# $Gl_n(R)$-invariant variational principles on frame bundles

J. Brajerčík

**Abstract.** Variational principles on frame bundles, given by the first and the second order Lagrangians invariant with respect to the structure group, are considered. Noether's currents, associated with the corresponding Lepage equivalents, are obtained. It is shown that for the first and the second order invariant variational problems, the system of the Euler-Lagrange equations for a frame field are equivalent with the lower order system of equations.

**M.S.C. 2000**: 49Q99, 49S05, 58A20, 58E30, 58J99.
**Key words**: Frame, variational principle, Lagrangian, Euler-Lagrange equations, invariant transformation, Noether's current.

## 1 Introduction

Let $FX$ be the frame bundle over an $n$-dimensional manifold $X$, and let $J^rFX$ be the $r$-jet prolongation of $FX$. We shall consider $J^rFX$ with the canonical prolongation of the right action of the general linear group $Gl_n(R)$ on $FX$. For foundations of the variational theory in fibered space we refer to [5], [7], [8], [10], [14], and the notions related to the frame bundles and invariance can be found in [6], [9], [11], [12], [15]. In this paper we study the consequences of $Gl_n(R)$-invariance for variational problems on $J^1FX$ and $J^2FX$. In particular, we discuss the corresponding Noether's currents. The generators of invariant transformations are the fundamental vector fields of the $Gl_n(R)$-action. Then the Noether's theorem gives us a conservation law for each one of $n^2$ linearly independent fundamental vector fields. Our main object is to study how the Noether's currents can be used to simplify the Euler-Lagrange equations for a frame field. We show that in case of first order invariant Lagrangian, the system of $n^2$ second order Euler-Lagrange equations is equivalent with the system of the same number of first order equations. Analogously, for the second order Lagrangian, the system of fourth order Euler-Lagrange equations is equivalent to the system of third order equations coming from the corresponding Noether's currents.

For variational problems on principal fiber bundles there are several different concepts of invariance. Castrillón, García, Ratiu and Shkoller [3], [4] consider invariance of

Balkan Journal of Geometry and Its Applications, Vol.13, No.1, 2008, pp. 11-19.

the first order Lagrangians on principal fiber bundle $P$, which determine constrained variational problems on the bundle $C(P)$ of connections of $P$. Muñoz and Rosado [13] study first order variational problems, invariant under diffeomorphisms of the base manifold (first order *natural* variational problems in the sense of Krupka [7]).

# 2 Invariant Lagrange structures

In this section we recall basic notions of the theory of invariant Lagrangians, and introduce our notation. For a more complete discussion we refer to [2].

If $Y$ is a fibered manifold over an $n$-dimensional manifold $X$, of dimension $n+m$, we denote by $J^rY$ the *r-jet prolongation* of $Y$, and $\pi^{r,s} : J^rY \to J^sY$, $\pi^r : J^rY \to X$ are the *canonical jet projections*. The *r-jet* of a section $\gamma$ of $Y$ at a point $x \in X$, is denoted $J^r_x\gamma$; and $x \to J^r\gamma(x) = J^r_x\gamma$ is the *r-jet prolongation* of $\gamma$. Any *fibered chart* $(V, \psi)$, $\psi = (x^i, y^\sigma)$, on $Y$, where $1 \le i \le n$, $1 \le \sigma \le m$, induces the *associated charts* on $X$ and on $J^rY$, $(U, \varphi)$, $\varphi = (x^i)$, and $(V^r, \psi^r)$, $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_r})$, respectively; here $V^r = (\pi^{r,0})^{-1}(V)$, and $U = \pi(V)$. Recall that the *formal derivative operator* is defined by

$$d_i = \frac{\partial}{\partial x^i} + y^\sigma_i \frac{\partial}{\partial y^\sigma} + y^\sigma_{i_1 i} \frac{\partial}{\partial y^\sigma_{i_1}} + \dots + y^\sigma_{i_1 i_2 \dots i_r i} \frac{\partial}{\partial y^\sigma_{i_1 i_2 \dots i_r}}.$$

For any open set $W \subset Y$, $\Omega^r_0 W$ denotes the ring of smooth functions on $W^r$. The $\Omega^r_0 W$-module of differential $q$-forms on $W^r$ is denoted by $\Omega^r_q W$, and the exterior algebra of forms on $W^r$ is denoted by $\Omega^r W$. The module of $\pi^{r,0}$-horizontal ($\pi^r$-horizontal) $q$-forms is denoted by $\Omega^r_{q,Y} W$ ($\Omega^r_{q,X} W$, respectively).

The *horizontalization* is the exterior algebra morphism $h : \Omega^r W \to \Omega^{r+1} W$, defined, in any fibered chart $(V, \psi)$, $\psi = (x^i, y^\sigma)$, by

$$hf = f \circ \pi^{r+1,r}, \quad hdx^i = dx^i, \quad hdy^\sigma_{j_1 j_2 \dots j_p} = y^\sigma_{j_1 j_2 \dots j_p k} dx^k,$$

where $f : W^r \to R$ is a function, and $0 \le p \le r$. A form $\eta \in \Omega^r_k W$ is *contact*, if $h\eta = 0$. For any fibered chart $(V, \psi)$, $\psi = (x^i, y^\sigma)$, the 1-forms

$$\omega^\sigma_{j_1 j_2 \dots j_p} = dy^\sigma_{j_1 j_2 \dots j_p} - y^\sigma_{j_1 j_2 \dots j_p k} dx^k,$$

where $0 \le p \le r-1$, are examples of contact 1-forms. $\eta$ is $\pi^r$-*horizontal* if and only if $(\pi^{r+1,r})^*\eta = h\eta$.

A *Lagrangian* (of *order* $r$) for $Y$ is any $\pi^r$-horizontal $n$-form on some $W^r$. A differential form $\rho \in \Omega^s_n W$, where $n = \dim X$, is called a *Lepage form*, if $p_1 d\rho$ is $\pi^{s+1,0}$-horizontal, i.e. $p_1 d\rho \in \Omega^{s+1}_{n+1,Y} W$. A Lepage form $\rho$ is a *Lepage equivalent* of a Lagrangian $\lambda \in \Omega^r_{n,X} W$, if $h\rho = \lambda$ (possibly up to a jet projection).

In a fibered chart $(V, \psi)$, $\psi = (x^i, y^\sigma)$, denote

$$\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n, \quad \omega_k = i_{\partial/\partial x^k} \omega_0.$$

In this fibered chart, a Lagrangian, defined on $V^r = (\pi^{r,0})^{-1}(V)$, has an expression

$$\lambda = \mathcal{L}\omega_0, \tag{2.1}$$

where $\mathcal{L} : V^r \to R$ is the *Lagrange function* associated with $\lambda$ and $(V, \psi)$. A pair $(Y, \lambda)$, consisting of a fibered manifold $Y$ and a Lagrangian $\lambda$ of order $r$ for $Y$ is called a *Lagrange structure* (of order $r$).

For our purpose we give the following examples of Lepage equivalents.

(1) Every first order Lagrangian $\lambda \in \Omega^1_{n,X}W$ has a unique Lepage equivalent $\Theta_\lambda \in \Omega^1_{n,Y}W$ whose order of contactness is $\leq 1$. If $\lambda$ is expressed by (2.1), then

$$\Theta_\lambda = \mathcal{L}\omega_0 + \frac{\partial \mathcal{L}}{\partial y^\sigma_i}\omega^\sigma \wedge \omega_i. \tag{2.2}$$

$\Theta_\lambda$ is the *Poincaré-Cartan equivalent* of $\lambda$, or the *Poincaré-Cartan form.*

(2) Formula

$$\Theta_\lambda = \mathcal{L}\omega_0 + \left(\frac{\partial \mathcal{L}}{\partial y^\sigma_i} - d_p\frac{\partial \mathcal{L}}{\partial y^\sigma_{pi}}\right)\omega^\sigma \wedge \omega_i + \frac{\partial \mathcal{L}}{\partial y^\sigma_{ji}}\omega^\sigma_j \wedge \omega_i \tag{2.3}$$

generalizes the Poincaré-Cartan form to *second order* Lagrangians $\lambda \in \Omega^2_{n,X}W$.

If $\rho$ is a Lepage equivalent of a Lagrangian $\lambda \in \Omega^r_{n,X}W$, $\lambda = \mathcal{L}\omega_0$, then by a direct calculation $p_1 d\rho = E_\sigma(\mathcal{L})\omega^\sigma \wedge \omega_0$, where

$$E_\sigma(\mathcal{L}) = \sum_{k=0}^{r}(-1)^k d_{i_1}d_{i_2}\dots d_{i_k}\frac{\partial \mathcal{L}}{\partial y^\sigma_{i_1 i_2 \dots i_k}}$$

are the *Euler-Lagrange expressions.* The $(n+1)$-form

$$E_\lambda = p_1 d\rho$$

is the *Euler-Lagrange form* associated with $\lambda$.

By an *automorphism* of $Y$ we mean a diffeomorphism $\alpha : W \to Y$, where $W \subset Y$ is an open set, such that there exists a diffeomorphism $\alpha_0 : \pi(W) \to X$ such that $\pi\alpha = \alpha_0\pi$. If $\alpha_0$ exists, it is unique, and is called the *$\pi$-projection* of $\alpha$. The *$r$-jet prolongation* of $\alpha$ is an automorphism $J^r\alpha : W^r \to J^rY$ of $J^rY$, defined by

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1}).$$

If $\xi$ is a $\pi$-projectable vector field on $Y$, and $\alpha_t$ is the local one-parameter group of $\xi$ with projection $\alpha_{(0)t}$, we define the *$r$-jet prolongation* of $\xi$ to be the vector field $J^r\xi$ on $J^rY$ whose local one-parameter group is $J^r_{\alpha_t}$. Thus,

$$J^r\xi(J^r_x\gamma) = \left\{\frac{d}{dt}J^r_{\alpha_{(0)t}(x)}(\alpha_t\gamma\alpha^{-1}_{(0)t})\right\}_0.$$

The chart expression for $J^r\xi$ can be found in [8] or [9].

We now compute the Lie derivative $\partial_{J^r\xi}\lambda$. Choose to this purpose a Lepage equivalent $\rho$ of $\lambda$, and denote by $s$ the *order* of $\rho$. Since $\lambda = h\rho$, or, which is the same, $J^r\gamma^*\lambda = J^s\gamma^*\rho$ for all sections $\gamma$, we obtain

$$J^r\gamma^*\partial_{J^r\xi}\lambda = J^s\gamma^*\partial_{J^s\xi}\rho = J^s\gamma^*(i_{J^s\xi}d\rho + di_{J^s\xi}\rho).$$

Omitting $\gamma$ and using the Euler-Lagrange form we get

$$\partial_{J^r\xi}\lambda = hi_{J^{s+1}\xi}E_\lambda + hdi_{J^s\xi}\rho.$$

This is the *differential first variation formula*; the first term on the right is the *Euler-Lagrange term*, and the second one is the *boundary term*.

An automorphism $\alpha : W \to Y$ of $Y$ is said to be an *invariant transformation* of a form $\eta \in \Omega_p^s W$, if

$$J^s\alpha^*\eta = \eta.$$

We say that a $\pi$-projectable vector field $\xi$ is the *generator* of invariant transformations of $\eta$, if

$$\partial_{J^s\xi}\eta = 0.$$

The following simple consequence of the first variation formula is known as the *Noether's theorem.* Let $\lambda \in \Omega_{n,X}^r W$ be a Lagrangian, let $\rho \in \Omega_n^s W$ be a Lepage equivalent of $\lambda$, and let $\gamma$ be an extremal. Then for any generator $\xi$ of invariant transformations of $\lambda$,

$$dJ^s\gamma^* i_{J^s\xi}\rho = 0.$$

An $(n-1)$-form $i_{J^s\xi}\rho$ is called the *Noether's current* associated with a Lepage form $\rho$ and a vector field $\xi$.

## 3 Frame bundle and its second jet prolongation

Let $X$ be an $n$-dimensional smooth manifold, and let $\mu : FX \to X$ be the *frame bundle* over $X$. $FX$ has the structure of a right principal $Gl_n(R)$-bundle. Recall that for every chart $(U, \varphi)$, $\varphi = (x^i)$, on $X$, the *associated chart* on $FX$, $(V, \psi)$, $\psi = (x^i, x_j^i)$, is defined by $V = \mu^{-1}(U)$, and

$$x^i(\Xi) = x^i(\mu(\Xi)), \quad \Xi_j = x_j^i(\Xi)\left(\frac{\partial}{\partial x^i}\right)_x,$$

where $\Xi \in V$, $x = \mu(\Xi)$, and $\Xi = (x, \Xi_j)$. We denote by $y_k^j$ the *inverse matrix* of $x_j^i$. The right action $FX \times Gl_n(R) \ni (\Xi, A) \to R_A(\Xi) = \Xi \cdot A \in FX$ is given by the equations

$$\bar{x}^i = x^i \circ R_A = x^i, \quad \bar{x}_j^i = x_j^i \circ R_A = x_k^i a_j^k,$$

where $A = a_j^i$ is an element of the group $Gl_n(R)$.

For the formulation of variational principles on the frame bundles in this paper we need the *r-jet prolongations* of $FX$, the manifolds $J^rFX$, where $r = 1, 2, 3, 4$. These manifolds are constructed from sections of the frame bundle $FX$ in a standard way. We introduce basic concepts for $J^2FX$, more general description of $J^rFX$ is available in [1]. To the charts $(U, \varphi)$, and $(V, \psi)$, introduced above, we associate a chart $(V^2, \psi^2)$, $\psi^2 = (x^i, x_j^i, x_{j,k}^i, x_{j,kl}^i)$, as follows. We denote by $V^2$ the set of 2-jets of smooth frame fields $U \ni x \to \gamma(x) \in V \subset FX$. If $J_x^2\gamma \in V^2$, we set

$$x^i(J_x^2\gamma) = x^i(x), \quad x_j^i(J_x^2\gamma) = x_j^i(\gamma(x)),$$

$$x_{j,k}^i(J_x^2\gamma) = D_k(x_j^i\gamma\varphi^{-1})(\varphi(x)), \quad x_{j,kl}^i(J_x^2\gamma) = D_kD_l(x_j^i\gamma\varphi^{-1})(\varphi(x)).$$

As usual, the 2-jets are equivalence classes of frame fields, which have the contact up to the second order, and have the canonical *jet prolongation* $x \to J^2\gamma(x) = J^2_x\gamma$ of any frame field $\gamma$. The general linear group acts on $J^2FX$ on the right by the formula $J^2_x\gamma \cdot A = J^2_x(\gamma \cdot A)$; the action is expressed by the equations

$$\bar{x}^i = x^i, \quad \bar{x}^i_j = x^i_k a^k_j, \quad \bar{x}^i_{j,k} = x^i_{m,k} a^m_j, \quad \bar{x}^i_{j,kl} = x^i_{m,kl} a^m_j. \tag{3.1}$$

It is easy to determine the orbits of the action $(J^2_x\gamma, A) \to J^2_x(\gamma \cdot A)$. Denoting

$$\Gamma^i_{kp} = -y^m_p x^i_{m,k}, \quad \Gamma^i_{klp} = -y^m_p x^i_{m,kl},$$

we obtain $Gl_n(R)$*-invariant functions* on $J^2FX$, and *equations of* $Gl_n(R)$*-orbits*

$$\Gamma^i_{kp} = c^i_{kp}, \quad \Gamma^i_{klp} = c^i_{klp},$$

where $c^i_{kp}, c^i_{klp} \in R$ are arbitrary numbers. The functions $\Gamma^i_{klp}$ are symmetric in $k, l$. We have the following result.

**Lemma 1.** *Every* $Gl_n(R)$*-invariant function on* $J^2FX$ *depends on* $x^i$, $\Gamma^i_{kp}$, $\Gamma^i_{klp}$.

In other words Lemma 1 says that $Gl_n(R)$-invariant functions coincide with the functions on the *bundle of second order connections* $C^2X = J^2FX/Gl_n(R)$ over $X$. From equations (3.1) we can obtain an extension of Lemma 1 to differential forms.

**Lemma 2.** *A k-form* $\eta$ *on* $J^2FX$ *is* $Gl_n(R)$*-invariant if and only if it has an expression*

$$\begin{aligned}\eta = \Delta_0 &+ y^{q_1}_{r_1} dx^{p_1}_{q_1} \wedge \Delta^{r_1}_{p_1} + y^{q_1}_{r_1} y^{q_2}_{r_2} dx^{p_1}_{q_1} \wedge dx^{p_2}_{q_2} \wedge \Delta^{r_1 r_2}_{p_1 p_2} \\ &+ \ldots + y^{q_1}_{r_1} y^{q_2}_{r_2} \ldots y^{q_k}_{r_k} dx^{p_1}_{q_1} \wedge dx^{p_2}_{q_2} \ldots \wedge dx^{p_k}_{q_k} \wedge \Delta^{r_1 r_2 \ldots r_k}_{p_1 p_2 \ldots p_k},\end{aligned}$$

*where* $\Delta_0$, $\Delta^{r_1}_{p_1}$, $\Delta^{r_1 r_2}_{p_1 p_2}$, $\ldots$, $\Delta^{r_1 r_2 \ldots r_k}_{p_1 p_2 \ldots p_k}$ *are arbitrary forms defined on* $C^2X$.

Let

$$\xi_0 = \xi^i_j \left( \frac{\partial}{\partial a^i_j} \right)_e$$

be a vector belonging to the Lie algebra $gl_n(R)$. Then the corresponding fundamental vector field on $J^2FX$ is given by

$$J^2\xi = \xi^i_s \left( x^t_i \frac{\partial}{\partial x^t_s} + x^t_{i,k} \frac{\partial}{\partial x^t_{s,k}} + x^t_{i,kl} \frac{\partial}{\partial x^t_{s,kl}} \right). \tag{3.2}$$

# 4 Reduction of the Euler-Lagrange equations

Using the results of Section 3, we determine in this section $Gl_n(R)$-invariant Lagrangians on $J^1FX$ and $J^2FX$. Then we give explicit expressions of the Euler-Lagrange forms, and the Noether's currents associated with the Lepage equivalents $\Theta_\lambda$ of these Lagrangians. Then we discuss consequences of $Gl_n(R)$-invariance of these

Lagrangians for the Euler-Lagrange equations. Our main tool is the first variation formula (Section 2).

Let us denote by $\Psi_{\lambda,\xi}$ the Noether's current associated with the Lepage form $\Theta_\lambda$ (2.2), (2.3) and a vector field $\xi$, and by $\omega^i_j$ the contact forms defined by

$$\omega^i_j = dx^i_j - x^i_{j,m}dx^m = dx^i_j + x^p_j\Gamma^i_{mp}dx^m.$$

**Lemma 3.** *Let $\lambda \in \Omega^1_{n,X}FX$ be a Lagrangian expressed by $\lambda = \mathcal{L}\omega_0$.*

(a) *$\lambda$ is $Gl_n(R)$-invariant if and only if $\mathcal{L}$ depends on $x^i$, $\Gamma^i_{kj}$ only.*

(b) *The Euler-Lagrange form of a $Gl_n(R)$-invariant Lagrangian has an expression*

$$E_\lambda = y^j_l\left(-\Gamma^p_{qi}\frac{\partial\mathcal{L}}{\partial\Gamma^p_{ql}} + \Gamma^l_{pq}\frac{\partial\mathcal{L}}{\partial\Gamma^i_{pq}} + \frac{\partial^2\mathcal{L}}{\partial x^p\partial\Gamma^i_{pl}}\right.$$
$$\left.+(\Gamma^k_{mpr} + \Gamma^k_{mq}\Gamma^q_{pr})\frac{\partial^2\mathcal{L}}{\partial\Gamma^k_{mr}\partial\Gamma^i_{pl}}\right)\omega^i_j\wedge\omega_0.$$

(c) *If $\lambda$ is $Gl_n(R)$-invariant, then the Noether's current associated with the Poincaré-Cartan form of $\lambda$ and any fundamental vector field $\xi$ is given by*

$$\Psi_{\lambda,\xi} = -\xi^m_j y^j_l x^i_m\frac{\partial\mathcal{L}}{\partial\Gamma^i_{kl}}\omega_k. \tag{4.1}$$

Let $X$ be an $n$-dimensional manifold, let $FX$ be the bundle of frames over $X$, and let $\mu$ be the bundle projection. Suppose that we have a Lagrangian $\lambda \in \Omega^1_{n,X}FX$ and a *$\mu$-vertical* vector field $\xi$ on $FX$. Then in our standard notation

$$\partial_{J^1\xi}\lambda = i_{J^2\xi}E_\lambda + hdi_{J^1\xi}\Theta_\lambda, \tag{4.2}$$

where $\Theta_\lambda$ is the Poincaré-Cartan equivalent of $\lambda$.

**Theorem 1.** *Let $\lambda \in \Omega^1_{n,X}FX$ be a $Gl_n(R)$-invariant Lagrangian, let $n \geq 2$, and let $\gamma$ be a section of $FX$. The following conditions are equivalent.*

(a) *$\gamma$ satisfies the Euler-Lagrange equations,*

$$E_\lambda \circ J^2\gamma = 0.$$

(b) *For any chart $(U,\varphi)$, $\varphi = (x^i)$, on $X$, and all $j$, $k$, there exist $(n-2)$-forms $\eta^j_k$ such that*

$$J^1\gamma^*\left(y^j_l x^i_k\frac{\partial\mathcal{L}}{\partial\Gamma^i_{ml}}\omega_m - d\eta^j_k\right) = 0.$$

**Proof.** By hypothesis, for any fundamental vector field $\xi$ on $FX$, $\partial_{J^1\xi}\lambda = 0$. Consequently, since $\xi$ is always $\mu$-vertical, the first variation formula (4.2) reduces to

$$i_{J^2\xi}E_\lambda + hd\Psi_{\lambda,\xi} = 0. \tag{4.3}$$

We can write this identity in a chart $(U, \varphi)$, $\varphi = (x^i)$, on $X$. Using (3.2) we have, according to Lemma 3,

$$i_{J^2\xi} E_\lambda = E_i^j(\mathcal{L}) \xi_j^k x_k^i \omega_0,$$

where

$$E_i^j(\mathcal{L}) = \left( -\Gamma^p_{qi} \frac{\partial \mathcal{L}}{\partial \Gamma^p_{ql}} + \Gamma^l_{pq} \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pq}} + \frac{\partial^2 \mathcal{L}}{\partial x^p \partial \Gamma^i_{pl}} + (\Gamma^k_{mps} + \Gamma^k_{mq}\Gamma^q_{ps}) \frac{\partial^2 \mathcal{L}}{\partial \Gamma^k_{ms} \partial \Gamma^i_{pl}} \right) y_l^j,$$

and the Noether's current $\Psi_{\lambda,\xi}$ is given by (4.1). It is convenient to denote

$$\psi_m^j = y_l^j x_m^i \frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}} \omega_k.$$

Then $\Psi_{\lambda,\xi} = -\xi_j^m \psi_m^j$, and the first variation formula (4.3) can equivalently be written in the form $E_i^j(\mathcal{L}) \xi_j^k x_k^i \omega_0 - \xi_j^k h d\psi_k^j = 0$. But the numbers $\xi_j^k \in R$ are arbitrary, so we have

$$E_i^j(\mathcal{L}) x_k^i \omega_0 - h d\psi_k^j = 0. \tag{4.4}$$

Suppose now that a section $\gamma$ satisfies the Euler-Lagrange equations. Then the form $E_i^j(\mathcal{L}) x_k^i \omega_0$ vanishes along $J^2\gamma$, so we have $J^2\gamma^* d\psi_k^j = dJ^2\gamma^* \psi_k^j = 0$. Integrating we can find an $(n-2)$-form $\eta_k^j$ on $U$ such that

$$J^1\gamma^* \psi_k^j = d\eta_k^j. \tag{4.5}$$

Conversely, if a section $\gamma$ satisfies condition (4.5), then by (4.4), $\gamma$ is necessarily an extremal. $\square$

For second order Lagrangians on $FX$ we have the following results.

**Lemma 4.** *Let $\lambda \in \Omega^2_{n,X} FX$ be a Lagrangian expressed by $\lambda = \mathcal{L}\omega_0$.*
(a) *$\lambda$ is $Gl_n(R)$-invariant if and only if $\mathcal{L}$ depends on $x^i$, $\Gamma^i_{kj}$, $\Gamma^i_{klj}$ only.*
(b) *The Euler-Lagrange form of a $Gl_n(R)$-invariant Lagrangian has an expression*

$$E_\lambda = y_l^j \left( -\Gamma^p_{qi} \frac{\partial \mathcal{L}}{\partial \Gamma^p_{ql}} - \Gamma^p_{qmi} \frac{\partial \mathcal{L}}{\partial \Gamma^p_{qml}} + \Gamma^l_{pq} \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pq}} + d_p \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pl}} - \Gamma^l_{pqm} \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pqm}} \right.$$
$$\left. -2\Gamma^l_{pt} \left( \Gamma^t_{qm} \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pqm}} + d_q \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pqt}} \right) - d_p d_q \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pql}} \right) dx_j^i \wedge \omega_0.$$

(c) *If $\lambda$ is $Gl_n(R)$-invariant, then the Noether's current associated with the Lepage form (2.3) and any fundamental vector field $\xi$ is given by*

$$\Psi_{\lambda,\xi} = \xi_j^m y_l^j x_m^i \left( -\frac{\partial \mathcal{L}}{\partial \Gamma^i_{kl}} + \Gamma^q_{pi} \frac{\partial \mathcal{L}}{\partial \Gamma^q_{pkl}} + \Gamma^l_{pq} \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pkq}} + d_p \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pkl}} \right) \omega_k.$$

**Theorem 2.** *Let $\lambda \in \Omega^2_{n,X} FX$ be a $Gl_n(R)$-invariant Lagrangian, let $n \geq 2$, and let $\gamma$ be a section of $FX$. The following conditions are equivalent.*

(a) $\gamma$ *satisfies the Euler-Lagrange equations,*

$$E_\lambda \circ J^4\gamma = 0.$$

(b) *For any chart* $(U, \varphi)$, $\varphi = (x^i)$, *on* $X$, *and all* $j$, $k$, *there exist* $(n-2)$*-forms* $\eta^j_k$ *such that*

$$J^3\gamma^* \left( y^j_l x^i_k \left( \frac{\partial \mathcal{L}}{\partial \Gamma^i_{ml}} - \Gamma^q_{pi} \frac{\partial \mathcal{L}}{\partial \Gamma^q_{pml}} - \Gamma^l_{pq} \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pmq}} - d_p \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pml}} \right) \omega_m - d\eta^j_k \right) = 0.$$

**Proof.** The first variation formula for a second order Lagrangian $\lambda$ has the form

$$\partial_{J^2\xi}\lambda = i_{J^4\xi}E_\lambda + hdi_{J^3\xi}\Theta_\lambda, \tag{4.6}$$

where $\Theta_\lambda$ is the Lepage equivalent of $\lambda$ given by (2.3). Again, left hand side vanishes and formula (4.6) reduces to

$$i_{J^4\xi}E_\lambda + hd\Psi_{\lambda,\xi} = 0,$$

where the forms $E_\lambda$ and $\Psi_{\lambda,\xi}$ are given by Lemma 4. The rest of the proof is analogous to the Proof of Theorem 1. □

**Acknowledgments.** The author is grateful to the Ministry of Education of the Slovak Republic (Grant VEGA No. 1/3009/06) and to the Czech Grant Agency (Grant No. 201/06/0922).

## References

[1] J. Brajerčík, *Higher order invariant variational principles on frame bundles*, Ph.D. Thesis, Masaryk University, Brno, 2005.

[2] J. Brajerčík and D. Krupka, *Variational principles for locally variational forms*, J. Math. Phys. 46, 052903 (2005), 1-15.

[3] M. Castrillón, P.L. García and T.S. Ratiu, *Euler-Poincaré reduction on principal bundles*, Lett. Math. Phys. 58 (2001), 167-180.

[4] M. Castrillón, T.S. Ratiu and S. Shkoller, *Reduction in principal fiber bundles: covariant Euler-Poincaré equations*, Proc. Amer. Math. Soc. 128 (2000), 2155-2164.

[5] P.L. García, *The Poincaré-Cartan invariant in the calculus of variations*, Symposia Mathematica 14 (1974), 219-246.

[6] I. Kolář, P.W. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer Verlag, Berlin, 1993.

[7] D. Krupka, *A geometric theory of ordinary first order variational problems in fibered manifolds II. Invariance*, J. Math. Anal. Appl. 49 (1975), 469-476.

[8] D. Krupka, *Natural Lagrange structures*, Differential Geometry, Banach Center Publications, Polish Scientific Publishers, Warsaw 12 (1984), 185-210.

[9] D. Krupka, *Some Geometric Aspects of the Calculus of Variations in Fibered Manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis, University J.E. Purkyně, Brno, 14, 1973.

[10] D. Krupka, *The geometry of Lagrange structures*, Preprint series in Global Analysis GA 7/1997, Silesian University, Mathematical Institute, Opava, 1997.

[11] D. Krupka and J. Janyška, *Lectures on Differential Invariants*, Folia Fac. Sci. Nat. UJEP Brunensis, University J.E. Purkyně, Brno, 1990.

[12] D. Krupka and M. Krupka, *Jets and contact elements*, Proc. Semin. on Diff. Geom., Mathematical Publications Vol. 2, Silesian University, Mathematical Institute, Opava (2000), 39-85.

[13] J. Muñoz Masqué and M. Eugenia Rosado María, *Invariant variational problems on linear frame bundles*, J. Phys. A: Math. Gen. 25 (2002), 2013-2036.

[14] D.J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series 142, Cambridge University Press, New York, 1989.

[15] A. Trautman, *Invariance of Lagrangian systems*, in: General Relativity, Papers in honor of J. L. Synge, Clarendon Press, Oxford (1972), 85-99.

*Author's address:*

Ján Brajerčík
Department of Mathematics, University of Prešov,
Ul. 17. novembra 1, 081 16 Prešov, Slovakia.
E-mail: brajerci@unipo.sk