Hypersurfaces with constant scalar or mean curvature in a unit sphere

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Abstract. Let M be an $n(n \geq 3)$-dimensional complete connected hypersurface in a unit sphere $S^{n+1}(1)$. In this paper, we show that (1) if M has non-zero mean curvature and constant scalar curvature $n(n-1)r$ and two distinct principal curvatures, one of which is simple, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, $c^2 = \frac{n-2}{nr}$ if $r \geq \frac{n-2}{n-1}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$. (2) if M has non-zero constant mean curvature and two distinct principal curvatures, one of which is simple, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, $c^2 = \frac{n-2}{nr}$ if one of the following conditions is satisfied: (i) $r \geq \frac{n-2}{n-1}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$; or (ii) $r > 1 - \frac{2}{n}$, $r \neq \frac{n-2}{n-1}$ and $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, where S is the squared norm of the second fundamental form of M.

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1 Introduction

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. If scalar curvature $n(n-1)r$ of M is constant and $r \geq 1$. S. Y. Cheng and Yau [1] and Li [5] obtained some characterization theorems in terms of the sectional curvature or the squared norm of the second fundamental form of M respectively. We should notice that the condition $r \geq 1$ plays an essential role in their proofs of theorems. On the other hand, for any $0 < c < 1$, by considering the standard immersions $S^{n-1}(c) \subset R^n, S^1(\sqrt{1-c^2}) \subset R^2$ and taking the Riemannian product immersion $S^1(\sqrt{1-c^2}) \times S^{n-1}(c) \hookrightarrow R^2 \times R^n$, we obtain a hypersurface $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, where $r = \frac{n-2}{nc^2} > 1 - \frac{2}{n}$. Hence, not all Riemannian products $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ appear in the results of [1] and [5]. Since the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ has only two distinct principal curvatures and its scalar curvature $n(n-1)r$ is constant and satisfies $r > 1 - \frac{2}{n}$,

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Cheng[2] asked the following interesting problem:

Problem 1.1 ([2]). *Let M be an n-dimensional complete hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r > 1 - \frac{2}{n}$ and*

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

then is M isometric to either a totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$?

Cheng [2] said that when $r = \frac{n-2}{n-1}$, he answered the Problem 1.1 affirmatively. For the general case, Problem 1.1 is still open.

In this paper, we try to solve Problem 1.1 and give a partial affirmative answer. We obtain the following:

Theorem 1.1. *Let M be an $n(n \geq 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with non-zero mean curvature and constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, one of which is simple. If $r \geq \frac{n-2}{n-1}$ and*

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = \frac{n-2}{nr}$.

If M has constant mean curvature, we can obtain the following:

Theorem 1.2. *Let M be an $n(n \geq 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with non-zero constant mean curvature and with two distinct principal curvatures, one of which is simple. If $r \geq \frac{n-2}{n-1}$ and*

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = \frac{n-2}{nr}$.

Remark 1.1. We shall note that in [9], the author had given a topological answer to problem 1.1 when M is compact. In [3] and [4], Cheng and the author and Suh had given a partial affirmative answer to problem 1.1 when M is compact.

On the other hand, Cheng [2] also proved the following theorem:

Theorem 1.3 ([2]). *Let M be an n-dimensional complete hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, one of which is simple. Then $r > 1-\frac{2}{n}$ and M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ if $r \neq \frac{n-2}{n-1}$ and*

$$S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

where $c^2 = \frac{n-2}{nr}$.

If M has constant mean curvature, we can obtain the following:

Theorem 1.4. *Let M be an $n(n \geq 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with constant mean curvature and with two distinct principal curvatures, one of which is simple. If $r > 1 - \frac{2}{n}$, $r \neq \frac{n-2}{n-1}$ and*

$$S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = \frac{n-2}{nr}$.

Remark 1.2. Recently, the authors also studied the complete hypersurfaces in a hyperbolic space with constant scalar curvature or with constant k-th mean curvature and with two distinct principal curvatures, one can see [7] and [8]. On the study of stable spacelike hyersurfaces with constant scalar curvature, one can see [6].

2 Preliminaries

Let M be an n-dimensional hypersurface in $S^{n+1}(1)$. We choose a local orthonormal frame $e_1, \cdots, e_{n+1}$ in $S^{n+1}(1)$ such that $e_1, \cdots, e_n$ are tangent to M. Let $\omega_1, \cdots, \omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n+1; \quad 1 \leq i, j, k, \cdots \leq n.$$

The structure equations of $S^{n+1}(1)$ are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.3}$$

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}. \tag{2.4}$$

Restricting to M,

$$\omega_{n+1} = 0, \tag{2.5}$$

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.6}$$

The structure equations of M are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.8}$$

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.9}$$

$$R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2.10}$$

$$n(n-1)r = n(n-1) + n^2H^2 - S, \tag{2.11}$$

where $n(n-1)r$ is the scalar curvature, H is the mean curvature and S is the squared norm of the second fundamental form of M.

3 Proof of theorem

Let M be an $n(n \geq 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, one of which is simple. Without lose of generality, we may assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu, \tag{3.1}$$

where λ_i for $i = 1, 2, \cdots, n$ are the principal curvatures of M. From (2.11) and (3.1), we have

$$n(n-1)(r-1) = (n-1)(n-2)\lambda^2 + 2(n-1)\lambda\mu. \tag{3.2}$$

If $\lambda = 0$ at a point of M, then from above equation, we obtain that $r = 1$ at this point. Since the scalar curvature $n(n-1)r$ is constant, we obtain $r \equiv 1$ on M. Since the principal curvatures λ and μ are continuous on M, by the same assertion we can deduce from (3.2) that $\lambda \equiv 0$ on M. By (2.9), we know that the sectional curvature of M is not less than 1. Therefore, we know that M is compact by use of Bonnet-Myers Theorem. According to Theorem 2 in Cheng and Yau [1], we have M^n is totally umbilical. This is impossible, therefore, we get $\lambda \neq 0$. From (3.2), we have

$$\mu = \frac{n(r-1)}{2\lambda} - \frac{n-2}{2}\lambda, \tag{3.3}$$

Since

$$\lambda - \mu = n\frac{\lambda^2 - (r-1)}{2\lambda} \neq 0,$$

we know that $\lambda^2 - (r-1) \neq 0$. If $\lambda^2 - (r-1) < 0$, we deduce that $r > 1$ and $\lambda^2 - \lambda\mu = \frac{n}{2}[\lambda^2 - (r-1)] < 0$. Therefore $\lambda\mu > \lambda^2$. From (2.9), we obtain the sectional curvature of M is not less than 1. Therefore, we know that M is compact by use of Bonnet-Myers Theorem. According to Theorem 2 in Cheng and Yau [1], we have M^n is totally umbilical. This is impossible, therefore, we get $\lambda^2 - (r-1) > 0$.

Let $\varpi = [\lambda^2 - (r-1)]^{-\frac{1}{n}}$. Cheng [2] proved the following:

Proposition 3.1 ([2]). *Let M be an $n(n \geq 3)$-dimensional connected hypersurface with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, and the space of principal vectors corresponding to one of them is of one* *dimension. Then M is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$, along which the principal curvature λ of multiplicity $n-1$ is constant and which is locally isometric to an $(n-1)$-dimensional sphere $S^{n-1}(c(s)) = E^n(s) \cap S^{n+1}(1)$ of constant curvature and $\varpi = [\lambda^2 - (r-1)]^{-\frac{1}{n}}$ satisfies the ordinary differential equation of order 2*

$$\frac{d^2\varpi}{ds^2} - \varpi\left(\frac{n-2}{n}\varpi^{-n} - r\right) = 0, \tag{3.4}$$

where $E^n(s)$ is an n-dimensional linear subspace in the Euclidean space R^{n+2} which is parallel to a fixed $E^n(s_0)$.

The following Lemma 3.1 in Wei and Suh [10] is important to us.

Lemma 3.1 ([10]). *Equations (3.4) is equivalent to its first order integral*

$$\left(\frac{d\varpi}{ds}\right)^2 + r\varpi^2 + \frac{1}{\varpi^{n-2}} = C, \tag{3.5}$$

where C is a constant; for a constant solution equal to ϖ_0, one has that $r > 0$ and $\varpi_0^n = \frac{n-2}{2r}$, so

$$C_0 = \frac{n}{2}\left(\frac{2r}{n-2}\right)^{(n-2)/n}. \tag{3.6}$$

Moreover, the constant solution of (3.4) corresponds to $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = \frac{n-2}{nr}$.

In [10], Wei and Suh proved the following:

Proposition 3.2 ([10]). *Let M be an $n(n \geq 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, one of which is simple. If $\lambda\mu + 1 \leq 0$, then M is isometric to $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = \frac{n-2}{nr}$.*

By the same method in [10], we can prove the following:

Proposition 3.3. *Let M be an $n(n \geq 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, one of which is simple. If $\lambda\mu + 1 \geq 0$, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = \frac{n-2}{nr}$.*

Proof. From (3.3), we have

$$\lambda\mu + 1 = \frac{n(r-1)}{2} - \frac{n-2}{2}\lambda^2 + 1. \tag{3.7}$$

If $\lambda\mu + 1 \geq 0$, we have from (3.7), $\frac{n-2}{2}[\lambda^2 - (r-1)] \leq r$, it follows that

$$\frac{n-2}{2}\varpi^{-n} - r \leq 0. \tag{3.8}$$

From (3.4), we have $\frac{d^2\varpi}{ds^2} \leq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, $\varpi(s)$ must be monotonic when s tends to infinity. We see from (3.5) that the positive function of $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and is monotonic when s tends infinity, we find that both $\lim_{s\to-\infty}\varpi(s)$ and $\lim_{s\to+\infty}\varpi(s)$ exist and then we have

$$\lim_{s\to-\infty}\frac{d\varpi(s)}{ds} = \lim_{s\to+\infty}\frac{d\varpi(s)}{ds} = 0. \tag{3.9}$$

By the monotonicity of $\frac{d\varpi}{ds}$, we see that $\frac{d\varpi}{ds} \equiv 0$ and $\varpi(s)$ is a constant. Then, by Lemma 3.1, it is easily see that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = \frac{n-2}{nr}$.

Since M has two distinct principal curvatures, we know that M has no umbilical points. From (3.1), we have

$$(n-1)\lambda + \mu = nH, \quad S = (n-1)\lambda^2 + \mu^2. \tag{3.10}$$

From (3.10) and (2.11), we have

$$\lambda\mu = (n-1)(r-1) - (n-2)H^2 + (n-2)H\sqrt{H^2-(r-1)}, \tag{3.11}$$

or

$$\lambda\mu = (n-1)(r-1) - (n-2)H^2 - (n-2)H\sqrt{H^2-(r-1)}. \tag{3.12}$$

From (2.11), we obtain

$$\begin{aligned}\lambda\mu =& (r-1) - \frac{(n-2)}{n^2}[S-n(r-1)] \\ &+ \frac{(n-2)}{n^2}\sqrt{[S+n(n-1)(r-1)][S-n(r-1)]},\end{aligned} \tag{3.13}$$

or

$$\begin{aligned}\lambda\mu =& (r-1) - \frac{(n-2)}{n^2}[S-n(r-1)] \\ &- \frac{(n-2)}{n^2}\sqrt{[S+n(n-1)(r-1)][S-n(r-1)]}.\end{aligned} \tag{3.14}$$

Proof of theorem 1.1. If there exists a point x on M such that (3.13) and (3.14) hold at x, that is, we have $S = -n(n-1)(r-1)$ or $S = n(r-1)$ at x. If $S = -n(n-1)(r-1)$ at x, from (2.11), we have $H = 0$ at x, this is a contradiction to $H \neq 0$ on M. If $S = n(r-1)$ at x, from (2.11) we have $S = nH^2$ at x, that is, x is a umbilical point on M, this is a contradiction to M has no umbilical points. Therefore, we only consider two cases:

Case (1). If (3.13) holds on M, since $r \geq \frac{n-2}{n-1}$, then we get $r-1 \geq -\frac{1}{n-1}$ and $n(r-1)+2 \geq \frac{n-2}{n-1}$. From

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

we have

$$\begin{aligned}(3.15)\qquad & n+n(r-1)-\frac{n-2}{n}[S-n(r-1)]\\ &\geq n+2(n-1)(r-1)-\frac{n-2}{n}\left[(n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}\right]\\ &=n+2(n-1)(r-1)-\frac{n-1}{n}[n(r-1)+2]-\frac{(n-2)^2}{n}\frac{1}{n(r-1)+2}\\ &=\frac{n^2-2(n-1)}{n}+(n-1)(r-1)-\frac{(n-2)^2}{n}\frac{1}{n(r-1)+2}\\ &\geq\frac{n^2-2(n-1)}{n}-1-\frac{(n-2)^2}{n}\frac{n-1}{n-2}=0,\end{aligned}$$

From (3.13) and (3.15), obviously, we have $\lambda\mu+1\geq 0$. By Proposition 3.3, we obtain that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2})\times S^{n-1}(c)$, where $c^2=\frac{n-2}{nr}$.

Case (2). If (3.14) holds on M, since $S\leq(n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}$, we know that this is equivalent to

$$\begin{aligned}(3.16)\qquad & \{n+n(r-1)-\frac{n-2}{n}[S-n(r-1)]\}^2\\ &\geq\frac{(n-2)^2}{n^2}\{n(n-1)(r-1)+S\}\{S-n(r-1)\}.\end{aligned}$$

Let $f^2=\sum_i(\lambda_i-H)^2=S-nH^2$. Obviously, by (2.11), we have $n(n-1)(r-1)+S\geq 0$ and $f^2=\frac{n-1}{n}[S-n(r-1)]$. Therefore, we know that $S-n(r-1)\geq 0$. From (3.15) and (3.16), we have

$$\begin{aligned}(3.17)\qquad & n+n(r-1)-\frac{n-2}{n}[S-n(r-1)]\\ &\geq\frac{n-2}{n}\sqrt{[n(n-1)(r-1)+S][S-n(r-1)]}.\end{aligned}$$

Therefore, (3.14) and (3.17) imply that $\lambda\mu+1\geq 0$. By Proposition 3.3, we obtain that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2})\times S^{n-1}(c)$, where $c^2=\frac{n-2}{nr}$. This completes the proof of Theorem 1.1.

In order to prove Theorem 1.2 and Theorem 1.4, we need the following Propositions due to [11].

Proposition 3.4 ([11]). *Let M be an $n(n\geq 3)$-dimensional connected hypersurface with constant mean curvature H and with two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. Then M is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$ along which the principal curvature λ of multiplicity $n-1$ is constant and which is locally isometric to an $(n-1)$-dimensional sphere $S^{n-1}(c(s))=E^n(s)\cap S^{n+1}(1)$ of constant curvature and $\varpi=|\lambda-H|^{-\frac{1}{n}}$ satisfies the ordinary differential equation of order 2*

$$(3.18)\qquad \frac{d^2\varpi}{ds^2}+\varpi[1+H^2+(2-n)H\varpi^{-n}+(1-n)\varpi^{-2n}]=0,$$

for $\lambda-H>0$ or

$$\frac{d^2\varpi}{ds^2} + \varpi[1 + H^2 + (n-2)H\varpi^{-n} + (1-n)\varpi^{-2n}] = 0, \tag{3.19}$$

for $\lambda - H < 0$, where $E^n(s)$ is an n-dimensional linear subspace in the Euclidean space R^{n+2} which is parallel to a fixed $E^n(s_0)$.

Lemma 3.2 ([11]). *Equation (3.18) or (3.19) is equivalent to its first order integral*

$$\left(\frac{d\varpi}{ds}\right)^2 + (1+H^2)\varpi^2 + 2H\varpi^{2-n} + \varpi^{2-2n} = C, \tag{3.20}$$

for $\lambda - H > 0$ or

$$\left(\frac{d\varpi}{ds}\right)^2 + (1+H^2)\varpi^2 - 2H\varpi^{2-n} + \varpi^{2-2n} = C, \tag{3.21}$$

for $\lambda - H < 0$, where C is a constant. Moreover, the constant solution of (3.18) or (3.19) corresponds to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$.

We can prove the following:

Proposition 3.5. *Let M be an $n(n \geq 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. If $\lambda\mu + 1 \geq 0$, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$.*

Proof. Let λ and μ be the two distinct principal curvatures of M with multiplicities $(n-1)$ and 1, respectively. Then, from $nH = (n-1)\lambda + \mu$, we have $\lambda\mu = nH\lambda - (n-1)\lambda^2$. Let $\varpi = |\lambda - H|^{-\frac{1}{n}}$. Then we have $\lambda = H + \varpi^{-n}$ for $\lambda - H > 0$ and $\lambda = H - \varpi^{-n}$ for $\lambda - H < 0$. If $\lambda - H > 0$, we have

$$\lambda\mu + 1 = 1 + H^2 + (2-n)H\varpi^{-n} + (1-n)\varpi^{-2n},$$

and if $\lambda - H < 0$, we have

$$\lambda\mu + 1 = 1 + H^2 + (n-2)H\varpi^{-n} + (1-n)\varpi^{-2n}.$$

Therefore, if $\lambda\mu + 1 \geq 0$, we obtain

$$1 + H^2 + (2-n)H\varpi^{-n} + (1-n)\varpi^{-2n} \geq 0,$$

for $\lambda - H > 0$ and

$$1 + H^2 + (n-2)H\varpi^{-n} + (1-n)\varpi^{-2n} \geq 0,$$

for $\lambda - H < 0$. From (3.18) and (3.19), we have $\frac{d^2\varpi}{ds^2} \leq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, $\varpi(s)$ must be monotonic when s tends to infinity. We see from (3.20) and (3.21) that the positive function of $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and is monotonic when s tends infinity, we find that both $\lim_{s\to-\infty} \varpi(s)$ and $\lim_{s\to+\infty} \varpi(s)$ exist and then we have

$$\lim_{s\to-\infty}\frac{d\varpi(s)}{ds}=\lim_{s\to+\infty}\frac{d\varpi(s)}{ds}=0. \tag{3.22}$$

By the monotonicity of $\frac{d\varpi}{ds}$, we see that $\frac{d\varpi}{ds}\equiv 0$ and $\varpi(s)$ is a constant. Then, by Lemma 3.2, it is easily see that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2})\times S^{n-1}(c)$. This completes the proof of Proposition 3.5.

On the other hand, if $\lambda\mu+1\le 0$, from above, we can obtain $\frac{d^2\varpi}{ds^2}\ge 0$. We see from (3.20) and (3.21) that the positive function of $\varpi(s)$ is bounded. Combining $\frac{d^2\varpi}{ds^2}\ge 0$ with the boundedness of $\varpi(s)$, similar to the proof of Proposition 3.5, we know that $\varpi(s)$ is constant. Then, by Lemma 3.2, it is easily see that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2})\times S^{n-1}(c)$. Therefore, we have the following:

Proposition 3.6. *Let M be an $n(n\ge 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. If $\lambda\mu+1\le 0$, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2})\times S^{n-1}(c)$.*

Proof of theorem 1.2. Since M has non-zero mean curvature, by the same assertion in the proof of Theorem 1.1, we only have two cases: (3.13) holds on M or (3.14) holds on M. If (3.13) holds on M, from the proof of Theorem 1.1, we have $\lambda\mu+1\ge 0$. By Proposition 3.5, we obtain that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2})\times S^{n-1}(c)$, where $c^2=\frac{n-2}{nr}$. If (3.14) holds on M, from the proof of Theorem 1.1, we have $\lambda\mu+1\le 0$. By Proposition 3.6, we obtain that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2})\times S^{n-1}(c)$, where $c^2=\frac{n-2}{nr}$. This completes the proof of Theorem 1.2.

Proof of theorem 1.4. We firstly prove that $H\ne 0$. In fact, if $H=0$, from (2.11) we have $S=-n(n-1)(r-1)$ on M. Since $S\ge(n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}$ is equivalent to

$$\frac{(n-2)^2}{n^2}[S+n(n-1)(r-1)][S-n(r-1)]$$
$$\ge\{n+n(r-1)-\frac{(n-2)}{n}[S-n(r-1)]\}^2,$$

we have from $S=-n(n-1)(r-1)$ that

$$0\ge\{n+n(n-1)(r-1)\}^2. \tag{3.23}$$

From (2.23), we have $r=\frac{n-2}{n-1}$, this is a contradiction to the assumption that $r\ne\frac{n-2}{n-1}$.

If there exists a point x on M such that (3.13) and (3.14) hold at x, that is, we have $S=-n(n-1)(r-1)$ or $S=n(r-1)$ at x. If $S=-n(n-1)(r-1)$ at x, from (2.11), we have $H=0$ at x, this is a contradiction to $H\ne 0$ on M. If $S=n(r-1)$ at x, from (2.11) we have $S=nH^2$ at x, that is, x is a umbilical point on M, this is a contradiction to M has no umbilical points. Therefore, we only consider two cases:

Case (1). If (3.13) holds on M, we can prove $\lambda\mu+1\ge 0$ on M. In fact, if $\lambda\mu+1<0$ at a point of M, then at this point

$$\frac{(n-2)}{n^2}\sqrt{[S+n(n-1)(r-1)][S-n(r-1)]}$$
$$< -1-(r-1)+\frac{(n-2)}{n^2}[S-n(r-1)].$$

Therefore, we have at this point

$$\frac{(n-2)^2}{n^2}[S+n(n-1)(r-1)][S-n(r-1)]$$
$$< \{n+n(r-1)-\frac{(n-2)}{n}[S-n(r-1)]\}^2,$$

this is equivalent to $S < (n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}$ at this point, we have a contradiction to $S \geq (n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}$ on M. Therefore, in case (1) we have $\lambda\mu+1\geq 0$. By Proposition 3.5, we obtain that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2})\times S^{n-1}(c)$, where $c^2=\frac{n-2}{nr}$.

Case (2). If (3.14) holds on M, next we shall prove that $\lambda\mu+1\leq 0$ on M. We consider three subcases:

(i) If $1+(r-1)-\frac{(n-2)}{n^2}[S-n(r-1)]\leq 0$ on M, then from (3.14), it is obvious that $\lambda\mu+1\leq 0$ on M.

(ii) If $1+(r-1)-\frac{(n-2)}{n^2}[S-n(r-1)]>0$ on M, from $S\geq (n-1)\frac{n(r-1)+2}{n-2}+\frac{n-2}{n(r-1)+2}$, we have

$$(n-2)[n(r-1)+2]S \geq (n-1)n^2(r-1)^2+4n(n-1)(r-1)+n^2,$$

that is

$$(n-2)\{4n(n-1)(r-1)+2n^2+(n-2)^2n(r-1)\}S$$
$$\geq \{2n(n-1)(r-1)+n^2\}^2+(n-2)^2n^2(n-1)(r-1)^2.$$

Hence

$$\{n+n(r-1)-\frac{n-2}{n}[S-n(r-1)]\}^2 \leq \frac{(n-2)^2}{n^2}\{n(n-1)(r-1)+S\}\{S-n(r-1)\}. \tag{3.24}$$

Since $1+(r-1)-\frac{(n-2)}{n^2}[S-n(r-1)]>0$ on M, from (3.24), we have

$$n+n(r-1)-\frac{n-2}{n}[S-n(r-1)] \leq \frac{(n-2)}{n}\sqrt{[n(n-1)(r-1)+S][S-n(r-1)]}. \tag{3.25}$$

From (3.14), we infer that $\lambda\mu+1\leq 0$ on M.

(iii) If $1+(r-1)-\frac{(n-2)}{n^2}[S-n(r-1)]\leq 0$ at a point p of M and $1+(r-1)-\frac{(n-2)}{n^2}[S-n(r-1)]>0$ at other points of M, in this case, from (i) and (ii), we have at point p, $\lambda\mu+1\leq 0$ and at other points of M, also $\lambda\mu+1\leq 0$. Therefore, we obtain $\lambda\mu+1\leq 0$ on M.

Therefore, we know that if (3.14) holds on M, then $\lambda\mu + 1 \leq 0$ on M. By Proposition 3.6, we obtain that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = \frac{n-2}{nr}$. This completes the proof of Theorem 1.4.

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