The number of limit cycles of a quintic Hamiltonian system with perturbation

Ali Atabaigi, Nemat Nyamoradi, Hamid R.Z. Zangeneh

Abstract. We consider number of limit cycles of perturbed quintic Hamiltonian system with perturbation in the form of $(2n+2m)$ or $(2n+2m+1)$ degree polynomials. We show that the perturbed system has at most $n+2m$ limit cycles. For $m=1$ and $n=1$ we showed that the perturbed system can have at most one limit cycles. If $m=1$ and $n=2$ we give some general conditions based on coefficients of the perturbed terms for the number of existing limit cycles.

M.S.C. 2000: 34C07, 34C08, 37G15, 34M50.
Key words: Zeros of Abelian integrals, Hilbert's 16th problem, Limit cycles.

1 Introduction

The second part of Hilbert's 16th problem concerned with the existence, number and distribution of the limit cycles of planar polynomial differential equations of degree n. This problem is still unsolved even for $n=2$. Therefore several similar problems which appear to be simpler proposed by Smale [8]. One of such problems is:

Weakened Hilbert's 16th problem: Let $H(x,y)$ be a real polynomial of degree n and let $P(x,y)$ and $Q(x,y)$ be real polynomials of degree m. Now consider the perturbed Hamiltonian system in form

$$\begin{aligned} \dot{x} &= H_y + \varepsilon P(x,y), \\ \dot{y} &= -H_x + \varepsilon Q(x,y), \end{aligned} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ and the level curves $H(x,y)=h$ of the Hamiltonian system $(1.1)|_{\varepsilon=0}$ contain at least a family Γ_h of closed orbits for $h \in (h_1,h_2)$, where h_1 and h_2 are real numbers and $H(x,y)$ is the hamiltonian of the unperturbed system $(1.1)|_{\varepsilon=0}$. Let

$$A(h) = \int_{\Gamma_h} Pdy - Qdx = \int\int_{H\le h} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy. \tag{1.2}$$

The function $A(h)$ is called Abelian integral in [11]. There is a close relation between number of zeros of Abelian integral $A(h)$ and number of limit cycles of system (1.1).

Balkan Journal of Geometry and Its Applications, Vol.13, No.2, 2008, pp. 1-11.

For a fixed H of degree $n \geq 3$, taking P and Q in (1.1) as arbitrary polynomials of degree m many authors [1, 3, 4, 5, 7] showed that the upper bound for number of isolated zeros of $A(h)$ is a linear function in n and m. Yang and Wang [10] showed that the system

$$\frac{dx}{dt} = y + \varepsilon \sum_{k=0}^{l} a_k x^k, \quad \frac{dy}{dt} = -x \tag{1.3}$$

has at most n limit cycle when $0 < \varepsilon \ll 1$, where $l = 2n+1$ or $2n+2$. For $n = 2$ and $n = 3$, for the bifurcation, location and stability of limit cycles, they obtained the conditions depending on the coefficients of the polynomials . Zhao [12] investigated the perturbed cubic Hamiltonian system

$$\frac{dx}{dt} = y + \varepsilon P_n(x, y), \quad \frac{dy}{dt} = -x - x^3 + \varepsilon Q_n(x, y), \tag{1.4}$$

where $P_n(x, y), Q_n(x, y)$ are polynomials of degree n, and proved that the upper bound for the number of isolated zeros of the Abelian integral corresponding to (1.4) is $3[(n-1)/2], n \geq 3$. Han [2] proved that the system

$$x' = y, \quad y' = -(x^3 + bx - x) - \varepsilon(a_1 + a_2 x + a_3 x^2)y \tag{1.5}$$

has four limit cycles for $0 < \varepsilon \ll 1$. Cheng-qiang Wu, Yonghui Xia [9] considered the system

$$\frac{dx}{dt} = y + \varepsilon \sum_{j=0}^{l} a_j x^j \mid y \mid^{2m-1}, \frac{dy}{dt} = -x - x^3, \tag{1.6}$$

where $0 < \varepsilon \ll 1$, $l = 2n+1$ or $2n+2$, n and m are arbitrarily positive integers and $a_0, a_1, ..., a_l$ are real, and showed that this system has at most $n+m$ limit cycles. They proved that system (1.6) has at most n limit cycles when $m = 1$.

In this paper, we consider the following perturbed Hamiltonian system

$$\frac{dx}{dt} = y + \varepsilon P(x, y), \quad \frac{dy}{dt} = -x - x^5, \tag{1.7}$$

where $P(x, y) = \sum_{j=0}^{l} a_j x^j \mid y \mid^{2m-1}$, $0 < \varepsilon \ll 1$, $l = 2n+1$ or $l = 2n+2$, n and m are arbitrarily positive integers and $a_0, a_1, ..., a_l$ are real. In §2 we derive a formula for $A(h)$ and show that an upper bound for number of zeros of $A(h)$ is $n + 2m$. Also by considering behavior of (1.7) about its equilibrium at infinity we show that (1.7) has at most $n+2m$ limit cycles. In §3 we consider number of zeros of $A(h)$ for $n = m = 1$ and $n = 1$, $m = 2$ and show that in the first case (1.7) can have at most one limit cycle and in the latter (1.7) can have at most two limit cycles in a parameter region Ω.

2 Abelian integral computation and main result

We calculate the Abelian integrals $A(h)$ corresponding to system (1.7) using equation (2). Clearly the unperturbed system of $(1.7)|_{\varepsilon=0}$ has the first integral

$$\Gamma_h : H(x,y) = \frac{x^2}{2} + \frac{x^6}{6} + \frac{y^2}{2} = h, \quad h > 0, \tag{2.1}$$

Which extends as h increases. Therefore from (2) we have

$$A(h) = \int\int_{D_h} \sum_{j=1}^{l} a_j j x^{j-1} \mid y \mid^{2m-1} dydx, \tag{2.2}$$

where D_h is the area surrounded by Γ_h, and $l = 2n+1$ or $2n+1$. First we notice that from (2.1), we have $y_{1,2} = \pm\sqrt{2h - x^2 - \frac{x^6}{3}}$. By symmetry of D_h with respect to $x = 0$ and simple calculation, For $l = 2n+1$, we have

$$\begin{aligned} A(h) &= \int\int_{D_h} (\sum_{j=1}^{l} j a_j x^{j-1}) \mid y \mid^{2m-1} dydx = 2\int_{x_1}^{x_2}\int_0^{y_2} \sum_{j=1}^{l} j a_j x^{j-1} y^{2m-1} dydx \\ &= \frac{1}{m}\int_{x_1}^{x_2} \left[\sum_{j=1}^{n+1}(2j-1)a_{2j-1}x^{2j-2} + \sum_{j=1}^{n+1}(2j)a_{2j}x^{2j-1}\right] y_2^{2m} dx, \end{aligned}$$

where x_1, x_2, are zeros of $3x^2 + x^6 - 6h = 0$. Now by Cardano's formulae for roots of cubic equation , we have

$$x_1 = -\sqrt{\frac{\sqrt[3]{(3h + \sqrt{9h^2+1})^2} - 1}{\sqrt[3]{3h + \sqrt{9h^2+1}}}}, \quad x_2 = -x_1.$$

Therefore by symmetry of D_h with respect to $y = 0$, $\int_{x_1}^{x_2} y_2^{2m} \sum_{j=1}^{n+1}(2j)a_{2j}x^{2j-1}dx = 0$, therefore

$$A(h) = \frac{2}{m}\sum_{j=1}^{n+1} a_{2j-1}(2j-1)\int_0^{x_2} x^{2j-2}(2h - x^2 - \frac{x^6}{3})^m dx. \tag{2.3}$$

To write $A(h)$ as a polynomial in h, first we notice that

$$(2h - x^2 - \frac{x^6}{3})^m = \sum_{s=0}^{m} C_m^s(-1)^s(2h)^{m-s}x^{2s}(1 + \frac{x^4}{3})^s, \quad (1 + \frac{x^4}{3})^s = \sum_{i=0}^{s} C_s^i(\frac{1}{3^i})x^{4i},$$

where $C_n^m = \frac{m!}{n!(m-n)!}$. Therefore (2.3) becomes

$$\begin{aligned} A(h) &= \frac{2}{m}\sum_{j=1}^{n+1} a_{2j-1}(2j-1)\left[\sum_{s=0}^{m}(-1)^s C_m^s B(j,s,x_2)(2h)^{m-s}x_2^{2s+2j-1}\right] \\ &= \frac{2x_2^{-1}}{m}\sum_{j=1}^{n+1} a_{2j-1}(2j-1)x_2^{2j}\left[\sum_{s=0}^{m}(-1)^s C_m^s B(j,s,x_2)(2h)^{m-s}x_2^{2s}\right], \end{aligned} \tag{2.4}$$

where

$$B(j,s,x_2)=\sum_{i=0}^{s}\frac{C_s^i}{3^i}\frac{x_2^{4i}}{4i+2s+2j-1}.$$

To simplify $A(h)$, let $\lambda^3=3h+\sqrt{9h^2+1}$, then $h=\frac{1}{6}(\lambda^3-\frac{1}{\lambda^3})$ and $x_2=\sqrt{\lambda-\frac{1}{\lambda}}$. To simplify equation (2.4) more we set $\mu=\lambda-\frac{1}{\lambda}$. It is clear for every positive μ there exist a unique $\lambda>1$ and $x_2^2=\mu$, $h=\frac{1}{6}\mu(\mu^2+3)$. Therefore for every positive h there correspond a unique $\mu>0$. Using this change of variable equation (2.4) becomes
(2.5)

$$A(h)=\frac{2x_2^{-1}\mu^{m+1}}{m}\underbrace{\sum_{j=1}^{n+1}a_{2j-1}(2j-1)\mu^{j-1}[\sum_{s=0}^{m}C_m^s(-1)^s(\frac{1}{3})^{m-s}(\mu^2+3)^{m-s}B(j,s,\sqrt{\mu})]}_{I(\mu)},$$

which $I(\mu)$ is a polynomial of degree $(n+2m)$ in μ. Since $h=\frac{1}{6}\mu(\mu^2+3)$, we have $\frac{dh}{d\mu}=\frac{1}{2}+\frac{1}{2}\mu^2>0$, therefore $\frac{d\mu}{dh}>0$. On the other hand let $h_*\in(0,\infty)$ be a zero of $A(h)=0$, then it is clear that $I(\mu_*)=0$ where $h_*=\frac{1}{6}\mu_*(\mu_*^2+3)$ and

$$A'(h_*)=\frac{2}{m}((\mu_*^{m+1/2})I'(\mu_*))\frac{2}{1+\mu_*^2},$$

Therefore we obtain:

Lemma 1. *(i) $A(h)$ has at most $n+2m$ zero for all $h\in(0,\infty)$.*
(ii) There is a $h^*\in(0,+\infty)$ such that $A(h^*)=0$ if and only if there is a $\mu^*\in(0,+\infty)$ such that $I(\mu^*)=0$, where $h_*=\frac{1}{6}\mu_*(\mu_*^2+3)$
(iii) $A(h^*)=0$ and $A'(h^*)>0(<0)$ if and only if $I(\mu^*)=0$ and $I'(\mu^*)>0(<0)$.

From (2.5) lemma 1 and lemma 1 in [9], we know that (1.7) has at most $n+2m$ limit cycle in the finite plane.

Remark 1. *From the above discussion we see that even power terms in (1.7) has no effect on number of limit cycles of system (1.7). So we may assume they are zero.*

2.1 Behavior at infinity

In this part we prove that system (1.7) has no limit cycle about equilibrium points at infinity. First let us denote

$$\begin{aligned}y+\varepsilon\sum_{j=0}^{l}a_jx^j|y|^{2m-1} &:= P_1(x,y)+P_2(x,y)+\cdots+P_N(x,y),\\ -x-x^5 &:= Q_1(x,y)+Q_2(x,y)+\cdots+Q_N(x,y),\end{aligned}$$

where P_j and Q_j are homogeneous polynomial of degree j. Then equilibrium point of (1.7) at infinity satisfy the following equation [6]

$$G(X,Y):=XQ_N(X,Y)-YP_N(X,Y)=0,\quad X^2+Y^2=1,\ Z=0, \tag{2.6}$$

where (X,Y) denotes the coordinate on the equator of the Poincaré sphere $\mathbb{S}^2=\{(X,Y,Z):X^2+Y^2+Z^2=1\}$ and $x=\frac{X}{Z}$, $y=\frac{Y}{Z}$ gives the relation between a point in xy-plane and a point on the sphere $\mathbb{S}^2$. Flow on the equator of $\mathbb{S}^2$ will be determined by the sign of $G(X,Y)$. It will be clockwise if $G<0$ and it is counterclockwise if $G>0$. Now we consider the following case separately.

(i) m=n=1. If $l=2n+1$ then $N=5$ and from (2.6) we get

$$G(X,Y)=X(-X^5)=0,\quad X^2+Y^2=1.$$

Therefore (1.7) have two equilibrium $(0,\pm 1,0)$ on the equator of $\mathbb{S}^2$. In this case $G<0$ the flow on the equator is clockwise, and since $(\pm 1,0,0)$ is not a critical points of (1.7), by ([6]) the behavior of the flow about the equilibrium of (1.7) at infinity is topologically equivalent to

$$\dot{x}=z^4(1+x^2)+x^6+\varepsilon\sum_{j=0}^{3}a_jx^jz^{4-j}sgnz,\quad \dot{z}=xz(z^4+x^4).$$

If $l=2n+2$, again $N=5$ and equilibrium points of (1.7) on the equator of $\mathbb{S}^2$ are

$$\begin{aligned}&(0,\pm 1,0)\quad\text{and}\quad(\pm\sqrt{\frac{\varepsilon a_4}{1+\varepsilon a_4}},-\sqrt{\frac{1}{1+\varepsilon a_4}},0),\quad\text{if } a_4>0,\\ &(0,\pm 1,0)\quad\text{and}\quad(\pm\sqrt{\frac{-\varepsilon a_4}{1-\varepsilon a_4}},\sqrt{\frac{1}{1-\varepsilon a_4}},0),\quad\text{if } a_4<0.\end{aligned}$$

Since $(\pm 1,0,0)$ is not a critical point of (1.7) on the equator of $\mathbb{S}^2$, the flow about the equilibrium of (1.7) at infinity is topologically equivalent to

$$\dot{x}=z^4(1+x^2)+x^6+\varepsilon\sum_{j=0}^{4}a_jx^jz^{4-j}sgnz,\quad \dot{z}=xz(z^4+x^4).$$

We notice that $z=0$ is a trajectory of (18) and therefore along two characteristic directions $\theta=0$ and π, there are orbits of (18) approaching, $(0,0)$, the unique singular point of (18) on the x-axis. Moreover the behavior at the antipodal points on the equator of the Poincaré sphere will be topologically equivalent. Therefore (1.7) has no limit cycle at infinity.

Similarly, if $l=2n+2$ the behavior of the flow about the equilibrium of (1.7) at infinity is given by (2.7) A similar proof shows that, in this case also, (1.7) has no limit cycle at infinity. Now we consider the case $m+n>2$. In this case the critical points of (1.7) at infinity are zeros of equation $-\varepsilon a_lY^{2m}X^lsgn(Y)=0$ on the equator of the Poincaré sphere. If $a_l\neq 0$, then the zeros are $(0,\pm 1,0),\ (\pm 1,0,0)$. Let $l=2n+1, a_l>0$ (the other cases can be done similarly). Therefore according to [6] the flow on the equator about $(0,\pm 1,0),\ (\pm 1,0,0)$ are topologically equivalent to

$$\begin{aligned}
(2.7) \qquad \dot{x} &= z^{2m+l-2}(x^2+1) + x^6 z^{2m+l-6} + \varepsilon \sum_{j=0}^{l} a_j x^j z^{l-j} sgnz, \\
\dot{z} &= x z^{2m+l-5}(x^4+z^4), \quad \text{and} \\
(2.8) \qquad \dot{y} &= -y^2 z^{2m+l-2} - z^{2m+l-6}(1+z^4) - \varepsilon \sum_{j=0}^{l} a_j y^{2m} z^{l-j} sgnzsgny, \\
\dot{z} &= -y z^{2m+l-1} - \varepsilon \sum_{j=0}^{l} a_j z^{l-j} |z||y|^{2m-1},
\end{aligned}$$

respectively. However, since $\dot{z} = 0$ on the x-axis or y-axis in (2.7)and (2.8), respectively. Similar to the case $m + n = 2$, (1.7) has no limit cycle at infinity. If $a_l = 0$, then $z = 0$ is the singular line of (2.7) and (2.8), respectively. But this implies that (1.7) has no limit cycle about equilibrium at infinity. Therefore by Lemma 2 and the above discussion, we have proved

Theorem 1. *The perturbed system (1.7) has at most $n + 2m$ limit cycles.*

3 The number of limit cycles of (1.7) when $m = 1$

Consider the perturbed system (1.7) with $m = 1$

$$\frac{dx}{dt} = y + \varepsilon \sum_{j=0}^{l} a_j x^j |\, y \,|, \quad \frac{dy}{dt} = -x - x^5, \tag{3.1}$$

where $l = 2n + 1$ or $2n + 2$. From (2.4), and by direct calculation, we have

$$A(h) = 2x_2^{-1} \sum_{j=1}^{n+1} a_{2j-1}(2j-1) \left(\frac{2hx_2^{2j}}{2j-1} - \frac{x_2^{2j+2}}{2j+1} - \frac{x_2^{2j+6}}{3(2j+5)} \right). \tag{3.2}$$

Now, by replacing $x_2^2 = \mu$, $h = \frac{1}{6}\mu(\mu^2+3)$, in equation (3.2)and after little simplification, we have

$$A(h) = 4\mu^{3/2} \sum_{j=1}^{n+1} a_{2j-1} \left(\frac{\mu^{j+1}}{2j+5} + \frac{\mu^{j-1}}{2j+1} \right),$$

Therefore $A(h) = 4x_2^{-1}\mu^2 I_{n+2}(\mu)$, where

$$I_{n+2}(\mu) = \frac{a_{2n+1}}{2n+7}\mu^{n+2} + \frac{a_{2n-1}}{2n+5}\mu^{n+1} + \frac{(a_{2n-3}+a_{2n+1})}{2n+3}\mu^n + ... + \frac{a_3}{5}\mu + \frac{a_1}{3}. \tag{3.3}$$

To consider number of zeros of $A(h)$ we may consider two special cases:
(i) n is even and $\frac{a_1}{a_{2n+1}} < 0$, (ii) n is odd and $\frac{a_1}{a_{2n+1}} > 0$.
It is clear that in this two cases $I_{n+2}(\mu)$ has at least one negative zero and therefore $A(\mu)$ has at most $n + 1$ zeros.

3.1 The number of limit cycles of (8) when $m = 1, n = 1$

For $n = 1$ and $m = 1$ the perturbed system (8) is

$$\frac{dx}{dt} = y + \varepsilon(a_0 + a_1x + a_2x^2 + a_3x^3) \mid y \mid, \quad \frac{dy}{dt} = -x - x^5. \tag{3.4}$$

Theorem 2. *Consider the perturbed system (3.4),*
(1) If $a_3a_1 > 0$, then system (3.4) has no limit cycle.
(2) If $a_3a_1 \leq 0$, then system (3.4) has a unique limit cycle, which is stable (unstable) when $a_3 < 0$ ($a_3 > 0$).

Proof. From (3.3) we have

$$\begin{aligned} I_3(\mu) &= \frac{1}{9}a_3\mu^3 + \frac{1}{7}a_1\mu^2 + \frac{1}{5}a_3\mu + \frac{1}{3}a_1, \\ I_3'(\mu) &= \frac{1}{3}a_3\mu^2 + \frac{2}{7}a_1\mu + \frac{1}{5}a_3. \end{aligned} \tag{3.5}$$

Number of positive zeros of $I_3(\mu)$ will be determined by position of its critical points and sign of $I_3'(\mu)$ at these points. But $I_3(\mu) = 0$ if and only if

$$\hat{I}_3(\mu) = \mu^3 + \frac{9a_1}{7a_3}\mu^2 + \frac{9}{5}\mu + \frac{3a_1}{a_3} = 0.$$

Critical points of $\hat{I}_3(\mu)$ are

$$\mu_\pm == -\frac{3a_1}{7a_3} \pm \Delta_1, \text{ if } \Delta_1 > 0, \text{ where } \Delta_1 = \sqrt{\frac{9a_1^2}{49a_3^2} - \frac{3}{5}}.$$

From the previous expression, it is clear that $\mu_- < 0 < \mu_+$. Also μ_+ is local minimum and μ_- is local maximum of (3.5). Now we consider cases $\frac{a_1}{a_3} > 0$ and $\frac{a_1}{a_3} \leq 0$ separately.
i) $\frac{a_1}{a_3} > 0$, in this case

$$\hat{I}_3'(\mu) = 3\mu^2 + \frac{18a_1}{7a_3}\mu + \frac{9}{5}$$

which is positive for all $\mu \geq 0$. On the other hand $\hat{I}_3(0) = \frac{3a_1}{a_3} > 0$, therefore $\hat{I}_3(\mu) > \hat{I}(0) > 0$ for all $\mu \geq 0$. Therefore $\hat{I}_3(\mu)$ has no positive root and system (3.4) has no limit cycle.
ii) $\frac{a_1}{a_3} \leq 0$, it is clear that $\hat{I}_3(\mu)$ has at least one positive zero, since $\hat{I}(0) = \frac{3a_1}{a_3} < 0$ and $\lim_{\mu\to\infty} \hat{I}_3(\mu) = \infty$. We notice that $\hat{I}'(\mu)$ have no zero if $\Delta_1 < 0$ and therefore $\hat{I}_3(\mu)$ has a unique zero. In case $\Delta_1 \geq 0$, $\hat{I}_3(\mu)$ has no zero in $(0, \mu_+)$, since $\hat{I}_3(0) < 0$ and it is decreasing in this interval and since it is increasing in (μ_+, ∞), $\hat{I}_3(\mu)$ has at most one zero in (μ_+, ∞) . Now let μ^* be a zero of $\hat{I}_3(\mu) = 0$. By above discussion it is clear that $\hat{I}_3'(\mu_*) > 0$. Therefore $I_3'(\mu^*)$ is negative when a_3 is negative and it positive when a_3 is positive. So the stability of the corresponding limit cycle follows Theorem 1 and Lemma 1. □

3.2 The number of limit cycles of (1.7) when $m = 1, n = 2$

For $m = 1$ and $n = 2$ the perturbed system (1.7) becomes

$$\begin{aligned} \frac{dx}{dt} &= y + \varepsilon(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5) \mid y \mid, \\ \frac{dy}{dt} &= -x - x^5, \end{aligned} \tag{3.6}$$

where $0 < \varepsilon \ll 1$. In this case (3.3) becomes

$$I_4(\mu) = \frac{1}{11}a_5\mu^4 + \frac{1}{9}a_3\mu^3 + \frac{1}{7}(a_1 + a_5)\mu^2 + \frac{1}{5}a_3\mu + \frac{1}{3}a_1.$$

To simplify our notation let $B = \frac{11}{42}(\frac{a_1}{a_5} + 1)$ and $A = -\frac{11a_3}{36a_5}$. It is easy to see that $I_4(\mu) = 0$ if and only if

$$\hat{I}_4(\mu) = \mu^4 - 4A\mu^3 + 6B\mu^2 - (36/5)A\mu + 14B - 11/3 = 0. \tag{3.7}$$

To consider number of zeros of $\hat{I}_4(\mu)$, first we consider critical points of

$$\hat{I}'_4(\mu) = 4\mu^3 - 12A\mu^2 + 12B\mu - (36/5)A. \tag{3.8}$$

The critical points of $\hat{I}'_4(\mu)$ are $\mu_\pm = A \pm \Delta$, where $\Delta = \sqrt{A^2 - B}$ if $\Delta > 0$. Number of positive zeros of $\hat{I}'_4(\mu)$ and the values of $\hat{I}'_4(\mu_\pm)$ play an important role in determining number of zeros of $I_4(\mu)$. After some computations we have

$$\hat{I}'_4(\mu_\pm) = 4(-2A^3 \mp 2\Delta^3 + 3AB - 9/5A) := I_\pm(A, B). \tag{3.9}$$

Let μ_* be a positive zero of $\hat{I}'_4(\mu) = 0$. For our purpose it is also necessary to find $\hat{I}_4(\mu_*)$. First we notice that

$$\mu_*^3 = 3A\mu_*^2 - 3B\mu_* + 9/5A, \tag{3.10}$$

$$\mu_*^3 - 3A\mu_*^2 = -3B\mu_* + 9/5A, \tag{3.11}$$

Using (3.7) and above relations we have

$$\begin{aligned} \hat{I}_4(\mu_*) &= \mu_*(\mu_*^3 - 3A\mu_*^2) - A\mu_*^3 + 6B\mu_*^2 - (36/5)A\mu_* + 14B - 11/3 \\ &= \alpha\mu_*^2 + \beta\mu_* + \gamma := P_2(\mu_*), \end{aligned} \tag{3.12}$$

where $\alpha = -3\Delta^2$, $\beta = 3A(B - 9/5)$ and $\gamma = -9/5A^2 + 14B - 11/3$. Now let us define the discriminant of the cubic equation $\hat{I}'_4(\mu) = 0$

$$\Delta_2 =: 16(12)^3[B^3 - \frac{3}{4}B^2A^2 + \frac{9}{5}A^4 - \frac{27}{10}A^2B + \frac{81}{100}A^2]. \tag{3.13}$$

By Cardano's method, if $\Delta_2 > 0$, the above cubic equation have a unique real zero μ_* defined by

(3.14)

$$\mu_* = A - (\frac{\Delta^2}{u} + u) \text{ where } u = \sqrt[3]{q/2 + \sqrt{q^2/4 - \Delta^6}} \text{ and } q = A(-2A^2 + 3B - 9/5).$$

Therefore $P_2(\mu_*)$ is a function of A and B and we denote it by

$$P_2(\mu_*) =: \zeta(A, B). \tag{3.15}$$

To consider number of limit cycles that can bifurcate from period annulus of (3.6) we partition A-B parametric region as follows:

$$\begin{aligned}
\Omega_1 &:= \{(A,B) : B \geq 11/42,\ A \leq 0\}, \quad \Omega_2 := \{(A,B) : B < 0,\ A > 0\},\\
\Omega_3 &:= \{(A,B) : 0 \leq B \leq 11/42,\ A < 0\}, \quad \Omega_4 := \{(A,B) : 0 < B < 11/42,\ A \geq 0\},\\
\Omega_5' &:= \{(A,B) : B > 11/42,\ A > 0\}, \quad \Omega_6' := \{(A,B) : B < 0,\ A < 0\}.
\end{aligned}$$

Now we prove the following lemmas:

Lemma 2. *(3.6) has no limit cycle in Ω_1.*

Proof. By equation (3.9), it is clear that $\hat{I}_4(\mu) > 0$ for $\mu > 0$. □

Lemma 3. *(3.6) has a unique limit cycle in $\bigcup_{i=2}^{i=4} \Omega_i$.*

Proof. We consider (3.6) in $\Omega_i,\ i = 2, 3, 4$ separately:

In Ω_2. In this case $\mu_- \leq 0 < \mu_+$. Since μ_+ is a local minimum of $\hat{I}_4'(\mu)$ and $\hat{I}_4'(0) < 0$. This implies that $\hat{I}_4'(\mu)$ has a unique positive zero which is a local minimum of $\hat{I}_4(\mu)$. But $\hat{I}_4^(0) < 0$ therefore $\hat{I}_4(\mu)$ has exactly one positive zero.

In Ω_3. By equation (3.8) it is easy to see that $\hat{I}_4'(\mu) > 0$ for $\mu > 0$. Therefore $\hat{I}_4(\mu)$ can have at most one zero. But $\hat{I}_4(0) < 0$ and $\lim_{\mu\to\infty} \hat{I}_4(\mu) = \infty$. Therefore $\hat{I}_4(\mu)$ have unique positive zero.

In Ω_4. If $B > A^2$ then $\hat{I}_4'(\mu)$ has no critical point and has a unique positive zero. Therefore $\hat{I}_4(\mu)$ has unique positive local minimum. But $\hat{I}_4(0) = 14B - 11/3 < 0$, therefore $\hat{I}_4(\mu)$ will have unique positive zero. If $B < A^2$ then $0 < \mu_- < \mu_+$ and by (3.9)

$$\hat{I}_4'(\mu_-) < 4(-2(A^3 - \Delta^3) - \frac{71}{70}A) < 0,$$

since $0 < B < \frac{11}{42}$ and $A \geq 0$. But $\hat{I}_4'(0) < 0$, therefore $\hat{I}_4'(\mu)$ has unique positive zero and $\hat{I}_4(\mu)$ has a unique positive local minimum. Therefore as previous case $\hat{I}_4(\mu)$ has unique positive zero. □

Now let us define the following region:

$$\Omega_5 \quad := \quad \{(A,B) \in \Omega_5' : \Delta_2 > 0\}$$

We notice that Graph of $\Delta_2 = 0$ and $B = A^2$ intersect each other at a unique point $(A, B) = (\frac{3}{\sqrt{5}}, \frac{9}{5})$ and the region $\{(A,B) \in \Omega_5' : B > A^2\}$ included in Ω_5.

Lemma 4. *System (3.6)has at most two limit cycles in Ω_5.*

Proof. In Ω_5, $\hat{I}_4'(\mu) = 0$ has a unique zero. Since $\hat{I}_4'(0) = -\frac{36A}{5} < 0$, This zero is positive and therefore $\hat{I}_4(\mu) = 0$ has at most two positive zero. Now Let us denote this point by μ_*. By above discussion it is clear that if $\zeta(A, B)$ defined by (3.15) is negative then $\hat{I}_4(\mu)$ will have two positive zero and system (3.6) will have two hyperbolic limit cycles and if it is positive then $\hat{I}_4(\mu)$ will be positive for all positive μ and system (3.6) will not have any limit cycle. Therefore we expect saddle node bifurcation of limit cycle when $\zeta(A, B) = 0$. Also $B = 11/42$ $(a_1 = 0)$ is a Hopf line and as we change a_1 from negative to positive a periodic orbit bifurcated from origin in system (3.6). □

In all cases in Lemma 4, $\hat{I}'_4(\mu)$ has unique zero. Now let us define

$$\begin{aligned}\Omega_{61} &:= \{(A,B) \in \Omega'_6 : A > -\frac{\sqrt{168}}{3} \text{ or } B > B_* \text{ where } B_* \gtrsim -30.96\},\\ \Omega_{62} &:= \{(A,B) \in \Omega'_6 : \Delta_2 > 0\},\ \ \Omega_6 := \Omega_{61} \cup \Omega_{62}.\end{aligned}$$

Lemma 5. *(3.6) has unique limit cycle in* Ω_6.

Proof. We consider (3.6) in regions Ω_{61} and Ω_{62} separately.

In Ω_{61}. Let μ_* be a zero of $\hat{I}'_4(\mu) = 0$ and consider $P_2(\mu_*)$ defined by (3.12). It is clear that in this case $\alpha < 0$, $\beta > 0$, $\gamma < 0$ and the discriminant of $P_2(\mu_*)$ is

$$\begin{aligned}(3.16)\quad \Delta_{P_2} &= B^2(9A^2 - 168) + (786/5A^2 + 43)B - 371/25A^2 - 108/5A^4\\ &= -108/5A^4 + A^2(-371/25 + 786/5B + 9B^2) - 168b^2 + 43B.\end{aligned}$$

But $A^2 \leq 168/9$ and $B < 0$. From first equation of (3.16) it is clear that $\Delta_{P_2} < 0$ since all involving terms are negative. Also using the second equation of (3.16) it is easy to check that $\Delta_{P_2} < 0$ for all $A < 0$ and $B_* < B < 0$. Since $\alpha < 0$ this implies that $P_2(\mu_*) < 0$, and $\hat{I}_4(\mu)$ has exactly one positive zero.

In Ω_{62} In this region $\hat{I}'_4(\mu)$ has a unique zero. But this zero is negative, since $\hat{I}'(0) = -\frac{36A}{5} > 0$. This implies that $\hat{I}'_4(\mu)$ has no positive zero and therefore $\hat{I}_4(\mu)$ has at most one positive zero in this region. On the other hand $\hat{I}_4(\mu)$ has at least one zero in Ω_6, since $\hat{I}_4(0) = 14B - 11/3 < 0$. □

Remark 2. *With more detailed approximation it is possible to enlarge the parameter region where Lemmas 4 and 5 holds. Numerical computation indicates that results of these lemmas 4 and 5 holds in* Ω'_5 *and* Ω'_6 *respectively.*

Now let us define $\Omega = \bigcup_{i=1}^{i=6} \Omega_i$. Using above lemmas we have the following theorem

Theorem 3. *For* $n = 1$ *and* $m = 2$ *and system (3.6) has at most two limit cycles in* Ω. *Further more if it has two limit cycles, smaller one is stable (unstable) and the larger one is unstable(stable) if* $a_5 > 0$ *(*$a_5 < 0$*).*

Proof. The first part of theorem is clear from lemmas $2 - 5$. Now let $0 < \mu_{*,1} < \mu_{*,2}$ are the positive zeros of $\hat{I}_4(\mu)$. From the proof of these lemmas it clear that in all cases $\hat{I}'_4(\mu_{*,1})$ is negative while $\hat{I}'_4(\mu_{*,2})$ is positive, therefore $I'_4(\mu_{*,1})$ $(I'_4(\mu_{*,2}))$ is positive (negative) if $a_5 > 0$ and is negative (positive) if $a_5 < 0$. Now stability of bifurcated limit cycles from period annulus at $h = h_{*,1} = \frac{1}{6}\mu_{*,1}(\mu_{*,1}^2 + 3)$ and $h = h_{*,2} = \frac{1}{6}\mu_{*,2}(\mu_{*,2}^2 + 3)$ follows from lemma 1 and theorem 1. Also let μ_* be the unique positive zero of $\hat{I}_4(\mu) = 0$ where $\hat{I}'_4(\mu_*) \neq 0$. Again from the proof of above lemma it is clear that in all cases $\hat{I}'_4(\mu_*) > 0$. Therefore $I'_4(\mu_*)$ is positive if $a_5 > 0$ and it is negative if $a_5 < 0$. Now stability of the unique limit cycle which is bifurcated at $h = h_* = \frac{1}{6}\mu_*(\mu_*^2 + 3)$ from the periodic annulus follows from lemma 1 and theorem 1. □

Example 4. *Consider system (3.6) with* $B = 3.5$ *and change the value of* A *between* 2.2187377 *and* 2.2187378. *Then number of zeros of* $I_4(\mu)$ *will change between zero and two. This shows that system (3.6) has two hyperbolic limit cycles close to level curves* $H(x, y) = h_{*,1} = 17.81125878$ *and* $H(x, y) = h_{*,2} = 17.81589918$. *The smaller limit cycles is stable while the larger one is unstable. Also we expect to have a non-hyperbolic limit cycle for a* $h_* \in (h_{*,1}, h_{*,2})$.

Acknowledgement. Authors thank Isfahan University of Technology (CEAMA) for support.

References

[1] L. Gavrilov, *Abelian integrals related to Morse polynomials and perturbations of plane Hamiltonian vector fields,* Ann. Inst. Fourier 49 (1999), 611-652.

[2] M. Han, *On the number and distributions of limit cycles in a cubic system*, China Ann. Math. 23A (2) (2002), 143-152.

[3] E. Horozov, I. D. Iliev, *On the number of limit cycles in perturbations of quadratic Hamiltonian systems,* Proc. London Math. Soc. 69 (1994), 198-224.

[4] C. Li, W. Li, J. Llibre, Z. Zhang, *Linear estimation of the number of zeros of Abelian integrals for quadratic isochronous centers,* Nonlinearity 13 (2000), 1775-1800.

[5] C. Li, J. Llibre, Z. Zhang, *Linear estimatation of the number of zeros of Abelian integrals for a class of integrable non- hamiltonian systems,* Nonlinear Anal. 47(2001), 4527-4531.

[6] L. Perko, *Differential Equations and Dynamical Systems,* Springer Verlag, New York 2002.

[7] C. Rousseau and H. Zoladek, *Zeros of complete elliptic integrals for 1:2 resonance,* J. Diff. Eqns. 94 (1991), 42-54.

[8] S. Smale, *M*athematical problems for the next century, The Math. Intell. 20 (2), 7-15; Mathematics: Frontiers and Perspective (2000) 271-294.

[9] C. q. Wu, Y. Xia, *The number of limit cycles of cubic Hamiltonian system with perturbation,* Nonlinear Anal., Real World Appl., in press 2006.

[10] C. X. Yang, R. Q. Wang, *Number and location of limit cycles in a class of perturbed polynomial system,* Acta Math. Appl. Sin. (English Series) 20 (1) (2004), 155-166.

[11] Y. Q. Ye, *The Theory of limit cycles,* Transaction on Mathematical Monographs, Vol. 66, American Mathematical society, New York 1986.

[12] Y. L. Zhao, *Ablian integrals for cubic Hamilton vector fileds,* Ph.D. Thesis in Peking University, 1998.

Authors' address:

Ali Atabaigi, Nemat Nyamoradi and Hamid R. Z. Zangeneh
Department of Mathematical Sciences,
Isfahan University of Technology,
Isfahan, 84156-83111, Iran.
E-mail addresses: aelmi@math.iut.ac.ir, nyamoradi@math.iut.ac.ir, hamidz@math.iut.ac.ir