# Multitime linear-quadratic regulator problem based on curvilinear integral

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**Abstract.** This paper interrelates the performance criteria involving path independent curvilinear integrals, the multitime maximum principle, the multitime Hamilton-Jacobi-Bellman PDEs and the multitime dynamic programming, to study the linear-quadratic regulator problems and to characterize the optimal control by means of multitime variant of the Riccati PDE that may be viewed as a feedback law.

Section 1 recalls the theory of an optimal control problem with curvilinear integral cost functional, the notion of maximum value function and the multitime Hamilton-Jacobi-Bellman PDEs. It explains also the connections between dynamic programming and the multitime maximum principle. Section 2 solves the linear-quadratic regulator problem via multitime maximum principle. Section 3 describes the linear-quadratic regulator problem via multitime Hamilton-Jacobi-Bellman PDEs.

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## 1 Multitime optimal control problem and Hamilton-Jacobi-Bellman PDEs

We introduce a multitime dynamic programming method based on multitime Hamilton-Jacobi-Bellman PDEs. These PDEs are equivalent to multitime Hamilton PDEs system and the multitime maximum principle.

### 1.1 Optimal control problem with running cost and terminal cost

The cost functionals of mechanical work type are very important for applications, but few researchers refer to them. In spite of mathematical difficulties, a systematic study of this kind of functionals was realized recently by our research group [9]-[21].

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A multitime optimal control problem, where the cost functional is the sum between a path independent curvilinear integral and a function of the final event, and the evolution PDE is an $m$-flow, has the form [20]

$$\max_{u(\cdot)} P(u(\cdot)) = \int_{\Gamma_{0,t_0}} X^0_\alpha(t, x(t), u(t))dt^\alpha + g(x(t_0))$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t), u(t)), i = 1, ..., n; \alpha = 1, ..., m,$$
$$u(t) \in \mathcal{U},\ t \in \Omega_{0,t_0};\ x(0) = x_0, x(t_0) = x_{t_0}.$$

This problem requires the following data: the *multitime* (multi-parameter of evolution) $t = (t^\alpha) \in R^m_+$; an arbitrary $C^1$ curve $\Gamma_{0,t_0}$ joining the diagonal opposite points $0 = (0, ..., 0)$ and $t_0 = (t^1_0, ..., t^m_0)$ in the parallelepiped $\Omega_{0,t_0} = [0, t_0]$ (multi-time interval) in $R^m_+$; a $C^2$ *state vector* $x : \Omega_{0,t_0} \to R^n$, $x(t) = (x^i(t))$; a $C^1$ *control vector* $u : \Omega_{0,t_0} \to U \subset R^k$, $u(t) = (u^a(t)),\ a = 1, ..., k$; a *running cost* $X^0_\alpha(t, x(t), u(t))dt^\alpha$ as a nonautonomous closed (completely integrable) *Lagrangian 1-form*, i.e., it satisfies $D_\beta X^0_\alpha = D_\alpha X^0_\beta$ ($D_\alpha$ is the total derivative operator) or

$$\left(\frac{\partial X^0_\alpha}{\partial u^a}\delta^\gamma_\beta - \frac{\partial X^0_\beta}{\partial u^a}\delta^\gamma_\alpha\right)\frac{\partial u^a}{\partial t^\gamma} = X^i_\alpha\frac{\partial X^0_\beta}{\partial x^i} - X^i_\beta\frac{\partial X^0_\alpha}{\partial x^i} + \frac{\partial X^0_\beta}{\partial t^\alpha} - \frac{\partial X^0_\alpha}{\partial t^\beta};$$

the *terminal cost functional* $g(x(t_0))$; the $C^1$ vector fields $X_\alpha = (X^i_\alpha)$ satisfying the complete integrability conditions ($m$-flow type problem), i.e., $D_\beta X_\alpha = D_\alpha X_\beta$ or

$$\left(\frac{\partial X_\alpha}{\partial u^a}\delta^\gamma_\beta - \frac{\partial X_\beta}{\partial u^a}\delta^\gamma_\alpha\right)\frac{\partial u^a}{\partial t^\gamma} = [X_\alpha, X_\beta] + \frac{\partial X_\beta}{\partial t^\alpha} - \frac{\partial X_\alpha}{\partial t^\beta},$$

where $[X_\alpha, X_\beta]$ means the *bracket* of vector fields. Some of the previous hypothesis select the set of all admissible controls (satisfying the complete integrability conditions, eventually a.e.)

$$\mathcal{U} = \left\{u : R^m_+ \to U \,\middle|\, D_\beta X^0_\alpha = D_\alpha X^0_\beta,\ D_\beta X_\alpha = D_\alpha X_\beta,\ a.e.\right\}.$$

The previous PDE evolution system is equivalent to the path-independent curvilinear integral equation

$$x(t) = x(0) + \int_{\gamma_{0,t}} X_\alpha(s, x(s), u(s))ds^\alpha,$$

where $\gamma_{0,t}$ is an arbitrary piecewise $C^1$ curve joining the opposite diagonal points 0 and $t$ of the parallelepiped $\Omega_{0,t} = [0, t] \subset \Omega_{0,t_0} = [0, t_0]$.

It is possible to show that in the multitime optimal control problems it is enough to use increasing curves.

**Definition**. A piecewise $C^1$ curve $\gamma_{0,t_0} : s^\alpha = s^\alpha(\tau),\ \tau \in [\tau_0, \tau_1],\ s(\tau_0) = 0, s(\tau_1) = t_0$ is called *increasing* if the tangent vector $(\dot{s}^\alpha)$ satisfies $(\dot{s}^\alpha) \geq 0$, where the equality is true only at isolated points.

If we use the *control Hamiltonian 1-form*

$$H_\alpha(t,x(t),u(t),p(t)) = X^0_\alpha(t,x(t),u(t)) + p_i(t)X^i_\alpha(t,x(t),u(t)),$$

we can formulate the *simplified multitime maximum principle* [20].

**Theorem 1**. *Suppose that the previous problem, with $X^0_\alpha, X^i_\alpha$ of class $C^1$, has an interior solution $\hat{u}(t) \in \mathcal{U}$ which determines the $m$-sheet of state variable $x(t)$. Then there exists a $C^1$ costate $p(t) = (p_i(t))$ defined over $\Omega_{0,t_0}$ such that the relations*

$$\frac{\partial p_j}{\partial t^\alpha}(t) = -\frac{\partial H_\alpha}{\partial x^j}(t,x(t),\hat{u}(t),p(t)),\ \forall t \in \Omega_{0,t_0};\ p_j(t_0) = 0$$

$$\frac{\partial x^j}{\partial t^\alpha}(t) = \frac{\partial H_\alpha}{\partial p_j}(t,x(t),\hat{u}(t),p(t)),\ \forall t \in \Omega_{0,t_0};\ x(0) = x_0$$

*and*

$$H_{\alpha u^a}(t,x(t),\hat{u}(t),p(t)) = 0,\ \forall t \in \Omega_{0,t_0}$$

*hold.*

## 1.2 Maximum value functions and multitime Hamilton-Jacobi-Bellman PDEs

Let us analyze the previous optimal control problem from Bellman point of view. We vary the starting multitime and the initial points (all possible choices of starting times and all possible initial points); for convenience, let say, $t \in \Omega_{0,t_0},\ x \in R^n$. We obtain a family of similar problems

$$\frac{\partial x^i}{\partial s^\alpha}(s) = X^i_\alpha(s,x(s),u(s)),\ x(t) = x,\ s \in \Omega_{t,t_0} \subset R^m_+$$

with the terminal cost

$$P_{x,t}(u(\cdot)) = \int_{\Gamma_{t,t_0}} X^0_\alpha(s,x(s),u(s))ds^\alpha + g(x(t_0)).$$

Suppose the cost defines the *maximum value function*

$$v(x,t) = \max_{u(\cdot)\in\mathcal{U}} P_{x,t}(u(\cdot)),\ x \in R^n,\ t \in \Omega_{0,t_0},$$

which satisfies the terminal condition $v(x,t_0) = g(x)$. If the maximum value function $v(x,t)$ satisfies some regularity conditions, then it is solution of special nonlinear PDEs system.

**Theorem 2**. *Suppose $v(x,t)$ is a $C^2$ function. Then it is the solution of the multitime Hamilton-Jacobi-Bellman PDEs system*

$$(mtHJB) \qquad \frac{\partial v}{\partial t^\beta}(x,t) + \max_{u\in U}\left\{\frac{\partial v}{\partial x^i}(x,t)X^i_\beta(t,x,u) + X^0_\beta(t,x,u)\right\} = 0$$

*with the terminal condition*

$$(t_0) \qquad v(x,t_0) = g(x), \; x \in R^n.$$

The proof of this Theorem will be given in a furthercoming paper.

**Remarks**. 1) The $(mtHJB)$ PDEs system is in fact of the form

$$\frac{\partial v}{\partial t^\beta}(x,t) + H_\beta\left(t, x, \frac{\partial v}{\partial x}(x,t)\right) = 0, \; x \in R^n, \; t \in \Omega_{0,t_0},$$

where $H_\beta = p_i(t)X^i_\beta(t,x,u) + X^0_\beta(t,x,u)$ is the *control Hamiltonian* 1*-form.*

2) Since the initial evolution $PDE$ system is completely integrable, the $(mtHJB)$ system is completely integrable.

3) Complementary results regarding the Hamilton-Jacobi-Bellman PDEs system can be found in [1]-[8]. Also, some ideas from [22]-[24] can be involved in this theory.

### 1.3 Connections between dynamic programming and the multitime maximum principle

We start with the multitime evolutionary dynamics

$$(PDE) \qquad \frac{\partial x^i}{\partial s^\alpha}(s) = X^i_\alpha(s, x(s), u(s)), \; t \le s \le t_0$$

and the cost functional

$$(P) \qquad P_{x,t}(u(\cdot)) = \int_{\gamma_{t,t_0}} X^0_\beta(s, x(s), u(s))ds^\beta + g(x(t_0)).$$

Suppose the cost produces the maximum value function

$$v(x,t) = \max_{u(\cdot)\in\mathcal{U}} P_{x,t}(u(\cdot)).$$

The costate $p$ in the multitime maximum principle is in fact the gradient with respect to $x$ of the maximum value function $v$, taken along an optimal $m$-sheet.

**Theorem 3 (costate and gradient)**. *Suppose $u^*(\cdot)$, $x^*(\cdot)$ is a solution of the control problem (PDE), (P). If the maximum value function $v$ is of class $C^2$, then the costate $p^*(\cdot) = (p^*{}_i(\cdot))$, which appears in the multitime maximum principles, is given by*

$$p^*{}_i(s) = \frac{\partial v}{\partial x^i}(x^*(s), s), \quad t \le s \le t_0.$$

The proof of this Theorem will be given in a furthercoming paper.

## 2 Linear-quadratic regulator problem via multitime maximum principle

The theory of multitime optimal control is concerned with operating a PDEs dynamic system at minimum cost. The case where the evolution is described by a set of first order linear PDEs and the cost is described by a quadratic functional is called the *linear-quadratic regulator problem*. One of the main results is that the solution of the linear-quadratic regulator problem is based on a *feedback controller*.

Let us accept that the evolution is given by the multitime linear control system

$$\frac{\partial x}{\partial t^\alpha}(t) = M_\alpha(t)x(t) + N_\alpha(t)u_\alpha(t),$$

$$t = (t^\alpha) \in R^m_+;\ M_\alpha \in \mathcal{M}_{n\times n};\ N_\alpha \in \mathcal{M}_{n\times k},\ \alpha = 1,\dots,m,$$

under the piecewise complete integrability conditions for the system and for its associated homogeneous system. The objective is to maximize

$$P(u(\cdot)) = -\frac{1}{2}x(t_0)^T Sx(t_0) - \frac{1}{2}\int_{\gamma_{0,t_0}} \left(x(t)^T Q_\alpha(t)x(t) + u_\alpha(t)^T R(t)u_\alpha(t)\right) dt^\alpha,$$

where $T$ denotes transposition, $S$ is a constant symmetric positive semi-definite matrix, $R(t)$ is a symmetric positive definite matrix, $Q_\alpha(t)$ are symmetric positive semi-definite matrices and

$$\left(x(t)^T Q_\alpha(t)x(t) + u_\alpha(t)^T R(t)u_\alpha(t)\right) dt^\alpha$$

is a closed 1-form.

The Hamiltonian 1-form is

$$H_\alpha = -\frac{1}{2}(x(t)^T Q_\alpha(t)x(t) + u_\alpha(t)^T R(t)u_\alpha(t)) + p(t)^T(M_\alpha(t)x(t) + N_\alpha(t)u_\alpha(t)).$$

Then

$$\frac{\partial x}{\partial t^\alpha}(t) = \frac{\partial H_\alpha}{\partial p} = M_\alpha(t)x(t) + N_\alpha(t)u_\alpha(t)$$

$$\frac{\partial p}{\partial t^\alpha}(t) = -\nabla_x H_\alpha = Q_\alpha(t)x(t) - M_\alpha(t)^T p(t),\ p(t_0) = Sx(t_0)$$

$$H_{\alpha u_\beta} = (-R(t)u_\alpha(t) + N_\alpha(t)^T p(t))\delta_{\alpha\beta} = 0.$$

It follows

$$u_\alpha(t) = R(t)^{-1}N_\alpha(t)^T p(t)$$

and

$$\frac{\partial x}{\partial t^\alpha}(t) = M_\alpha(t)x(t) + N_\alpha(t)R(t)^{-1}N_\alpha(t)^T p(t).$$

We can justify the existence of a quadratic matrix $K(t)$ such that

$$p(t) = K(t)x(t).$$

This gives the feed-back control law

$$u^*_\alpha(t) = R(t)^{-1}N_\alpha(t)^T K(t)x(t).$$

We obtain

$$\frac{\partial x}{\partial t^\alpha}(t) = \left(M_\alpha(t) + N_\alpha(t)R(t)^{-1}N_\alpha(t)^T K(t)\right)x(t).$$

On the other hand

$$\frac{\partial p}{\partial t^\alpha}(t) = \frac{\partial K}{\partial t^\alpha}(t)x(t) + K(t)\frac{\partial x}{\partial t^\alpha}(t)$$

$$= \left(\frac{\partial K}{\partial t^\alpha}(t) + K(t)\left(M_\alpha(t) + N_\alpha(t)R(t)^{-1}N_\alpha(t)^T K(t)\right)\right)x(t).$$

But

$$\frac{\partial p}{\partial t^\alpha}(t) = (Q_\alpha(t) - M_\alpha(t)^T K(t))x(t).$$

Equating, we find

$$[\frac{\partial K}{\partial t^\alpha}(t) + K(t)N_\alpha(t)R(t)^{-1}N_\alpha(t)^T K(t) + K(t)M_\alpha(t) + M_\alpha(t)^T K(t)) - Q_\alpha(t)]x(t) = 0$$

On the other hand $x(t)$ is arbitrary, since this relation holds for arbitrary choice of initial state $x(0)$ and $K(t)$ does not depend upon the initial state vector. It follows the Riccati PDEs

$$\frac{\partial K}{\partial t^\alpha}(t) + K(t)N_\alpha(t)R(t)^{-1}N_\alpha(t)^T K(t) + K(t)M_\alpha(t) + M_\alpha(t)^T K(t)) - Q_\alpha(t) = 0$$

$$K(t_0) = -S.$$

This PDE can be solved backward in multitime from $t_0$ to 0 with the Kalman matrices $R(t)^{-1}N_\alpha(t)^T K(t)$ stored in order to obtain the feedback control law. The convexity of each $H_\alpha$ shows that $u^*_\alpha(t)$ is a unique maximizer.

The solution $K(t)$ of Riccati PDEs is symmetric since it and the transpose $K(t)^T$ satisfy the same PDE and the same terminal condition. If the matrix $S$ is positive definite, then the matrix $K(t_0)$ is positive definite and $K(t)$ is also positive definite for each $t \in [0, t_0]$.

**Theorem 4**. *Suppose the linear regulator problem is formulated for unbounded $u_\alpha(t)$, specified $t_0$, positive semidefinite matrices $S$, $Q_\alpha(t)$, and positive definite matrix $R(t)$. Then there exists a unique optimal feedback control*

$$u^*_\alpha(t) = R(t)^{-1}N_\alpha(t)^T K(t)x(t),$$

*where $K(t)$ is the unique solution of the Riccati PDEs satisfying the given boundary condition.*

**Remark**. Instead the matrix $R(t)$ we can use a set of matrices $R_\alpha(t)$, $\alpha = 1, ..., m$.

**Example**. Let us consider the dynamic PDEs system

$$\frac{\partial x}{\partial t^\alpha}(t) = -a_\alpha x(t) - u_\alpha(t),\ x(0) = x_0,\ \alpha = 1, 2$$

and the objective functional

$$P(u(\cdot)) = -\frac{1}{2}\int_{\gamma_{0,\infty}} \left(q_\alpha x(t)^2 + r_\alpha u_\alpha(t)^2\right) dt^\alpha,\ q_\alpha, r_\alpha > 0,\ \alpha = 1, 2,$$

under the complete integrability conditions

$$a_2 u_1 = a_1 u_2, \ \frac{\partial u_2}{\partial t^1} = \frac{\partial u_1}{\partial t^2}$$

$$-q_2 a_1 x^2 - q_2 u_1 + r_2 u_2 \frac{\partial u_2}{\partial t^1} = -q_1 a_2 x^2 - q_1 u_2 + r_1 u_1 \frac{\partial u_1}{\partial t^2}.$$

For the maximization, we use the Hamiltonian 1-form (with $t$ omitted)

$$H_\alpha = -\frac{1}{2}(q_\alpha x^2 + r_\alpha u_\alpha^2) + p(-a_\alpha x - u_\alpha).$$

The optimal policy, obtained from $\frac{\partial H_\alpha}{\partial u_\alpha} = 0$, is $u_\alpha^* = -\frac{p}{r_\alpha}$, $r_1 u_1^* = r_2 u_2^*$. Applying for $p(t) = kx(t)$, $k = \text{const} > 0$, i.e., $u_\alpha^* = -\frac{k}{r_\alpha} x(t)$, the Riccati PDEs system is reduced to an algebraic system

$$k^2 - 2 r_\alpha a_\alpha k - r_\alpha q_\alpha = 0, \ \alpha = 1, 2$$

with the solution

$$k = \frac{r_1 q_1 - r_2 q_2}{r_2 a_2 - r_1 a_1}, \ (r_1 q_1 - r_2 q_2)(r_2 a_2 - r_1 a_1) > 0.$$

Writing this solution in the form

$$k = a_\alpha r_\alpha + r_\alpha \sqrt{a_\alpha^2 + \frac{q_\alpha}{r_\alpha}}$$

and denoting $b_\alpha = \sqrt{a_\alpha^2 + \frac{q_\alpha}{r_\alpha}}$, we obtain

$$k = (a_\alpha + b_\alpha) r_\alpha, \ u_\alpha^* = -(a_\alpha + b_\alpha) x(t), \ \frac{\partial x}{\partial t^\alpha}(t) = -b_\alpha x(t)$$

and the optimal solution is

$$x^*(t) = x_0 e^{-b_\alpha t^\alpha}, \ u^*(t) = (a_\alpha + b_\alpha) x_0 e^{-b_\alpha t^\alpha}.$$

**Commentary**. Consider a country with a foreign debt of $x(t)$ dollars and a repayment policy $u(t)$ at "two time" $t = (t^1, t^2)$. We can formulate the previous optimal problem. It follows the optimal repayment policy and the resulting debt which decreases at the exponential rate $(b_1, b_2)$ over two-time.

# 3 Linear-quadratic regulator problem via multitime Hamilton-Jacobi-Bellman PDEs

We formulate again a linear-quadratic regulator problem using the matrices

$$M_\alpha, \, Q_\alpha, S \in \mathcal{M}_{n\times n}; \ N_\alpha \in \mathcal{M}_{n\times k}; \ R_\alpha \in \mathcal{M}_{r\times r},$$

where $Q_\alpha, R_\alpha, S$ are symmetric positive semi-definite matrices and $R_\alpha$ are invertible (positive definite) matrices. The idea is to minimize the quadratic cost functional

$$P(u(\cdot)) = \frac{1}{2}x(t_0)^T Sx(t_0) + \frac{1}{2}\int_{\gamma_{t,t_0}} \left(x(s)^T Q_\alpha(s)x(s) + u_\alpha(s)^T R_\alpha(s)u_\alpha(s)\right) ds^\alpha,$$

over the multitime linear dynamics

$$\frac{\partial x}{\partial s^\alpha}(s) = M_\alpha(s)x(s) + N_\alpha(s)u_\alpha(s),\ t \leq s \leq t_0,\, x(t) = x,$$

knowing that the complete integrability conditions for the curvilinear integral and for evolution PDEs are satisfied, and the control values $u_\alpha$ are unconstrained, i.e., the control parameter values can range over all $R^{r+m}$. In other words, we want to maximize the cost functional

$$P_{x,t}(u(\cdot)) = -\frac{1}{2}x(t_0)^T Sx(t_0)$$

$$-\frac{1}{2}\int_{\gamma_{t,t_0}} \left(x(s)^T Q_\alpha(s)x(s) + u_\alpha(s)^T R_\alpha(s)u_\alpha(s)\right) ds^\alpha,$$

with a normal linear PDEs system as constraint.

To design an optimal control we use the associated dynamic programming problem and its solution. Denoting

$$X_\alpha = M_\alpha x + N_\alpha u_\alpha,\, X_\alpha^0 = -x^T Q_\alpha x - u^T R_\alpha u_\alpha,\, g = -x^T Sx,$$

we build the multitime Hamilton-Jacobi-Bellman PDEs system (mtHJB)

$$\frac{\partial v}{\partial t^\alpha}(x,t) + \max_{u\in R^{r+m}} \left\{(\nabla v)^T N_\alpha u_\alpha - u_\alpha^T R_\alpha u_\alpha\right\} + (\nabla v)^T M_\alpha x - x^T Q_\alpha x = 0,$$

with the terminal condition $v(x,t_0) = -x(t_0)^T Sx(t_0)$.

**Maximization**. Having in mind that each matrix $R_\alpha(t)$ is positive semidefinite, the maximum

$$\max_{u\in R^{r+m}} \left\{(\nabla v)^T N_\alpha u_\alpha - u_\alpha^T R_\alpha u_\alpha\right\}$$

is attained at the point $u = (u_\alpha)$, where $u_\alpha$ is a critical point of the function

$$\psi_\alpha = (\nabla v)^T N_\alpha u_\alpha - u_\alpha^T R_\alpha u_\alpha.$$

Solving the equation $\dfrac{\partial \psi_\alpha}{\partial u_\alpha} = 0$, i.e., $(\nabla v)^T N_\alpha - u_\alpha^T R_\alpha = 0$, we find

$$u_\alpha = \frac{1}{2}R_\alpha^{-1}N_\alpha^T \nabla v.$$

This is the optimal control, under the hypothesis that there exist the function $v$ satisfying (mtHJB) PDEs with terminal condition $(t_0)$.

**Finding the maximum value function**. Replacing $u_\alpha = \dfrac{1}{2}R_\alpha^{-1}N_\alpha^T \nabla v$ into (mtHJB) PDEs, we obtain the problem

$$\frac{\partial v}{\partial t^\alpha} + \frac{1}{4}(\nabla v)^T N_\alpha R_\alpha^{-1} N_\alpha^T \nabla v + (\nabla v)^T M_\alpha x - x^T Q_\alpha x = 0$$

$$v(x, t_0) = -x^T(t_0)Sx(t_0).$$

Let us look for a solution of the form $v(x,t) = x^T K(t)x$, i.e., we try to find a symmetric $n \times n$ matrix of functions $K(t)$ such that $v(x,t)$ is a solution of the problem (mtHJB). Since $\frac{\partial v}{\partial t^\alpha} = x^T \frac{\partial K}{\partial t^\alpha} x$ and $\nabla_x v = 2K(t)x$, the (mtHJB) becomes

$$x^T \left( \frac{\partial K}{\partial t^\alpha}(t) + K(t)N_\alpha(t)R_\alpha^{-1}(t)N_\alpha^T(t)K(t) + 2K(t)M_\alpha(t) - Q_\alpha(t) \right) x = 0.$$

On the other hand,

$$2x^T K M_\alpha x = x^T K M_\alpha x + (x^T K M_\alpha x)^T = x^T K M_\alpha x + x^T M_\alpha^T K x.$$

Consequently,

$$x^T [\frac{\partial K}{\partial t^\alpha}(t) + K(t)N_\alpha(t)R_\alpha^{-1}(t)N_\alpha^T(t)K(t) + K(t)M_\alpha(t) + M_\alpha^T(t)K(t) - Q_\alpha(t)]x = 0.$$

Identifying after $x$, we find the multitime Riccati matrix PDEs

$$\frac{\partial K}{\partial t^\alpha}(t) + K(t)N_\alpha(t)R_\alpha^{-1}(t)N_\alpha^T(t)K(t) + K(t)M_\alpha(t) + M_\alpha^T(t)K(t) - Q_\alpha(t) = 0.$$

Since $v(x,t_0) = x^T(t_0)K(t_0)x(t_0) = -x^T(t_0)Sx(t_0)$, it appears the terminal condition $K(t_0) = -S$. If this last problem admits a solution $K(t)$, i.e., the Riccati PDEs satisfy the complete integrability conditions, then we can construct the optimal feedback control

$$u_\alpha = \frac{1}{2} R_\alpha^{-1} N_\alpha^T K(t)x(t).$$

**Theorem 5**. *Suppose the linear regulator problem is formulated for unbounded $u_\alpha(t)$, specified $t_0$, positive semidefinite matrices $S$, $Q_\alpha(t)$, and positive definite matrices $R_\alpha(t)$. Then there exists a unique optimal feed-back control*

$$u_\alpha^*(t) = R(t)_\alpha^{-1} N_\alpha(t)^T K(t)x(t),$$

*where $K(t)$ is the unique solution of the Riccati PDEs satisfying the given boundary condition.*

It remains to show that the multitime Riccati matrix PDEs does have a solution.

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