermore, from $\phi\xi = 0$, we get

$$B\varphi V + \sum_{i=1}^{k} \alpha_i \sum_{j=1}^{k} \lambda_{ij} N_j = 0,$$

which is equivalent to

$$\varphi V = 0, \ \sum_{i=1}^{k} \alpha_i \lambda_{ip} = 0.$$

□

Thus we have the following Theorems.

Theorem 4.3. *Let $\bar{M}$ be an invariant submanifold of a Sasakian manifold M. If ξ is tangent to $\bar{M}$, then the induced structure $(\varphi, V, \bar{\eta}, \upsilon, \bar{g})$ on $\bar{M}$ is a Sasakian structure.*

Proof. $V \neq 0$, that is, $\alpha_i = 0$, (or ξ is tangent to $\bar{M}$), then we have

$$\bar{\eta}(X) = \bar{g}(X,V), \bar{g}(\varphi X, \varphi Y) = \bar{g}(X,Y) - \bar{\eta}(X)\bar{\eta}(Y),$$
$$\bar{\eta}(V) = \bar{g}(V,V) = 1, \varphi V = 0, \bar{\eta}(\varphi X) = \bar{g}(V, \varphi X) = -\bar{g}(\varphi V, X) = 0.$$

Moreover, from (2.3), we conclude that

$$(\bar{\nabla}_X \varphi)Y = -\bar{g}(X,Y)V + \bar{\eta}(Y)X, \quad \bar{\nabla}_X V = \varphi X.$$

Thus $\bar{M}$ is a Sasakian manifold with Sasakian structure tensors $(\varphi, V, \bar{\eta}, \bar{g})$.
□

Theorem 4.4. *Let $\bar{M}$ be an immersed submanifold of a Sasakian manifold M. Then $\bar{M}$ is an invariant submanifold of the Sasakian manifold M, if and only if that the induced structure $(\varphi, V, \bar{\eta}, \bar{g})$ on $\bar{M}$ is a Sasakian structure.*

Proof. From Theorem 4.2 the necessity is obvious. Conversely, we assume that the induced structure $(\varphi, V, \bar{\eta}, \bar{g})$ is a Sasakian structure. Then we have $\upsilon_i(X)U_i = 0$, that is, $\upsilon_i(X) = 0$ $i = 1, 2, ..., k$ and from (3.7) we get $\alpha_i = 0$. Thus we get that $\bar{M}$ is an invariant submanifold of M and ξ is tangent to $\bar{M}$.
□

Theorem 4.5. *Let $\bar{M}$ be an invariant submanifold with Sasakian structure $(\varphi, V, \bar{\eta}, \upsilon, \bar{g})$ of Sasakian space form $M(c)$. Then $\bar{M}$ is a curvature-invariant submanifold.*

Proof. By using the equations (2.4) and (3.1), we find

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{1}{4}(c+3)\{g(Y,Z)X - g(X,Z)Y\} - \frac{1}{4}(c-1)\{\eta(Y)\eta(Z)X \\ &- \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi - g(\phi Y, Z)\phi X \\ &+ g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y \end{aligned}$$

and

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0,$$

for all $X, Y, Z \in \Gamma(T\bar{M})$, which proves our assertion.
□

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