$PDI\&PDE$-constrained optimization problems with curvilinear functional quotients as objective vectors

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Abstract. In this work we introduce and perform a study on the multi-time multi-objective fractional variational problem of minimizing a vector of quotients of path independent curvilinear integral functionals (MFP) subject to certain partial differential equations (PDE) and/or partial differential inequations (PDI), using a geometrical language. The paper is organized as follows: §1 formulates a $PDI\&PDE$-constrained optimization problem. §2 states and proves necessary conditions for the optimality of the problem (MP) of minimizing a vector of path independent curvilinear integral functionals constrained by $PDIs$ and $PDEs$. §3 analyzes necessary efficiency conditions for the problem (MFP), and §4 studies different types of dualities.

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1 $PDI\&PDE$-constrained optimization problem

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n, respectively. The local coordinates on T and M will be written $t = (t^\alpha)$ and $x = (x^i)$, respectively. Let $J^1(T, M)$ be the first order jet bundle associated to T and M.

To develop our theory, we recall the following relations between two vectors $v = (v^j)$ and $w = (w^j)$, $j = \overline{1, a}$:

$$\begin{aligned}
&v = w \ \Leftrightarrow\ v^j = w^j, \quad j = \overline{1, a}\,;\\
&v < w \ \Leftrightarrow\ v^j < w^j, \quad j = \overline{1, a}\,;\\
&v \leqq w \ \Leftrightarrow\ v^j \leq w^j, \quad j = \overline{1, a} \quad \text{(product order relation)}.\\
&v \leq w \ \Leftrightarrow\ v \leqq w \text{ and } v \neq w.
\end{aligned}$$

Using the product order relation on $\mathbf{R}^p$, the hyperparallelepiped $\Omega_{t_0,t_1} \subset \mathbf{R}^p$, with the diagonal opposite points $t_0 = (t_0^1, \ldots, t_0^p)$ and $t_1 = (t_1^1, \ldots, t_1^p)$, can be written as the interval $[t_0, t_1]$.

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Now, we introduce the C^∞-class Lagrange 1-forms densities

$$f_\alpha = (f_\alpha^\ell)\colon J^1(T,M) \to \mathbf{R}^r, \quad k_\alpha = (k_\alpha^\ell)\colon J^1(T,M) \to \mathbf{R}^r, \quad \ell = \overline{1,r}, \quad \alpha = \overline{1,p}.$$

Suppose that $D_\beta f_\alpha^\ell = D_\alpha f_\beta^\ell$, and $D_\beta k_\alpha^\ell = D_\alpha k_\beta^\ell$, $\alpha,\beta = \overline{1,p}$, $\alpha \neq \beta$, $\ell = \overline{1,r}$, where D_β is the total derivative (closeness conditions, complete integrability conditions) and

$$\int_{\gamma_{t_0,t_1}} k_\alpha^\ell(t, x(t), x_\gamma(t))\, dt^\alpha > 0,$$

where $x_\gamma(t) = \dfrac{\partial x}{\partial t^\gamma}(t)$, $\gamma = \overline{1,p}$, are partial velocities and γ_{t_0,t_1} is a piecewise C^1-class curve joining the points t_0 and t_1. The closed Lagrange 1-forms densities f_α^ℓ and k_α^ℓ will be used to define certain quotients of curvilinear integral functionals. Also we accept that the Lagrange matrix density

$$g = (g_a^b)\colon J^1(T,M) \to \mathbf{R}^{ms}, \quad a = \overline{1,s}, \quad b = \overline{1,m}, \quad m < n,$$

of C^∞-class defines the partial differential inequations (PDI) (of evolution)

$$g(t, x(t), x_\gamma(t)) \leqq 0, \quad t \in \Omega_{t_0,t_1}, \tag{1.1}$$

and the Lagrange matrix density

$$h = (h_a^b)\colon J^1(T,M) \to \mathbf{R}^{qs}, \quad a = \overline{1,s}, \quad b = \overline{1,q},\ q < n,$$

defines the partial differential equation (PDE) (of evolution)

$$h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0,t_1}. \tag{1.2}$$

The purpose of this work is to study the multitime multi-objective fractional variational problem of minimizing a vector of quotients of path independent curvilinear functionals

$$\left(\frac{\displaystyle\int_{\gamma_{t_0,t_1}} f_\alpha^1(t, x(t), x_\gamma(t))\, dt^\alpha}{\displaystyle\int_{\gamma_{t_0,t_1}} k_\alpha^1(t, x(t), x_\gamma(t))\, dt^\alpha}, \dots, \frac{\displaystyle\int_{\gamma_{t_0,t_1}} f_\alpha^r(t, x(t), x_\gamma(t))\, dt^\alpha}{\displaystyle\int_{\gamma_{t_0,t_1}} k_\alpha^r(t, x(t), x_\gamma(t))\, dt^\alpha} \right),$$

knowing that the function $x(t)$ satisfies the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$, or $x(t)|_{\partial\Omega_{t_0,t_1}} =$ given, the partial differential inequations of evolution (1.1), and the partial differential equation of evolution (1.2). Such a problem is called *PDI&PDE-constrained optimization problem* (see also [2]-[8], [10], [11], [19], [20]).

Introducing the notations

$$F^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f_\alpha^\ell(t, x(t), x_\gamma(t))\, dt^\alpha, \quad K^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} k_\alpha^\ell(t, x(t), x_\gamma(t))\, dt^\alpha,$$

the above-mentioned extremizing problem can be written

$$(MFP)\begin{cases}\min\limits_{x(\cdot)} & \left(\dfrac{F^1(x(\cdot))}{K^1(x(\cdot))},\dots,\dfrac{F^r(x(\cdot))}{K^r(x(\cdot))}\right)\\ & \text{subject to}\\ & x(t_0)=x_0,\quad x(t_1)=x_1,\\ & g(t,x(t),x_\gamma(t))\leqq 0,\quad t\in\Omega_{t_0,t_1},\\ & h(t,x(t),x_\gamma(t))=0,\quad t\in\Omega_{t_0,t_1}.\end{cases}$$

Let $C^\infty(\Omega_{t_0,t_1},M)$ be the space of all functions $x\colon \Omega_{t_0,t_1}\to M$ of C^∞-class, with the norm

$$\|x\|=\|x\|_\infty+\sum_{\alpha=1}^{p}\|x_\alpha\|_\infty.$$

The set

$$\mathcal{F}(\Omega_{t_0,t_1})=\{x\in C^\infty(\Omega_{t_0,t_1},M)\,|\,x(t_0)=x_0,\ x(t_1)=x_1,\ g(t,x(t),x_\gamma(t))\leqq 0,\\ h(t,x(t),x_\gamma(t))=0,\ t\in\Omega_{t_0,t_1}\}$$

is called the *set of all feasible solutions* of the problem (MFP).

Partial differential inequations/equations mathematically represent a multitude of natural phenomena, and in turn, applications in science and engineering ubiquitously give rise to problems formulated as $PDI\&PDE$-constrained optimization. The areas of research who strongly motivate the $PDI\&PDE$-constrained optimization include: shape optimization in fluid mechanics and medicine, material inversion - in geophysics, data assimilation in regional weather prediction modelling, structural optimization, and optimal control of processes. $PDI\&PDE$-constrained optimization problems are generally infinite dimensional in nature, large and complex. As a result, this class of optimization problems present significant reasoning and computational challenges, many of which have been studied in recent years in Germany, USA, Romania, etc. As computing power grows and optimization techniques become more advanced, one wonders whether there are enough commonalities among $PDI\&PDE$-constrained optimization problems from different fields to develop ratiocinations and algorithms for more than a single application. This question has been the topic of many papers, conferences and recent scientific grants.

The basic optimization problems of path independent curvilinear integrals with PDE constraints or with isoperimetric constraints, expressed by the multiple integrals or path independent curvilinear integrals, were stated for the first time in our works [12]-[18]. The papers [15], [17], [18] focuss on *multitime maximum principle* in multitime optimal control problems.

2 Necessary conditions of optimality

In order to obtain necessary conditions for the optimality of the problem (MFP), we start with a vector of path independent curvilinear functionals,

$$F(x(\cdot))=\int_{\gamma_{t_0,t_1}} f_\alpha(t,x(t),x_\gamma(t))\,dt^\alpha=\left(\int_{\gamma_{t_0,t_1}} f^1_\alpha(t,x(t),x_\gamma(t))\,dt^\alpha,\dots,\right.\\ \left.\int_{\gamma_{t_0,t_1}} f^r_\alpha(t,x(t),x_\gamma(t))\,dt^\alpha\right)=(F^1(x(\cdot)),\dots,F^r(x(\cdot))),$$

and we formulate a simplified $PDI\&PDE$-constrained minimum problem

$$(MP)\begin{cases}\min\limits_{x(\cdot)} \quad F(x(\cdot)) \\ \qquad \text{subject to} \\ \qquad x(t_0)=x_0, \quad x(t_1)=x_1, \\ \qquad g(t,x(t),x_\gamma(t)) \leqq 0, \quad t\in\Omega_{t_0,t_1}, \\ \qquad h(t,x(t),x_\gamma(t)) = 0, \quad t\in\Omega_{t_0,t_1}.\end{cases}$$

We are interested in finding necessary conditions for the optimality, respectively efficiency conditions, for the problem (MP) in the domain $\mathcal{F}(\Omega_{t_0,t_1})$.

Definition 2.1. A feasible solution $x^\circ(\cdot)\in\mathcal{F}(\Omega_{t_0,t_1})$ is called *efficient point* for the program (MP) if and only if for any feasible solution $x(\cdot)\in\mathcal{F}(\Omega_{t_0,t_1})$, the inequality $F(x(\cdot))\leqq F(x^\circ(\cdot))$ implies the equality $F(x(\cdot))=F(x^\circ(\cdot))$.

To analyze the previous problem, we start with the case of a single functional. Let $s=(s_\alpha)\colon J^1(\Omega_{t_0,t_1},M)\to\mathbf{R}^p$ be a closed Lagrange 1-form density of C^∞-class which produces the action $S(x(\cdot))=\displaystyle\int_{\gamma_{t_0,t_1}} s_\beta(t,x(t),x_\gamma(t))\,dt^\beta$. Consider the following $PDI\&PDE$-constrained variational problem

$$(SP)\begin{cases}\min\limits_{x(\cdot)} \quad S(x(\cdot)) \\ \qquad \text{subject to} \\ \qquad g(t,x(t),x_\gamma(t)) \leqq 0, \quad t\in\Omega_{t_0,t_1}, \\ \qquad h(t,x(t),x_\gamma(t)) = 0, \quad t\in\Omega_{t_0,t_1}.\end{cases}$$

We define the auxiliary Lagrange density 1-form $L=(L_\alpha)$ as

$$\begin{aligned}L_\alpha(t,x(t),x_\gamma(t),\lambda,\mu(t),\nu(t)) &= \lambda s_\alpha(t,x(t),x_\gamma(t)) + <\mu_\alpha(t),g(t,x(t),x_\gamma(t))> \\ &\quad + <\nu_\alpha(t),h(t,x(t),x_\gamma(t))>,\ \alpha=\overline{1,p},\end{aligned}$$

where λ is real number and $\mu(t)=(\mu_\alpha(t))=(\mu^a_{\alpha b}(t))$, $\nu=(\nu_\alpha(t))=(\nu^a_{\alpha b}(t))$ are Lagrange multipliers subject to the condition that the 1-form $L=(L_\alpha)$ is closed. Extending the results in [15], [18], the necessary conditions for the optimality of a feasible solution $x^\circ(\cdot)\in\mathcal{F}(\Omega_{t_0,t_1})$ in the problem (SP) are

$$\begin{cases}\dfrac{\partial L_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) - D_\gamma\dfrac{\partial L_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t))=0,\ \alpha=\overline{1,p} \quad \text{(Euler-Lagrange PDE)} \\ <\mu_\alpha(t),g(t,x^\circ(t),x^\circ_\gamma(t))>=0,\ t\in\Omega_{t_0,t_1},\ \alpha=\overline{1,p}, \\ \mu_\alpha(t)\geqq 0, \quad t\in\Omega_{t_0,t_1}, \quad \alpha=\overline{1,p}.\end{cases}$$

Definition 2.2. If $\lambda\neq 0$, the optimal feasible solution $x^\circ(\cdot)$ of the problem (SP) is called *normal*.

Without loss of generality, if $x^\circ(\cdot)$ is an optimal normal solution of the problem (SP), we can assume that $\lambda=1$.

The following Theorem describes the previous necessary optimality conditions in the language of [4], [15], [18], [19].

Theorem 2.3. *Let $s = (s_\alpha)$ be a closed 1-form of C^∞-class. If $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is a normal optimal solution of the problem (SP), then there exist the multipliers λ, $\mu(t)$, $\nu(t)$ satisfying the following conditions:*

$$(VC)\begin{cases} \quad \lambda \dfrac{\partial s_\alpha}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) + < \mu_\alpha(t), \dfrac{\partial g}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) > \\ + < \nu_\alpha(t), \dfrac{\partial h}{\partial x}(t, x^\circ(t), x^\circ_\gamma(t)) > - D_\gamma \Bigg(\lambda \dfrac{\partial s_\alpha}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) \\ + < \mu_\alpha(t), \dfrac{\partial g}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) > + < \nu_\alpha(t), \dfrac{\partial h}{\partial x_\gamma}(t, x^\circ(t), x^\circ_\gamma(t)) > \Bigg) = 0, \\ t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1,p} \quad \textit{(Euler-Lagrange PDEs)} \\ \quad < \mu_\alpha(t), g(t, x^\circ(t), x^\circ_\gamma(t)) >= 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\ \quad \mu_\alpha(t) \geqq 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\ \quad [\lambda = 1]. \end{cases}$$

Now we turn back to the vector problem (MP). To develop further our theory, we need the result in the following Lemma (for the single-time case, see [3]).

Lemma 2.4. *The function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is an efficient solution of the problem (MP) if and only if $x^\circ(\cdot)$ is an optimal solution of each scalar problem $P_\ell(x^\circ(\cdot))$, $\ell = \overline{1,r}$, where*

$$P_\ell(x^\circ)\begin{cases} \min\limits_{x(\cdot)} \quad F^\ell(x(\cdot)) \\ \qquad \text{subject to} \\ \qquad x(t_0) = x_0, \quad x(t_1) = x_1, \\ \qquad g(t, x(t), x_\gamma(t)) \leqq 0, \quad t \in \Omega_{t_0,t_1}, \\ \qquad h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0,t_1}, \\ \qquad F^j(x(\cdot)) \le F^j(x^\circ(\cdot)), \quad j = \overline{1,r}, \ j \ne \ell. \end{cases}$$

Proof. In order to prove the direct implication, we suppose that the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is an efficient solution of the problem (MP) and there is $k \in \{1, \ldots, r\}$ such that $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is not an optimal solution of the scalar problem $P_k(x^\circ(\cdot))$. Then there exists a function $y(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ such that

$$F^j(y(\cdot)) \le F^j(x^\circ(\cdot)), \quad j = \overline{1,r}, \ j \ne k; \ F^k(y(\cdot)) < F^k(x^\circ(\cdot)).$$

These relations contradict the efficiency of the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ for the problem (MP). Consequently, the point $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is an optimal solution for each program $P_\ell(x^\circ(\cdot))$, $\ell = \overline{1,r}$.

Conversely, let us consider that the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is an optimal solution of all problems $P_\ell(x^\circ(\cdot))$, $\ell = \overline{1,r}$. Suppose that $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is not an efficient solution of the problem (MP). Then there exists a function $y(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ such that $F^j(y(\cdot)) \le F^j(x^\circ(\cdot))$, $j = \overline{1,r}$, and there is $k \in \{1, \ldots, r\}$ such that $F^k(y(\cdot)) < F^k(x^\circ(\cdot))$. This is a contradiction to the assumption that the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ minimizes the functional $F^k(x(\cdot))$ on the set of all feasible solutions of problem $P_k(x^\circ(\cdot))$. Therefore, the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is an efficient solution of the problem (MP) ∎

Lemma 2.5. *Let ℓ be fixed between 1 and r. If the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is a [normal] optimal solution of the scalar problem $P_\ell(x^\circ(\cdot))$, then there exist the real vectors $(\lambda_{j\ell})$, $j = \overline{1,r}$, and the matrix functions μ_ℓ, ν_ℓ, such that the following conditions are satisfied*

$$\lambda_{j\ell}\frac{\partial f^j_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) + < \mu_{\ell\alpha}(t), \frac{\partial g}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) >$$

$$+ < \nu_{\ell\alpha}(t), \frac{\partial h}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > -D_\gamma\Bigg(\lambda_{j\ell}\frac{\partial f^j_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t))$$

$$+ < \mu_{\ell\alpha}(t), \frac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \nu_{\ell\alpha}(t), \frac{\partial h}{\partial x_\gamma}(t,x^\circ_\gamma(t),x^\circ_\gamma(t)) > \Bigg) = 0,$$

$$t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p} \quad \textit{(Euler-Lagrange PDEs)}$$

$$< \mu_{\ell\alpha}(t), g(t,x^\circ(t),x^\circ_\gamma(t)) >= 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p},$$

$$\mu_{\ell\alpha}(t) \geqq 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p},$$

$$\lambda_{j\ell} \geq 0 \quad [\lambda_{\ell\ell} = 1].$$

For a proof, see [9].

Definition 2.6. The function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is called *normal efficient* solution of the problem (MP) if it is normal optimal solution for at least one of the problems $P_\ell(x^\circ(\cdot))$, $\ell = \overline{1,r}$.

It follows the main result of this section.

Theorem 2.7. *If $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is a normal efficient solution of the problem (MP), then there exist a vector $\lambda^\circ \in \mathbf{R}^r$ and the smooth matrix functions $\mu^\circ(t) = (\mu^\circ_\alpha(t))$, $\nu^\circ(t) = (\nu^\circ_\alpha(t))$, which satisfy the following conditions*

$$(MV)\begin{cases} < \lambda^\circ, \dfrac{\partial f_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \mu^\circ_\alpha(t), \dfrac{\partial g}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > \\ + < \nu^\circ_\alpha(t), \dfrac{\partial h}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > -D\gamma\Bigg(< \lambda^\circ, \dfrac{\partial f_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > \\ + < \mu^\circ_\alpha(t), \dfrac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \nu^\circ_\alpha(t), \dfrac{\partial h}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > \Bigg) = 0, \\ \qquad t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p} \quad \textit{(Euler-Lagrange PDEs)} \\ < \mu^\circ_\alpha(t), g(t,x^\circ(t),x^\circ_\gamma(t)) >= 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\ \mu^\circ_\alpha(t) \geqq 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\ \lambda^\circ \geq 0, \\ < e, \lambda^\circ >= 1, \quad e = (1,\dots,1) \in \mathbf{R}^r. \end{cases}$$

Proof. If the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is a [normal] efficient solution of the problem (MP), according to Lemma 2.4, the point $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is a normal efficient solution of each scalar problem $P_\ell(x^\circ(\cdot))$, $\ell = \overline{1,r}$. According to Lemma 2.5, there exist the matrix $\lambda_{j\ell}$, $j, \ell = \overline{1,r}$, and the functions $\mu_{\ell\alpha}$, $\nu_{\ell\alpha}$, satisfying the following conditions

$$(FV)_\ell\begin{cases} \lambda_{j\ell}\dfrac{\partial f^j_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t))+<\mu_{\ell\alpha}(t),\dfrac{\partial g}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t))> \\ +<\nu_{\ell\alpha}(t),\dfrac{\partial h}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t))>-D_\gamma\Bigg(\lambda_{j\ell}\dfrac{\partial f^j_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) \\ +<\mu_{\ell\alpha}(t),\dfrac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t))>+<\nu_{\ell\alpha},\dfrac{\partial h}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t))>\Bigg)=0, \\ \qquad t\in\Omega_{t_0,t_1},\ \alpha=\overline{1,p}\quad \text{(Euler-Lagrange PDEs)} \\ <\mu_{\ell\alpha}(t),g(t,x^\circ(t),x^\circ_\gamma(t))>=0,\quad t\in\Omega_{t_0,t_1},\quad \alpha=\overline{1,p}, \\ \mu_{\ell\alpha}(t)\geqq 0,\quad t\in\Omega_{t_0,t_1},\quad \alpha=\overline{1,p}, \\ \lambda_{j\ell}\ge 0\quad [\lambda_{\ell\ell}=1]. \end{cases}$$

Making the sum of all relations $(FV)_\ell$ from $\ell=1$ to $\ell=r$ and denoting

$$\Lambda_j=\sum_{\ell=1}^r\lambda_{j\ell},\quad M_\alpha(t)=\sum_{\ell=1}^r\mu_{\ell\alpha}(t),\quad N_\alpha(t)=\sum_{\ell=1}^r\nu_{\ell\alpha}(t),$$

the following relations are obtained

$$(FV)\begin{cases} \Lambda_j\dfrac{\partial f^j_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t))+<M_\alpha(t),\dfrac{\partial g}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t))> \\ +<N_\alpha(t),\dfrac{\partial h}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t))>-D_\gamma\Bigg(\Lambda_j\dfrac{\partial f^j_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) \\ +<M_\alpha(t),\dfrac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t))>+<N_\alpha(t),\dfrac{\partial h}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t))>\Bigg)=0, \\ \qquad t\in\Omega_{t_0,t_1},\ \alpha=\overline{1,p}\quad \text{(Euler-Lagrange PDEs)} \\ <M_\alpha(t),g(t,x^\circ(t),x^\circ_\gamma(t))>=0,\quad t\in\Omega_{t_0,t_1},\quad \alpha=\overline{1,p}, \\ M_\alpha(t)\geqq 0,\quad t\in\Omega_{t_0,t_1},\quad \alpha=\overline{1,p}, \\ \lambda_j\ge 0,\ j=\overline{1,r}. \end{cases}$$

We divide the relations (FV) by $S=\sum_{j=1}^r\Lambda_j\ge 1$ and we denote

$$\Lambda^\circ_j=\frac{\Lambda_j}{S},\quad M^\circ_\alpha(t)=\frac{M_\alpha(t)}{S},\quad N^\circ_\alpha(t)=\frac{N_\alpha(t)}{S}.$$

Thus we obtain the relations from the statement ■

3 Necessary efficiency conditions for the problem (MFP)

Consider $x^\circ(\cdot)\in\mathcal{F}(\Omega_{t_0,t_1})$ being a feasible solution of the problem (MFP) and for each index j between 1 and r, let us introduce the real number

$$R^j_\circ(x^\circ(\cdot))=\frac{F^j(x^\circ(\cdot))}{K^j(x^\circ(\cdot))}.$$

For a fixed index ℓ, we consider the following pair of extremizing problems

$$(FPR)_\ell \begin{cases} \min\limits_{x(\cdot)} \quad \dfrac{F^\ell(x(\cdot))}{K^\ell(x(\cdot))} \\ \quad \text{subject to} \\ \quad x(t_0) = x_0, \quad x(t_1) = x_1 \\ \quad g(t, x(t), x_\gamma(t)) \leqq 0, \quad t \in \Omega_{t_0,t_1}, \\ \quad h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0,t_1}, \\ \quad F^j(x(\cdot)) - R^j_\circ(x^\circ(\cdot))K^j(x(\cdot)) \leqq 0, \quad j = \overline{1,r},\ j \neq \ell. \end{cases}$$

$$(SPR)_\ell \begin{cases} \min\limits_{x(\cdot)} \quad F^\ell(x(\cdot)) - R^\ell_\circ(x^\circ(\cdot))K^\ell(x(\cdot)) \\ \quad \text{subject to} \\ \quad x(t_0) = x_0, x(t_1) = x_1 \\ \quad g(t, x(t), x_\gamma(t)) \leqq 0, \quad t \in \Omega_{t_0,t_1}, \\ \quad h(t, x(t), x_\gamma(t)) = 0, \quad t \in \Omega_{t_0,t_1}, \\ \quad F^j(x(\cdot)) - R^j_\circ(x^\circ(\cdot))K^j(x(\cdot)) \leqq 0, \quad j = \overline{1,r},\ j \neq \ell, \end{cases}$$

With the statements of Jagannathan [3], we have

Lemma 3.1. *The function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is optimal in $(FPR)_\ell$ if and only if it is optimal in $(SPR)_\ell$, $\ell = \overline{1,r}$.*

Using Lemma 2.4 and Lemma 3.1, we can formulate

Theorem 3.2. *The function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is an efficient solution for the problem (MFP) if and only if it is an optimal solution for each problem $(SPR)_\ell$, $\ell = \overline{1,r}$.*

Proof. We shall prove this statement using the double implication.

THE NECESSITY. Let us suppose that the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is efficient for the problem (MFP). Then it is optimal for problem $(FPR)_\ell$, $\ell = \overline{1,r}$, according to Lemma 2.4. Also, for any $\ell = \overline{1,r}$, if the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is optimal for problem $(FPR)_\ell$, then it is optimal for problem $(SPR)_\ell$, $\ell = \overline{1,r}$ (according to Lemma 3.1).

THE SUFFICIENCY. Let us suppose that the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is efficient for the problem $(SPR)_\ell$, for all $\ell = \overline{1,r}$. Then it is optimal for problem $(FPR)_\ell$, $\ell = \overline{1,r}$, according to Lemma 3.1. Also, for any $\ell = \overline{1,r}$, the function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is optimal for problem $(FPR)_\ell$, therefore it is optimal for problem (MPF) (according to Lemma 2.4) ∎

Remark. The function $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is a normal efficient solution of the problem (MFP) if it is a normal optimal solution for at least one of the scalar problems $(FPR)_\ell$, $\ell = \overline{1,r}$.

Consider $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbf{R}^r$ and the matrix functions $\mu\colon \Omega_{t_0,t_1} \to \mathbf{R}^{msp}$, $\nu\colon \Omega_{t_0,t_1} \to \mathbf{R}^{qsp}$ such that the auxiliary Lagrange 1-form $L = (L_\alpha)$,

$$\begin{aligned} L_\alpha(t, x(t), x_\gamma(t), \lambda, \mu(t), \nu(t)) = \lambda_j \left(f^j_\alpha(t, x(t), x_\gamma(t)) - R^j_\circ(x^\circ(\cdot))k^j_\alpha(t, x(t), x_\gamma(t))\right) \\ + < \mu_\alpha(t), g(t, x(t), x_\gamma(t)) > \\ + < \nu_\alpha(t), h(t, x(t), x_\gamma(t)) >, \quad \alpha = \overline{1,p}, \end{aligned}$$

be closed. Having in mind the background introduced above, we can state the main results of this section. First of all, we shall introduce our necessary efficiency conditions.

Theorem 3.3 (Necessary efficiency conditions). *Let the function* $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ *be a normal efficient solution of problem* (MFP). *Then there exist* $\Lambda^{1\circ}$, $\Lambda^{2\circ} \in \mathbf{R}^r$ *and the smooth functions* $M^\circ\colon \Omega_{t_0,t_1} \to \mathbf{R}^{msp}$, $N^\circ\colon \Omega_{t_0,t_1} \to \mathbf{R}^{spq}$, *such that we have*

$$(MFV)\begin{cases} \quad \Lambda_j^{1\circ}\dfrac{\partial f_\alpha^j}{\partial x}(t,x^\circ(t),x_\gamma^\circ(t)) - \Lambda_j^{2\circ}\dfrac{\partial k_\alpha^j}{\partial x}(t,x^\circ(t),x_\gamma^\circ(t)) \\ + < M_\alpha^\circ(t), \dfrac{\partial g}{\partial x}(t,x^\circ(t),x_\gamma^\circ(t)) > + < N_\alpha^\circ(t), \dfrac{\partial h}{\partial x}(t,x^\circ(t),x_\gamma^\circ(t)) > \\ -D_\gamma\Big\{\Lambda_j^{1\circ}\dfrac{\partial f_\alpha^j}{\partial x_\gamma}(t,x^\circ(t),x_\gamma^\circ(t)) - \Lambda_j^{2\circ}\dfrac{\partial k_\alpha^j}{\partial x_\gamma}(t,x^\circ(t),x_\gamma^\circ(t)) \\ + < M_\alpha^\circ(t,x^\circ(t),x_\gamma^\circ(t)), \dfrac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x_\gamma^\circ(t)) > \\ + < N_\alpha^\circ(t), \dfrac{\partial h}{\partial x_\gamma}(t,x^\circ(t),x_\gamma^\circ(t)) > \Big\} = 0, \\ \qquad\qquad t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p} \quad \textit{(Euler-Lagrange PDEs)} \\ \quad < M_\alpha^\circ(t), g(t,x^\circ(t),x_\gamma^\circ(t)) >= 0, \quad t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p}, \\ \quad M_\alpha^\circ(t) \geqq 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\ \quad \Lambda^{1\circ} \geq 0, \qquad < e, \Lambda^{1\circ} >= 1, \quad e = (1,\ldots,1) \in \mathbf{R}^r. \end{cases}$$

Proof. There are $\lambda_{j\ell}^1$, $\lambda_{j\ell}^2$, $j = \overline{1,r}$, and the functions $\mu_{\ell\alpha}(t)$, $\nu_{\ell\alpha}(t)$, such that

$$(FV)_\ell\begin{cases} \quad \lambda_{j\ell}^1\dfrac{\partial f_\alpha^j}{\partial x}(t,x^\circ(t),x_\gamma^\circ(t)) - \lambda_{j\ell}^2\dfrac{\partial k_\alpha^j}{\partial x}(t,x^\circ(t),x_\gamma^\circ(t))\Big] \\ + < \mu_{\ell\alpha}(t), \dfrac{\partial g}{\partial x}(t,x^\circ(t),x_\gamma^\circ(t)) > + < \nu_{\ell\alpha}(t), \dfrac{\partial h}{\partial x}(t,x^\circ(t),x_\gamma^\circ(t)) > \\ -D_\gamma\Big\{\lambda_{j\ell}^1\dfrac{\partial f_\alpha^j}{\partial x_\gamma}(t,x^\circ(t),x_\gamma^\circ(t)) - \lambda_{j\ell}^2\dfrac{\partial k_\alpha^j}{\partial x_\gamma}(t,x^\circ(t),x_\gamma^\circ(t)) \\ + < \mu_{\ell\alpha}(t), \dfrac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x_\gamma^\circ(t)) > + < \nu_{\ell\alpha}(t), \dfrac{\partial h}{\partial x_\gamma}(t,x^\circ(t),x_\gamma^\circ(t)) > \Big\} = 0, \\ \qquad\qquad t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p} \quad \text{(Euler-Lagrange PDEs)} \\ \quad < \mu_{\ell\alpha}(t), g(t,x^\circ(t),x_\gamma^\circ(t)) >= 0, \quad t \in \Omega_{t_0,t_1},\ \alpha = \overline{1,p}, \\ \quad \mu_{\ell\alpha}(t) \geqq 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\ \quad \lambda_{j\ell} \geq 0 \quad [\lambda_{\ell\ell} = 1]. \end{cases}$$

We make the sum of all relations $(FV)_\ell$ after $\ell = \overline{1,r}$ and denoting

$$\bar{\Lambda}_j^1 = \sum_{\ell=1}^r \lambda_{j\ell}^1, \quad \bar{\Lambda}_j^2 = \sum_{\ell=1}^r \lambda_{j\ell}^2, \quad \bar{M}_\alpha(t) = \sum_{\ell=1}^r \mu_{\ell\alpha}(t), \quad \bar{N}_\alpha(t) = \sum_{\ell=1}^r \nu_{\ell\alpha}(t)$$

we obtain

$$(FV)\begin{cases} \bar\Lambda^1_j \dfrac{\partial f^j_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) - \bar\Lambda^2_j \dfrac{\partial k^j_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) \\ + < \bar M_\alpha(t), \dfrac{\partial g}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \bar N_\alpha(t), \dfrac{\partial h}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > \\ -D_\gamma\Big\{\bar\Lambda^1_j \Big[\dfrac{\partial f^j_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) - \bar\Lambda^2_j \dfrac{\partial k^j_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) \\ + < \bar M_\alpha(t), \dfrac{\partial g}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > + < \bar N_\alpha(t), \dfrac{\partial h}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) > \Big\} = 0, \\ \qquad t\in\Omega_{t_0,t_1},\ \alpha=\overline{1,p} \quad \text{(Euler-Lagrange PDEs)} \\ < \bar M_\alpha(t), g(t,x^\circ(t),x^\circ_\gamma(t)) >= 0, \quad t\in\Omega_{t_0,t_1},\ \alpha=\overline{1,p}, \\ \bar M_\alpha(t) \geqq 0, \quad t\in\Omega_{t_0,t_1}, \quad \alpha=\overline{1,p}, \\ \bar\Lambda^1_j \geq 0. \end{cases}$$

Dividing the relations (FV) by $S=\sum_{j=1}^r \bar\Lambda^1_j \geq 1$ and denoting

$$\Lambda^{1\circ}_j = \frac{\bar\Lambda^1_j}{S},\quad \Lambda^{2\circ}_j = \frac{\bar\Lambda^2_j}{S},\quad M^\circ_\alpha(t)=\frac{\bar M_\alpha(t)}{S},\quad N^\circ_\alpha(t)=\frac{\bar N_\alpha(t)}{S},$$

the relations (FV) take the form (MFV) ■

4 A dual program theory

Let ρ be a real number and $b : C^\infty(\Omega_{t_0,t_1},M)\times C^\infty(\Omega_{t_0,t_1},M)\to[0,\infty)$ a functional. Let $a=(a_\alpha)$ be a closed Lagrange 1-form. We associate the path independent curvilinear functional $A(x(\cdot)) = \int_{\gamma_{t_0,t_1}} a_\alpha(t,x(t),x_\gamma(t))\,dt^\alpha$. The definition of the quasiinvexity (see also [2], [7], [10], [11], [20]) helps us to state the results included in this section.

Definition 4.1. The functional A is called *[strictly]* (ρ,b)*-quasiinvex at the point* $x^\circ(\cdot)$ if there is a vector function η: $J^1(\Omega_{t_0,t_1},M)\times J^1(\Omega_{t_0,t_1},M)\to \mathbf{R}^n$, vanishing at the point $(t,x^\circ(t),x^\circ_\gamma(t),x^\circ(t),x^\circ_\gamma(t))$, and the functional θ: $C^\infty(\Omega_{t_0,t_1},M)\times C^\infty(\Omega_{t_0,t_1},M)\to\mathbf{R}^n$, such that for any $x(\cdot)$ $[x(\cdot)\neq x^\circ(\cdot)]$, the following implication holds

$$(A(x(\cdot))\leqq A(x^\circ(\cdot)))\to\Bigg(b(x(\cdot),x^\circ(\cdot))\int_{\gamma_{t_0,t_1}}\{< \eta(t,x(t),x_\gamma(t),x^\circ(t),x^\circ_\gamma(t))\ ,$$
$$\frac{\partial a_\alpha}{\partial x}(t,x^\circ(t),x^\circ_\gamma(t)) > + < D_\gamma\eta(t,x(t),x_\gamma(t),x^\circ(t),x^\circ_\gamma(t)),$$
$$\frac{\partial a_\alpha}{\partial x_\gamma}(t,x^\circ(t),x^\circ_\gamma(t)) >\}dt^\alpha [<] \leqq -\rho b(x(\cdot),x^\circ(\cdot))\|\theta(x(\cdot),x^\circ(\cdot))\|^2\Bigg).$$

We associate a multi-objective variational dual problem to the problem (MFP), preserving the same set of feasible solutions:

$$(MFD)\begin{cases} \max\limits_{y(\cdot)} \left(\dfrac{F^1(y(\cdot))}{K^1(y(\cdot))}, \ldots, \dfrac{F^r(y(\cdot))}{K^r(y(\cdot))} \right) \\ \quad \text{subject to} \\ \quad y(t_0) = x_0, \quad y(t_1) = x_1 \\ \quad \Lambda_\ell^{1\circ} \dfrac{\partial f_\alpha^\ell}{\partial y}(t, y(t), y_\gamma(t)) - \Lambda_\ell^{2\circ} \dfrac{\partial k_\alpha^\ell}{\partial y}(t, y(t), y_\gamma(t)) \\ + < \mu_\alpha(t), \dfrac{\partial g}{\partial y}(t, y(t), y_\gamma(t)) > + < \nu_\alpha(t), \dfrac{\partial h}{\partial y}(t, y(t), y_\gamma(t)) > \\ -D_\gamma \Big\{ \Lambda_\ell^{1\circ} \dfrac{\partial f_\alpha^\ell}{\partial y_\gamma}(t, y(t), y_\gamma(t)) - \Lambda_\ell^{2\circ} \dfrac{\partial k_\alpha^\ell}{\partial y_\gamma}(t, y(t), y_\gamma(t)) \\ + < \mu_\alpha(t), \dfrac{\partial g}{\partial y_\gamma}(t, y(t), y_\gamma(t)) > + < \nu_\alpha(t), \dfrac{\partial h}{\partial y_\gamma}(t, y(t), y_\gamma(t)) > \Big\} = 0, \\ \hfill t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1,p} \\ \quad < \mu_\alpha(t), g(t, y(t), y_\gamma(t)) > + < \nu_\alpha(t), h(t, y(t), y_\gamma(t)) > \ \geqq 0, \\ \hfill \alpha = \overline{1,p}, \ t \in \Omega_{t_0,t_1} \\ \quad \Lambda^{1\circ} \geq 0, \ < e, \Lambda^{1\circ} >= 1, \ e = (1, \ldots, 1) \in \mathbf{R}^r. \end{cases}$$

To formulate our original results, we use the minimizing functional vector $\pi(x(\cdot))$ of the problem (MFD) at the point $x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ and the maximizing functional vector

$$\delta(y(\cdot), y_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu(\cdot), \nu(\cdot))$$

of the dual problem (MFD) at

$$(y(\cdot), y_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu(\cdot), \nu(\cdot)) \in \Delta,$$

where Δ is the domain of the problem (MFD).

Theorem 4.2 (Weak duality). *Let $x^\circ(\cdot)$ be a feasible solution of the problem (MFP) and $y(\cdot)$ be a normal efficient solution of the dual problem (MFD). Assume that the following conditions are fulfilled:*

a) $\Lambda_\ell^{1\circ} > 0$, $\Lambda_\ell^{2\circ} > 0$, $\ell = \overline{1,r}$, $\Lambda_\ell^{1\circ} F^\ell(y(\cdot)) - \Lambda_\ell^{2\circ} K^\ell(y(\cdot)) = 0$;

b) *for any $\ell = \overline{1,r}$, the functional $F^\ell(x(\cdot))$ is (ρ'^ℓ, b)-quasiinvex at the point $y(\cdot)$ and $-K^\ell(x(\cdot))$ is (ρ''^ℓ, b)-quasiinvex at the point $y(\cdot)$ with respect to η and θ;*

c) *the functional*

$$\int_{\gamma_{t_0,t_1}} [< \mu_\alpha(t), g(t, x(t), x_\gamma(t)) > + < \nu_\alpha(t), h(t, x(t), x_\gamma(t)) >] dt^\alpha$$

is (ρ''', b) - quasiinvex at $y(\cdot)$ with respect to η and θ;

d) *one of the functionals of* b), c) *is strictly (ρ'^ℓ, b)-quasiinvex;*

e) $\rho'^\ell \Lambda_\ell^{1\circ} + \rho''^\ell \Lambda_\ell^{2\circ} + \rho''' \geqq 0$.

Then, the inequality $\pi(x^\circ(\cdot)) \leq \delta(y(\cdot), y_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu(\cdot), \nu(\cdot))$ is false.

The proof will be given in a further paper (see also, [9]).

Theorem 4.3 (Direct duality). *Let $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ be a normal efficient solution of (MFP) and suppose that the hypotheses of Theorem 4.2 are satisfied. Then there are* *the vectors* $\Lambda^{1\circ}, \Lambda^{2\circ} \in \mathbf{R}^r$ *and the smooth functions* $\mu^\circ \colon \Omega_{t_0,t_1} \to \mathbf{R}^{msp}$, $\nu^\circ \colon \Omega_{t_0,t_1} \to \mathbf{R}^{qsp}$ *such that* $(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ *is an efficient solution of the dual* (MFD) *and* $\pi(x^\circ(\cdot)) = \delta(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$.

Proof. We take into account that the point $x^\circ(\cdot)$ is a normal efficient solution of the problem (MFP). Therefore, according to Theorem 3.3, there are $\Lambda^{1\circ}, \Lambda^{2\circ} \in \mathbf{R}^r$ and the smooth functions $\mu^\circ \colon \Omega_{t_0,t_1} \to \mathbf{R}^{msp}$, $\nu^\circ \colon \Omega_{t_0,t_1} \to \mathbf{R}^{qsp}$ satisfying the relations $(MFV)_C$. Also, $< \nu^\circ_\alpha(t), h(t, x^\circ(t), x^\circ_\gamma(t)) >= 0, \quad \alpha = \overline{1,p}$. Hence

$$(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot)) \in \Delta,\ \pi(x^\circ(\cdot)) = \delta(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$$

Therefore, the statements are proved. ∎

Theorem 4.4 (Converse duality). *Let* $(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ *be an efficient solution of the dual problem* (MFD). *Suppose the following conditions are fulfilled:*

a) $\bar{x}(\cdot)$ *is a normal efficient solution of the primal problem* (MFP)*;*

b) *for any* $\ell = \overline{1,r}$, $F^\ell(x^\circ(\cdot)) > 0$, $K^\ell(x^\circ(\cdot)) > 0$, $\Lambda^{1\circ}_\ell F^\ell(x^\circ(\cdot)) - \Lambda^{2\circ}_\ell K^\ell(x^\circ(\cdot)) = 0$*;*

c) *for any* $\ell = \overline{1,r}$, $F^\ell(x(\cdot))$ *is* (ρ'^ℓ, b)*-quasiinvex at the point* $x^\circ(\cdot)$ *and* $-K^\ell(x(\cdot))$ *is* (ρ''^ℓ, b)*-quasiinvex at the point* $x^\circ(\cdot)$*, with respect to* η *and* θ*;*

d) *the functional*

$$\int_{\gamma_{t_0,t_1}} [< \mu_\alpha(t), g(t, x(t), x_\gamma(t)) > + < \nu_\alpha(t), h(t, x(t), x_\gamma(t)) >] \, dt^\alpha$$

is (ρ''', b)*-quasiinvex at the point* x° *with respect to* η *and* θ*;*

e) *one of the functionals of* c), d) *is strictly* (ρ'^ℓ, b), (ρ''^ℓ, b) *or* (ρ'''^ℓ, b)*-quasiinvex with respect to* η *and* θ*, respectively;*

f) $\rho'^\ell \Lambda^{1\circ}_\ell + \rho''^\ell \Lambda^{2\circ}_\ell + \rho''' \geqq 0$.

Then $\bar{x}(\cdot) = x^\circ(\cdot)$ *and moreover,* $\pi(x^\circ(\cdot)) = \delta(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$.

Proof. Let us suppose that $\bar{x}(\cdot) \neq x^\circ(\cdot)$. According to Theorem 3.3, there are the vectors $\bar{\Lambda}^{1\circ}, \bar{\Lambda}^{2\circ} \in \mathbf{R}^r$ and the functions $\bar{\mu} \colon \Omega_{t_0,t_1} \to \mathbf{R}^{msp}$, $\bar{\nu} \colon \Omega_{t_0,t_1} \to \mathbf{R}^{qsp}$, satisfying the conditions $(MFV)_C$. It follows

$$< \bar{\mu}_\alpha(t), g(t, \bar{x}(\cdot), \bar{x}_\gamma(\cdot)) > + < \bar{\nu}_\alpha(t), h(t, \bar{x}(\cdot), \bar{x}_\gamma(\cdot)) >= 0, \alpha = \overline{1,p}$$

and therefore $(\bar{x}, \bar{x}_\gamma, \bar{\Lambda}^{1\circ}, \bar{\Lambda}^{2\circ}, \bar{\mu}, \bar{\nu}) \in \Delta$. Moreover, $\pi(\bar{x}) = \delta(\bar{x}, \bar{x}_\gamma, \bar{\Lambda}^{1\circ}, \bar{\Lambda}^{2\circ}, \bar{\mu}, \bar{\nu})$. According to Theorem 4.2, we have $\pi(\bar{x}(\cdot)) \nleq \delta(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$. Consequently, $\delta(\bar{x}(\cdot), \bar{x}_\gamma(\cdot), \bar{\Lambda}^{1\circ}, \bar{\Lambda}^{2\circ}, \bar{\mu}(\cdot), \bar{\nu}(\cdot)) \nleq \delta(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$. Then the maximal efficiency of the point $(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ is contradicted. Hence, $\bar{x}(\cdot) = x^\circ(\cdot)$ and $\pi(x^\circ(\cdot)) = \delta(x^\circ(\cdot), x^\circ_\gamma(\cdot), \Lambda^{1\circ}, \Lambda^{2\circ}, \mu^\circ(\cdot), \nu^\circ(\cdot))$ ∎

Remark 4.5. To make a computer aided study of $PDI\&PDE$-constrained optimization problems we can perform symbolic computations via Maple software package (see also [1], [16]).

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