Homogeneous Lorentzian 3-manifolds with a parallel null vector field

W. Batat, G. Calvaruso and B. De Leo

Abstract. Lorentzian three-manifolds admitting a parallel null vector field have been intensively studied, since they possess several interesting geometrical properties which do not have a Riemannian counterpart. We completely classify homogeneous structures on Lorentzian three-manifolds which admit a parallel null vector field. This leads to the full classification of locally homogeneous examples within this class of manifolds.

M.S.C. 2000: 53C50,53C20, 53C30.
Key words: Lorentzian manifolds, homogeneous structures.

1 Introduction

The existence of parallel vector fields has strong and interesting consequences on the geometry of a manifold. If a Riemannian manifold (M, g) admits such a vector field, then (M, g) is locally reducible. The same property remains true for a pseudo-Riemannian manifold admitting a parallel non-null vector field. However, in the pseudo-Riemannian framework, a peculiar phenomenon arises: it can exist a parallel null vector field.

The geometry of a Lorentzian three-manifolds admitting a parallel null vector field has been studied in the fundamental paper [3]. These manifolds possess several interesting geometrical properties which do not have analogues in Riemannian settings. They are described in terms of a suitable system of local coordinates (t, x, y) and form a quite large class, depending on an arbitrary two-variables function $f(x, y)$. The Levi-Civita connection and curvature of (M, g_f) are completely described and several geometric consequences are deduced. In particular, locally symmetric examples are classified in terms of the defining function f, as well as curvature homogeneous examples with diagonal Ricci operator. It is then natural to ask the following

QUESTION: *When (M, g_f) is locally homogeneous?*

The purpose of this paper is to answer the question above. It is worthwhile to remark that, differently from the Riemannian case, scalar curvature invariants do not help

Balkan Journal of Geometry and Its Applications, Vol.14, No.1, 2009, pp. 11-20.

here to determine the locally homogeneous examples. In fact, all scalar curvature invariants of a Lorentzian three-space (M, g_f) admitting a parallel null vector field vanish identically ([3], p.844).

In order to determine the locally homogeneous spaces of the form (M, g_f), we shall make use of the notion of homogeneous structure. Homogeneous pseudo-Riemannian structures were introduced by Gadea and Oubiña in [5], to obtain a characterization of reductive homogeneous pseudo-Riemannian manifolds, similar to the one given for homogeneous Riemannian manifolds by Ambrose and Singer [1] (see also [8]). In dimension three, as a consequence of the characterization proved by the second author in [2], a homogeneous Lorentzian space is necessarily reductive. Hence, the existence of a homogeneous structure on a Lorentzian three-space is a necessary and sufficient condition for local homogeneity. Recent complementary results about Lorentzian manifolds can be found in [6],[7].

In Section 2, we shall recall the description of Lorentzian three-manifolds (M, g_f) admitting a parallel null vector field and the definition and basic properties of homogeneous (pseudo-Riemannian) structures. Then, in Section 3 we shall write down and solve the system of partial differential equations determining a homogeneous structure on (M, g_f) in terms of its local components and we shall give the complete classification of locally homogeneous spaces (M, g_f).

2 Preliminaries

We start with a short description of Lorentzian three-manifolds admitting a parallel null vector field, referring to [3] for more details and further results. Such a manifold (M, g) admits a system of canonical local coordinates (t, x, y), adapted to a parallel plane field in such a way that $\frac{\partial}{\partial_t}$ is the parallel null vector field, and there exists a smooth function $f = f(x, y)$, such that the Lorentzian metric tensor $g = g_f$ is described in local coordinates as follows:

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}, \tag{2.1}$$

where $\varepsilon = \pm 1$. In the sequel, we shall denote by (M, g_f) this Lorentzian manifold. In [3], a general description was provided for the wider class of Lorentzian three-manifolds admitting a parallel degenerate line field, that is, having a null vector field u such that $\nabla u = \omega \otimes u$ for a suitable 1-form ω. In this more general case, the function f occurring in (2.1) also depends on t. Restricting ourselves to the case when $f = f(x, y)$, we have that the Levi-Civita connection ∇ of (M, g_f) is completely determined by

$$\nabla_{\partial_x}\partial_y = \frac{1}{2} f_x \partial_t, \qquad \nabla_{\partial_y}\partial_y = \frac{1}{2} f_y \partial_t - \frac{\varepsilon}{2} f_x \partial_x, \tag{2.2}$$

where we put $\partial_t = \frac{\partial}{\partial_t}$, $\partial_x = \frac{\partial}{\partial_x}$ and $\partial_y = \frac{\partial}{\partial_y}$.

Next, the only non-vanishing local components of the curvature tensor are described by

$$R(\partial_x, \partial_y)\partial_x = -\frac{1}{2} f_{xx} \partial_t, \qquad R(\partial_x, \partial_y)\partial_y = \frac{\varepsilon}{2} f_{xx} \partial_x. \tag{2.3}$$

Since M is three-dimensional, its curvature is completely determined by the Ricci tensor ϱ. By (2.3) it easily follows that the local components of the Ricci tensor are given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{\varepsilon}{2} f_{xx} \end{pmatrix}. \tag{2.4}$$

In particular, (2.4) yields that (M, g_f) is flat if and only if $f_{xx} = 0$.

We now recall the definition and basic properties of homogeneous pseudo-Riemannian structures.

Definition 2.1. **[5]** *Let* (M, g) *be a pseudo-Riemannian manifold. A* homogeneous pseudo-Riemannian structure *is a tensor field* T *of type* $(1, 2)$ *on* M*, such that the connection* $\tilde{\nabla} = \nabla - T$ *satisfies*

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} T = 0.$$

More explicitly, T is the solution of the following system of equations (known as *Ambrose-Singer equations*):

$$g(T_X Y, Z) + g(Y, T_X Z) = 0, \tag{2.5}$$

$$(\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}, \tag{2.6}$$

$$(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}, \tag{2.7}$$

for all vector fields X, Y, Z. The geometric meaning of the existence of a homogeneous pseudo-Riemannian structure is explained by the following

Theorem 2.2. **[5]** *A connected, simply connected and complete pseudo-Riemannian manifold* (M, g) *admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

If at least one of the hypotheses of connectedness, simply connectedness or completeness is lacking in Theorem 2.2, then the existence of a solution of (2.5)-(2.7) implies that (M, g) is locally isometric to a reductive homogeneous space (and so, locally homogeneous).

A three-dimensional homogeneous Lorentzian manifold is necessarily reductive. This was proved in [4] and also follows independently from the classification the second author gave in [2]. Therefore, the existence of a homogeneous structure on a Lorentzian three-manifold is a necessary and sufficient condition for its local homogeneity.

3 Homogeneous structures on (M, g_f)

In the sequel, we shall assume (M, g_f) non-flat, that is, $f_{xx} \neq 0$. A homogeneous Lorentzian structure T on (M, g_f) is uniquely determined by its local components T_{ij}^k

with respect to the coordinate vector fields $\{\partial_1 = \partial_t, \partial_2 = \partial_x, \partial_3 = \partial_y\}$. Functions T_{ij}^k are defined by

$$T(\partial_i, \partial_j) = \sum_{k=1}^{3} T_{ij}^k \partial_k,$$

for all indices i, j, k. Writing the Ambrose-Singer equations (2.5)-(2.7) for the coordinate vector fields ∂_i, we obtain the equivalent system of equations

$$T_{ij}^r g_{rk} + T_{ik}^r g_{rj} = 0, \tag{3.1}$$

$$\nabla_i \varrho_{jk} = -T_{ij}^r \varrho_{rk} - T_{ik}^s \varrho_{js}, \tag{3.2}$$

$$(\nabla_i T)_{\partial_j} = T_{\partial_i} T_{\partial_j} - T_{\partial_j} T_{\partial_i} - T_{T_{\partial_i} \partial_j}, \tag{3.3}$$

for all indices i, j, k. Note that in (3.2) we took into account the fact that the curvature is completely determined by the Ricci tensor ϱ. Using (2.1), (2.2) and (2.4), from (3.1) and (3.2) we get

$$\begin{cases} T_{\partial_t} \partial_t = 0, \\ T_{\partial_i} \partial_t = -T_{i3}^3 \partial_t, \quad i = 2, 3, \\ T_{\partial_i} \partial_x = -\varepsilon T_{i3}^2 \partial_t, \quad i = 1, 2, 3, \\ T_{\partial_t} \partial_y = T_{13}^2 \partial_x, \\ T_{\partial_i} \partial_y = -f T_{i3}^3 \partial_t + T_{i3}^2 \partial_x + T_{i3}^3 \partial_y, \quad i = 2, 3, \\ T_{23}^3 = -\frac{f_{xxx}}{2 f_{xx}}, \\ T_{33}^3 = -\frac{f_{xxy}}{2 f_{xx}}. \end{cases} \tag{3.4}$$

From (3.3), we obtain the following system

$$\begin{cases} \partial_t(T_{13}^2) = \partial_x(T_{13}^2) = \partial_y(T_{13}^2) = 0, \\ \partial_t(T_{23}^3) = \partial_x(T_{23}^3) = \partial_y(T_{23}^3) = 0, \\ \partial_t(T_{23}^2) = T_{13}^2(T_{23}^3 + \varepsilon T_{13}^2), \\ \partial_x(T_{23}^2) = \varepsilon T_{23}^2 T_{13}^2, \\ \partial_y(T_{23}^2) - \frac{f_x}{2} T_{13}^2 - \varepsilon \frac{f_x}{2} T_{23}^3 = T_{23}^3 T_{33}^2 - T_{33}^3 T_{23}^2 + \varepsilon T_{33}^2 T_{13}^2, \\ \partial_t(T_{33}^2) = T_{13}^2(T_{33}^3 - T_{23}^2), \\ \partial_x(T_{33}^2) - \frac{f_x}{2} T_{13}^2 = T_{33}^3 T_{23}^2 - 2 T_{23}^3 T_{33}^2 + f T_{23}^3 T_{13}^2 - \left(T_{23}^2\right)^2, \\ \partial_y(T_{33}^2) - \frac{f_y}{2} T_{13}^2 + \varepsilon \frac{f_x}{2}(T_{23}^2 - T_{33}^3) = f T_{13}^2 T_{33}^3 - T_{33}^2 T_{23}^2 - T_{33}^3 T_{33}^2, \\ \partial_t(T_{33}^3) = -T_{13}^2 T_{23}^3, \\ \partial_x(T_{33}^3) = -T_{23}^3(T_{23}^2 + T_{33}^3), \\ \partial_y(T_{33}^3) + \varepsilon \frac{f_x}{2} T_{23}^3 = -T_{33}^2 T_{23}^3 - \left(T_{33}^3\right)^2. \end{cases} \tag{3.5}$$

We note that T_{13}^2 and T_{23}^3 are real constants; hence, by the sixth equation in (3.4), $\frac{f_{xxx}}{f_{xx}}$ must be constant. Now, put

$$T_{13}^2 = \alpha, \quad T_{23}^3 = \beta = -\frac{f_{xxx}}{2f_{xx}}, \quad T_{23}^2 = U, \quad T_{33}^2 = V, \quad T_{33}^3 = W = -\frac{f_{xxy}}{2f_{xx}}, \tag{3.6}$$

where α, β are real constants and U, V, W are real-valued smooth functions on M. Then, (3.5) becomes

$$\begin{cases} \partial_t U = \alpha(\beta + \varepsilon\alpha), \\ \partial_x U = \varepsilon\alpha U, \\ \partial_y U = \frac{f_x}{2}(\alpha + \varepsilon\beta) + V(\varepsilon\alpha + \beta) - UW, \\ \partial_t V = \alpha(W - U), \\ \partial_x V = \alpha\frac{f_x}{2} + U(W - U) + \beta(\alpha f - 2V), \\ \partial_y V = \varepsilon\frac{f_x}{2}(W - U) + \alpha\frac{f_y}{2} + (\alpha f - V)W - UV, \\ \partial_t W = -\alpha\beta, \\ \partial_x W = -\beta(U + W), \\ \partial_y W = -\varepsilon\beta\frac{f_x}{2} - \beta V - W^2. \end{cases} \tag{3.7}$$

Since W depends only on x and y, it follows from (3.7) that $\alpha\beta = 0$, and (3.7) becomes

$$\begin{cases} \partial_t U = \varepsilon\alpha^2, \\ \partial_x U = \varepsilon\alpha U, \\ \partial_y U = \frac{f_x}{2}(\alpha + \varepsilon\beta) + V(\varepsilon\alpha + \beta) - UW, \\ \partial_t V = \alpha(W - U), \\ \partial_x V = \alpha\frac{f_x}{2} + U(W - U) - 2\beta V, \\ \partial_y V = \varepsilon\frac{f_x}{2}(W - U) + \alpha\frac{f_y}{2} + (\alpha f - V)W - UV, \\ \partial_t W = 0, \\ \partial_x W = -\beta(U + W), \\ \partial_y W = -\varepsilon\beta\frac{f_x}{2} - \beta V - W^2. \end{cases} \tag{3.8}$$

Next, we derive the first and the second equation of (3.8) for x and t, respectively, obtaining

$$\partial_x\partial_t U = 0, \qquad \partial_t\partial_x U = \alpha^3.$$

Therefore, $\alpha = 0$ and (3.8) reduces to

$$\begin{cases} \partial_t U = 0, \\ \partial_x U = 0, \\ \partial_y U = \frac{\varepsilon\beta}{2} f_x + \beta V - UW, \\ \partial_t V = 0, \\ \partial_x V = U(W - U) - 2\beta V, \\ \partial_y V = \frac{\varepsilon}{2} f_x(W - U) - V(W + U), \\ \partial_t W = 0, \\ \partial_x W = -\beta(U + W), \\ \partial_y W = -\frac{\varepsilon\beta}{2} f_x - \beta V - W^2. \end{cases} \tag{3.9}$$

We now calculate $\partial_x\partial_y U$ by the third equation of (3.9) and, taking into account the second equation and the expression of $\partial_x V$ and $\partial_x W$, we get

$$\beta\left(\frac{\varepsilon}{2} f_{xx} + 2UW - 2\beta V\right) = 0. \tag{3.10}$$

Moreover, deriving the last two equations of (3.9) respect to y and x, respectively and using the expression of $\partial_y U, \partial_x V, \partial_x W$ and $\partial_y W$, by the identity $\partial_x \partial_y W = \partial_y \partial_x W$ we get

$$\beta\left(-\frac{\varepsilon}{2} f_{xx} + U^2 + 2\beta V + W^2\right) = 0. \tag{3.11}$$

Summing up (3.10) and (3.11), then we obtain

$$\beta\,(U+W)^2 = 0.$$

This leads to distinguish two cases:

First Case: $\beta = 0$.
In this case, f_{xx} only depends on y and the system (3.9) reduces to

$$\left\{\begin{array}{lll} \partial_t U = 0, & \partial_x U = 0, & \partial_y U = -UW, \\ \partial_t V = 0, & \partial_x V = U(W-U), & \partial_y V = \frac{\varepsilon}{2} f_x (W-U) - V\,(W+U)\,, \\ \partial_t W = 0, & \partial_x W = 0, & \partial_y W = -W^2, \end{array}\right. \tag{3.12}$$

By the last equation of (3.12), we can distinguish two subcases:

I a): If $W = 0$: we have that f_{xx} is constant. Thus,

$$f(x,y) = \frac{\theta}{2} x^2 + F(y)x + G(y), \tag{3.13}$$

where $\theta \neq 0$ is a real constant and F, G are smooth functions. Moreover, $U = u_o$, with u_o a real constant, and the system (3.12) becomes

$$\left\{\begin{array}{l} \partial_x V = -u_o^2, \\ \partial_y V = -\frac{\varepsilon u_o}{2} f_x - u_o V. \end{array}\right. \tag{3.14}$$

Derive the first equation of (3.14) with respect to y and the second respect to x. Since $\partial_y \partial_x V = \partial_y \partial_y V$ we get

$$u_o\left(u_o^2 - \frac{\varepsilon}{2} f_{xx}\right) = 0.$$

If $u_o = 0$, we obtain $V = v_o$, with v_o a real constant, $U = W = 0$ and $f(x,y)$ determined by (3.13).
If $u_o \neq 0$, then $f_{xx} = 2\varepsilon u_o^2$ and so, in (3.13) we have $\theta = 2\varepsilon u_o^2$. The system (3.14) implies that $V(x,y) = -u_o^2 x + H(y)$, with

$$H'(y) + u_o H(y) + \frac{\varepsilon u_o}{2} F(y) = 0, \tag{3.15}$$

where H is a smooth function.

I b): If $W \neq 0$, the last equation of (3.12) gives

$$W(y) = \frac{1}{y+k}, \tag{3.16}$$

where k is a real constant. On the other hand, $W = -\frac{f_{xxy}}{2 f_{xx}}$, thus

$$f(x,y) = \frac{s}{2(y+k)^2}x^2 + P(y)x + Q(y), \tag{3.17}$$

with $s \neq 0$ a real constant (since $f_{xx} \neq 0$). Next, from the third equation of (3.12) and by (3.16), if $U \neq 0$, we deduce

$$U = \frac{r}{y+k}, \tag{3.18}$$

for a real constant $r \neq 0$. On the other hand, the case $U = 0$ can not occur. In fact, if $U = 0$ then by (3.12) we get

$$\begin{cases} \partial_t V = 0, \\ \partial_x V = 0, \\ \partial_y V = (\frac{\varepsilon}{2} f_x - V)W. \end{cases}$$

Deriving the last equation of the above system by x and using the fact that $V = V(y)$ and $W = W(y)$, we obtain $f_{xx}W = 0$, which is a contradiction, since (M, g) is not flat and $W \neq 0$. Now, from (3.12), we calculate

$$\begin{aligned} \partial_y \partial_x V &= 2UW(U-W), \\ \partial_x \partial_y V &= (U-W)\left(U(U+W) - \frac{\varepsilon}{2} f_{xx}\right), \end{aligned}$$

which, together with (3.16), (3.17) and (3.18), imply

$$(r-1)(2r^2 - 2r - \varepsilon s) = 0.$$

So, if $r = 1$ then $U(y) = W(y) = \frac{1}{y+k}$ and from (3.12) we easily get $V(y) = \frac{\sigma}{(y+k)^2}$, for a real constant σ, and f is given by (3.17).

If $r = \frac{1 \pm \sqrt{1+2\varepsilon s}}{2}$, with $1 + 2\varepsilon s \geq 0$ and $s \neq 0$, because $W(y) = \frac{1}{y+k}$, (3.12) becomes

$$\begin{cases} \partial_t V = 0, \\ \partial_x V = -\frac{\varepsilon s}{2(y+k)^2}, \\ \partial_y V = \frac{\varepsilon\left(1 \mp \sqrt{1+2\varepsilon s}\right)}{4(y+k)} f_x - \frac{\left(3 \pm \sqrt{1+2\varepsilon s}\right)}{2(y+k)} V. \end{cases} \tag{3.19}$$

The second equation in (3.19) gives

$$V(x,y) = -\frac{\varepsilon s}{2\,(y+k)^2}x + L(y), \tag{3.20}$$

where $L = L(y)$ is a smooth function. Next, deriving the above expression of V by y, we find

$$\partial_y V = \frac{\varepsilon s}{(y+k)^3}x + L'(y). \tag{3.21}$$

We compare (3.21) together with the third equation in (3.19). Taking into account the expression of f_x deduced by (3.17), we obtain

$$L'(y) + \frac{\left(3 \pm \sqrt{1+2\varepsilon s}\right)}{2\,(y+k)} L(y) = \frac{\varepsilon\left(1 \mp \sqrt{1+2\varepsilon s}\right)}{4\,(y+k)} P(y). \tag{3.22}$$

Then, V is given by (3.20), under the condition that (3.22) is satisfied.

Second Case: $\beta \neq 0$.
In this case $W = -U = -\frac{f_{xxy}}{2f_{xx}}$ and the system (3.9) becomes

$$\begin{cases} \partial_y U = \frac{\varepsilon\beta}{2} f_x + \beta V + U^2, \\ \partial_t V = 0, \\ \partial_x V = -2U^2 - 2\beta V, \\ \partial_y V = -\varepsilon f_x U. \end{cases} \tag{3.23}$$

Hence, U only depends on y. The first equation of (3.23), derived with respect to the x, gives

$$\partial_x V = -\frac{\varepsilon}{2} f_{xx}. \tag{3.24}$$

We compare the above formula with the third equation of (3.23), getting

$$V = \frac{1}{\beta}\left(\frac{\varepsilon}{4} f_{xx} - U^2\right).$$

Now, taking into account the first and the third equation of (3.23), and making use of the (3.24), we get

$$2U_y = \frac{\varepsilon}{2}(f_{xx} + 2\beta f_x). \tag{3.25}$$

Since $\frac{f_{xxx}}{2f_{xx}} = -\beta$, we can integrate with respect to the x and we obtain

$$f_{xx} = \pm e^{M(y)-2\beta x},$$

where $M = M(y)$ is a smooth function. Again integrating with respect to the x, we find

$$f_x(x,y) = \mp\frac{e^{M(y)-2\beta x}}{2\beta} + N(y), \tag{3.26}$$

with N a smooth function only depending on y. Taking into account (3.25), since $U = \frac{f_{xxy}}{2f_{xx}}$, it holds:

$$M''(y) = \varepsilon\beta N(y). \tag{3.27}$$

Thus, integrating again (3.26) with respect to the x, we obtain

$$f(x,y) = \pm\frac{e^{M(y)-2\beta x}}{4\beta^2} + \frac{\varepsilon}{\beta} M''(y)x + R(y), \tag{3.28}$$

where $R = R(y)$ is a smooth function. Therefore, calculations above lead to the following

Theorem 3.1. *Let (M, g_f) be a non-flat Lorentzian three-space admitting a parallel null vector field. (M, g_f) is locally homogeneous if and only for its defining function f, one of the following statements holds true:*

(i) (locally symmetric case) there exist a real constant $\theta \neq 0$ *and two one-variable smooth functions* F *and* G*, such that*

$$f(x,y) = \frac{\theta}{2}x^2 + F(y)x + G(y);$$

(ii) there exist a real constant $s \neq 0$ *and two one-variable smooth functions* P *and* Q*, such that*

$$f(x,y) = \frac{s}{2(y+k)^2}x^2 + P(y)x + Q(y);$$

(iii) there exist two one-variable smooth functions M *and* R*, such that*

$$f(x,y) = \pm\frac{e^{M(y)-2\beta x}}{4\beta^2} + \frac{\varepsilon}{\beta}M''(y)x + R(y).$$

Notice that in order to prove Theorem 3.1, we completely solved the system (3.1)-(3.3). Thus, we also obtained the complete classification of homogeneous structures on (M, g_f), summarized in the following

Proposition 3.2. *Let* (M, g_f) *be a non-flat locally homogeneous Lorentzian three-space admitting a parallel null vector field. All and the ones homogeneous structures on* (M, g_f) *are determined (through their local components* T_{ij}^k *with respect to the coordinate vector fields) by* (3.6)*, where* $\alpha = 0$*,* β *is a real constant and* U, V, W *are smooth real-valued functions for which one of the following statements holds true:*

(i) If f *satisfies* (3.13)*, then* $\beta = 0$*,* $W = 0$ *and*

- *either* $U = 0$ *and* $V = v_0$ *is constant, or*
- $U = u_0 \neq 0$ *is constant and* $V(x,y) = -u_0^2 x + H(y)$*, for a smooth function* H *satisfying* (3.15)*. In this case,* $\theta = 2\varepsilon u_0^2$ *in* (3.13)*.*

(ii) If f *satisfies* (3.17)*, then*

- *either there exist two real constants* k *and* σ*, such that* $U(y) = W(y) = \frac{1}{y+k}$ *and* $V(y) = \frac{\sigma}{(y+k)^2}$*, or*
- *there exist two real constants* k *and* $s \neq 0$*, such that* $U = \frac{1 \pm \sqrt{1+2\varepsilon s}}{2(y+k)}$*,* $W(y) = \frac{1}{y+k}$ *and* $V(x,y) = -\frac{\varepsilon s}{2(y+k)^2}x + L(y)$*, for a smooth function* L *satisfying* (3.22)*.*

(iii) If f *satisfies* (3.28)*, then* $\beta \neq 0$*,* $U(y) = -W(y) = \frac{M'(y)}{2}$ *and* $V(x,y) = \frac{1}{4\beta}(\pm\varepsilon e^{M(y)-2\beta x} - M'(y)^2)$*, for a smooth function* M*.*

Aknowledgements: The first author was supported by ENSET d'Oran. The second and the third authors were supported by MURST and the University of Salento.

References

[1] W. Ambrose and I.M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. 25 (1958), 647–669.

[2] G. Calvaruso, *Homogeneous structures on three-dimensional Lorentzian manifolds*, J. Geom. Phys. 57 (2007), 1279–1291.

[3] M. Chaichi, E. García-Río and M.E. Vázquez-Abal, *Three-dimensional Lorentz manifolds admitting a parallel null vector field*, J. Phys. A: Math. Gen. 38 (2005), 841–850.

[4] M.E. Fels and A.G. Renner, *Non-reductive homogeneous pseudo-Riemannian manifolds of dimension four*, Canad. J. Math. (2) 58 (2006), 282–311.

[5] P.M. Gadea and J.A. Oubina, *Homogeneous pseudo-Riemannian structrues and homogeneous almost para-Hermitian structures*, Houston J. Math. (3) 18 (1992), 449–465.

[6] I.E. Hirica and L. Nicolescu, *Conformal connections on Lyra manifolds*, Balkan J. Geom. Appl. (2) 13 (2008), 43–49.

[7] G. Shabbir, *Proper affine vector fields in Bianchi type I space-time*, Differ. Geom. Dyn. Syst. 9 (2007), 138–148.

[8] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lect. Notes 83, Cambridge Univ. Press, 1983.

Authors' addresses:

Wafaa Batat
École Normale Superieure de L'Enseignement Technique d'Oran,
Département de Mathématiques et Informatique,
Oran, Algérie.
E-mail: wafa.batat@enset-oran.dz

Giovanni Calvaruso
Dipartimento di Matematica "E. De Giorgi",
Università del Salento
Lecce, Italy.
E-mail: giovanni.calvaruso@unile.it

Barbara De Leo
Dipartimento di Matematica "E. De Giorgi",
Università del Salento,
Lecce, Italy.
E-mail: barbara.deleo@unile.it