# Horizontal forms on jet bundles

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*Dedicated to the 70-th anniversary*
*of Professor Constantin Udriste*

**Abstract.** We give an elementary proof that, if the order of a horizontal $s$-form on a jet bundle does not increase under the operation of the horizontal differential, then the coefficients of the form must be polynomial of degree $s$ in the highest-order coordinates.

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## 1 Introduction

Let $\pi : E \to M$ be a fibred manifold with $\dim M = m$ and $\dim E = m + n$, and let $J^k\pi$ denote the $k$-th order jet manifold for $0 \le k \le \infty$ with projections $\pi_{k,0} : J^k\pi \to E$, $\pi_k : J^k\pi \to M$. Taking a fibred chart $U$ on $E$ with coordinates $(x^i, u^a)$, where $1 \le i \le m$ and $1 \le a \le n$, the corresponding coordinates on $\pi_{k,0}^{-1}(U) \subset J^k\pi$ are$(x^i, u^a_I)$, where $I \in \mathbb{N}^m$ is a multi-index with length $0 \le |I| \le k$.

A differential form $\phi$ on $J^k\pi$ is said to be *horizontal* if it vanishes when contracted with any vector field vertical over $M$. If $f$ is a function on $J^k\pi$, the *total derivative*

$$d_i f = \frac{df}{dx^i} = \frac{\partial f}{\partial x^i} + \sum_{|I|=0}^{k} u^a_{I+1_i} \frac{\partial f}{\partial u^a_I}$$

is a function on $\pi_{k+1,0}^{-1}(U) \subset J^{k+1}\pi$. The *horizontal differential* $d_h$, given in coordinates by

$$d_h f = \frac{df}{dx^i} dx^i \, ,$$

is an operation on functions which incorporates the total derivatives and gives rise to a well-defined global horizontal 1-form on $J^{k+1}\pi$. The operation may be extended to act on horizontal $s$-forms $\phi$ by using using $d_h d = -dd_h$. Note that some authors use the notation $D, D_i$ instead of $d_h, d_i$, and use the terminology 'formal derivative' rather than 'total derivative'; indeed the coordinate formula for $d_i$ is simply a restatement of the chain rule in jet coordinates. A coordinate-free definition of $d_h$ may be found in, for example, [2].

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It is clear from the coordinate representation that total derivatives commute, so that $d_h^2 = 0$ and that

$$0 \to \Omega_h^0 J^k\pi \to \Omega_h^1 J^{k+1}\pi \to \Omega_h^2 J^{k+2}\pi \to \ldots \to \Omega_h^m J^{k+m}\pi$$

is a sequence. We are therefore led to ask about its exactness.

In the case $k = \infty$ it is known that the sequence is locally exact, and proofs of this are often given by embedding the sequence in a bicomplex known as the variational bicomplex. The proofs are not, however, straightforward. For instance, a proof given by Tulczyjew [3] involves intricate calculations, whereas one given (in a slightly different context) by Vinogradov [4] uses the heavyweight machinary of spectral sequences. More information about the various approaches to this problem may be found in a recent comprehensive review article by Vitolo [5].

The answer is different when $k$ is finite: in general,

$$0 \to \Omega_h^0 J^k\pi \to \Omega_h^1 J^{k+1}\pi \to \Omega_h^2 J^{k+2}\pi \to \ldots$$

is not exact, even locally, and the same is true for

$$\Omega_h^s E \to \Omega_h^{s+1} J^1\pi \to \Omega_h^{s+2} J^2\pi \to \ldots .$$

To see a simple example, take $M = \mathbb{R}^2$, $E = \mathbb{R}^2 \times \mathbb{R}^2$ and let

$$\phi = (u_1^1 u_2^2 - u_2^1 u_1^2)\, dx^1 \wedge dx^2 \in \Omega_h^2 J^1\pi\,;$$

then $d_h\phi = 0$, but if $\psi \in \Omega_h^1 E$ then $d_h\psi$ is linear in the first derivative coordinates and so cannot equal $\phi$. The difficulty arises because $d_h$ does not always increase the order of a horizontal form, even modulo $d_h$-exact forms. We are therefore led to define a horizontal form $\phi$ as having *stable order* if $\mathrm{o}(d_h\phi) \le \mathrm{o}(\phi)$, where $\mathrm{o}(\phi)$ denotes the order of $\phi \in \Omega_h^s J^k\pi$ and is defined by saying that $\mathrm{o}(\phi) = l \le k$ if $\phi$ is projectable to $J^l\pi$ but not to $J^{l-1}\pi$.

It is straightforward to find a sufficient condition for a horizontal form to have stable order. Suppose the $s$-form $\phi$ is given locally as a sum of terms

$$\widehat{\phi}{}^{I_1\cdots I_q}_{a_1\cdots a_q i_{q+1}\cdots i_s}\, d_h u^{a_1}_{I_1} \wedge \cdots \wedge d_h u^{a_q}_{I_q} \wedge dx^{i_{q+1}} \wedge \cdots \wedge dx^{i_s} \tag{1}$$

where $0 \le q \le s$, $|I_1| = \cdots = |I_q| = k-1$ and $\mathrm{o}\left(\widehat{\phi}{}^{I_1\cdots I_q}_{a_1\cdots a_q i_{q+1}\cdots i_s}\right) \le k-1$; then $\mathrm{o}(\phi) = k$ and $\mathrm{o}(d_h\phi) \le k$, so that $\phi$ has stable order. We may write $\phi$ as

$$\phi_{i_1\cdots i_s} dx^{i_1} \wedge \cdots \wedge dx^{i_s}\,,$$

and we may express the sufficient condition in terms of these coefficient functions by stating that $\phi$ has stable order when the $\phi_{i_1\cdots i_s}$ are polynomial of degree $s$ in the coordinates $u^a_J$ with $|J| = k$, and can be expressed as sums of determinants of these coordinates.

These latter conditions are also necessary: if $\mathrm{o}(\phi) = k$ and $\mathrm{o}(d_h\phi) \le k$ then it may be shown that the coefficients $\phi_{i_1\cdots i_s}$ are polynomials of degree $s$ in the highest-order coordinates, and $\phi$ is is given locally as a sum of terms of the form (1) above. The matter is discussed in Anderson's monograph [1], but the proof of necessity is, again, not straightforward.

In this note we give an elementary proof of the first necessary condition, about the polynomial structure of the coefficient functions:

**Theorem.** *If the horizontal $s$-form $\phi$ has order $k$ (where $0 \leq s < m$), and if the order of $d_h\phi$ does not exceed $k$, then the coefficients of $\phi$ must be polynomial of degree not exceeding $s$ in the $k$-th order derivative coordinates.*

The method we shall adopt is quite straightforward: we shall show that, whenever the coefficients are differentiated $s+1$ times, the result always equals zero. To demonstrate this, we shall make repeated use of a lemma which is derived directly from the order stability of $\phi$.

## 2 The fundamental lemma

If a function $f$ has order $k$ then necessarily the 1-form $d_h f$ has order $k+1$. Order stability applies only to $s$-forms $\phi$ with $s \geq 1$; it arises from skew-symmetry, so that the derivatives of the coefficients of an order-stable form $\phi$ with respect to the coordinates of order $k$ must satisfy a family of linear constraints.

**Fundamental Lemma.** *Let $\phi \in \Omega^s_h$ with $s < m$, and let the coordinate representation of $\phi$ be*

$$\phi = \phi_{i_1 i_2 \cdots i_s} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_s} \,,$$

*where the coefficient functions $\phi_{i_1 i_2 \cdots i_s}$ are skew-symmetric in all their indices. Suppose $d_h\phi \in \Omega^{s+1}_h$ satisfies*

$$\mathrm{o}(d_h\phi) \leq \mathrm{o}(\phi) = k \,.$$

*Then, for distinct indices $i_1, i_2, \ldots, i_s, j$ and any multi-index $J$ with $|J| = k$,*

$$\frac{\partial \phi_{i_1 i_2 \cdots i_s}}{\partial u^b_J} = \sum_{\substack{1 \leq q \leq s \\ J(i_q) > 0}} \frac{\partial \phi_{i_1 i_2 \cdots i_{q-1} j i_{q+1} \cdots i_s}}{\partial u^b_{J - 1_{i_q} + 1_j}} \,.$$

*Proof.* Write

$$d_h\phi = (d_j \phi_{i_1 i_2 \cdots i_s}) dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s}$$

so that, taking account of skew-symmetry, the coefficients of $d_h\phi$ satisfy

$$\mathrm{o}\left\{ (d_j \phi_{i_1 i_2 \cdots i_s} - \sum_{q=1}^{s} d_{i_q} \phi_{i_1 i_2 \cdots i_{q-1} j i_{q+1} \cdots i_s} \right\} \leq k \,;$$

thus, writing out the total derivatives explicitly,

$$\mathrm{o}\left\{ \frac{\partial \phi_{i_1 i_2 \cdots i_s}}{\partial x^j} - \sum_{q=1}^{s} \frac{\partial \phi_{i_1 i_2 \cdots i_{q-1} j i_{q+1} \cdots i_s}}{\partial x^{i_q}} \right.$$
$$\left. + \sum_{|I| \leq k} \left( u^a_{I+1_j} \frac{\partial \phi_{i_1 i_2 \cdots i_s}}{\partial u^a_I} - \sum_{q=1}^{s} u^a_{I+1_{i_q}} \frac{\partial \phi_{i_1 i_2 \cdots i_{q-1} j i_{q+1} \cdots i_s}}{\partial u^a_I} \right) \right\} \leq k \,.$$

Choose any coordinate $u^b_J$ where $|J| = k$, and any index $j$, and differentiate the coefficients of $d_h\phi$ with respect to the $(k+1)$-th order coordinate $u^b_{J+1_j}$. The first term in the sum over $I$ gives a non-zero result only when $I = J$, and each of the other terms in that sum gives a non-zero result only when $I + 1_{i_q} = J + 1_j$, occuring when $J(i_q) > 0$ and $I = J - 1_{i_q} + 1_j$. The terms outside that sum do not contribute. But overall the result must be zero, and so we obtain

$$\frac{\partial \phi_{i_1 i_2 \cdots i_s}}{\partial u^a_J} - \sum_{\substack{1 \le q \le s \\ J(i_q) > 0}} \frac{\partial \phi_{i_1 i_2 \cdots i_{q-1} j i_{q+1} \cdots i_s}}{\partial u^a_{J - 1_{i_q} + 1_j}} = 0$$

as required. □

**Corollary.** *If $J(i_q) = 0$ for $1 \le q \le s$ then*

$$\frac{\partial \phi_{i_1 i_2 \cdots i_s}}{\partial u^a_J} = 0\,.$$

□

# 3 An example

To see how the Fundamental Lemma can be used to prove that the coefficient functions must be polynomial, it is helpful to take an example. If the conditions of the Corollary are satisfied for a given function and for one of the coordinates with respect to which we are differentiating, then the result follows immediately. In general this will not be the case, and so the approach is to use the lemma to manipulate the derivatives until, eventually, the corollary can be applied to all the terms. We consider here the case $m = 3$, and take a 3rd-order 2-form

$$\phi = \phi_{12} dx^1 \wedge dx^2 + \phi_{23} dx^2 \wedge dx^3 + \phi_{31} dx^3 \wedge dx^1\,.$$

The third derivative of $\phi_{12}$ with respect to $u^a_{(123)}$, $u^b_{(222)}$ and $u^c_{(133)}$ then satisfies

$$\begin{aligned}
\frac{\partial^3 \phi_{12}}{\partial u^a_{(123)}\,\partial u^b_{(222)}\,\partial u^c_{(133)}} &= \frac{\partial^3 \phi_{32}}{\partial u^a_{(233)}\,\partial u^b_{(222)}\,\partial u^c_{(133)}} + \frac{\partial^3 \phi_{13}}{\partial u^a_{(133)}\,\partial u^b_{(222)}\,\partial u^c_{(133)}} \\
&= \frac{\partial^3 \phi_{31}}{\partial u^a_{(233)}\,\partial u^b_{(122)}\,\partial u^c_{(133)}} + \frac{\partial^3 \phi_{13}}{\partial u^a_{(133)}\,\partial u^b_{(222)}\,\partial u^c_{(133)}} \\
&= \frac{\partial^3 \phi_{31}}{\partial u^a_{(233)}\,\partial u^b_{(122)}\,\partial u^c_{(133)}} + \frac{\partial^3 \phi_{23}}{\partial u^a_{(133)}\,\partial u^b_{(222)}\,\partial u^c_{(233)}} \\
&\quad + \frac{\partial^3 \phi_{12}}{\partial u^a_{(133)}\,\partial u^b_{(222)}\,\partial u^c_{(123)}} \\
&= \frac{\partial^3 \phi_{21}}{\partial u^a_{(233)}\,\partial u^b_{(122)}\,\partial u^c_{(123)}} + \frac{\partial^3 \phi_{32}}{\partial u^a_{(233)}\,\partial u^b_{(122)}\,\partial u^c_{(233)}} \\
&\quad + \frac{\partial^3 \phi_{13}}{\partial u^a_{(133)}\,\partial u^b_{(122)}\,\partial u^c_{(233)}} + \frac{\partial^3 \phi_{32}}{\partial u^a_{(333)}\,\partial u^b_{(222)}\,\partial u^c_{(123)}} \\
&= 2\frac{\partial^3 \phi_{31}}{\partial u^a_{(333)}\,\partial u^b_{(122)}\,\partial u^c_{(123)}} + \frac{\partial^3 \phi_{31}}{\partial u^a_{(233)}\,\partial u^b_{(112)}\,\partial u^c_{(233)}} \\
&\quad + \frac{\partial^3 \phi_{12}}{\partial u^a_{(133)}\,\partial u^b_{(122)}\,\partial u^c_{(223)}} \\
&= 2\frac{\partial^3 \phi_{31}}{\partial u^a_{(333)}\,\partial u^b_{(122)}\,\partial u^c_{(122)}} + 3\frac{\partial^3 \phi_{32}}{\partial u^a_{(333)}\,\partial u^b_{(122)}\,\partial u^c_{(223)}} \\
&\quad + \frac{\partial^3 \phi_{21}}{\partial u^a_{(233)}\,\partial u^b_{(112)}\,\partial u^c_{(223)}} \\
&= 2\frac{\partial^3 \phi_{32}}{\partial u^a_{(333)}\,\partial u^b_{(122)}\,\partial u^c_{(222)}} + 4\frac{\partial^3 \phi_{31}}{\partial u^a_{(333)}\,\partial u^b_{(112)}\,\partial u^c_{(223)}} \\
&= 2\frac{\partial^3 \phi_{31}}{\partial u^a_{(333)}\,\partial u^b_{(112)}\,\partial u^c_{(222)}} + 4\frac{\partial^3 \phi_{21}}{\partial u^a_{(333)}\,\partial u^b_{(112)}\,\partial u^c_{(222)}} = 0,
\end{aligned}$$

where at each step we have applied the lemma to all the non-zero terms from the preceding step. In this case there are eight steps to the process. For each term we have a choice of three derivatives to which we might apply the lemma; the key, of course, is to make a good choice and to avoid going round in circles. In the proof of the theorem, we deescribe how to make this choice.

# 4 Proof of the theorem

We now return to the general case, and show that

$$\frac{\partial^{s+1}\phi_{i_1 i_2\cdots i_s}}{\partial u^b_J\,\partial u^{a_1}_{I_1}\cdots\partial u^{a_s}_{I_s}} = 0$$

for any indices $i_1,\ldots,i_s$ and any coordinate functions $u^{a_1}_{I_1},\ldots,u^{a_s}_{I_s},u^b_J$, where the multi-indices $I_1,\ldots,I_s,J$ all have length $k$. The strategy of the proof will be to develop an algorithm for applying the Fundamental Lemma to the initial term and then to all subsequent terms, and to develop a mechanism showing that, eventually, all the terms must vanish.

Start by choosing an index $j\notin\{i_1,\ldots,i_s\}$. If at any given step there is a term

$$\frac{\partial^{s+1}\phi_{i_1 i_2\cdots i_s}}{\partial u^b_{\tilde J}\,\partial u^{a_1}_{\tilde I_1}\cdots\partial u^{a_s}_{\tilde I_s}}$$

where $\tilde I_1,\tilde I_2,\ldots,\tilde I_s,J$ are multi-indices of length $k$ and where the index $j$ does *not* appear in the function being differentiated, then use the Fundamental Lemma to replace this by

$$\sum_{\substack{1\le q\le s\\ \tilde J(i_q)>0}}\frac{\partial^{s+1}\phi_{i_1 i_2\cdots i_{q-1} j i_{q+1}\cdots i_s}}{\partial u^b_{\tilde J-1_{i_q}+1_j}\,\partial u^{a_1}_{\tilde I_1}\cdots\partial u^{a_s}_{\tilde I_s}}\,.$$

In the resulting non-zero terms, the value of $\tilde J(j)$ has increased by one, whereas the other multi-indices are unchanged. On the other hand, if at any given step there is a term

$$\frac{\partial^{s+1}\phi_{i_1 i_2\cdots i_{p-1} j i_{p+1}\cdots i_s}}{\partial u^b_{\tilde J}\,\partial u^{a_1}_{\tilde I_1}\cdots\partial u^{a_s}_{\tilde I_s}}$$

where the index $j$ *does* appear in the function being differentiated, then again use the Fundamental Lemma to replace this by

$$\frac{\partial^{s+1}\phi_{i_1 i_2\cdots i_s}}{\partial u^b_{\tilde J}\,\partial u^{a_1}_{\tilde I_1}\cdots\partial u^{a_p}_{\tilde I_p-1_j+1_{i_p}}\cdots\partial u^{a_s}_{\tilde I_s}} + \sum_{\substack{1\le q\le s,\ q\ne p\\ \tilde I_p(i_q)>0}}\frac{\partial^{s+1}\phi_{i_1 i_2\cdots i_{q-1} i_p i_{q+1}\cdots i_{p-1} j i_{p+1}\cdots i_s}}{\partial u^b_{\tilde J}\,\partial u^{a_1}_{\tilde I_1}\cdots\partial u^{a_p}_{\tilde I_p-1_{i_q}+1_{i_p}}\cdots\partial u^{a_s}_{\tilde I_s}}$$

where the separate first term is taken as zero if if $\tilde I_p(j)=0$. In the resulting non-zero terms, the value of $\tilde I_p(i_p)$ has now increased by one, whereas the other multi-indices are unchanged. Note that in both cases the resulting terms have one of the two structures described, and so the procedure may be continued indefinitely.

Now associate with each non-zero term

$$\frac{\partial^{s+1}\phi_{i_1 i_2 \cdots i_s}}{\partial u^b_{\tilde{J}}\,\partial u^{a_1}_{\tilde{I}_1}\cdots\partial u^{a_s}_{\tilde{I}_s}} \quad \text{or} \quad \frac{\partial^{s+1}\phi_{i_1 i_2 \cdots i_{p-1} j i_{p+1} \cdots i_s}}{\partial u^b_{\tilde{J}}\,\partial u^{a_1}_{\tilde{I}_1}\cdots\partial u^{a_s}_{\tilde{I}_s}}$$

the natural number

$$N = \tilde{J}(j) + \sum_{q=1}^{s} \tilde{I}_q(i_q)\,.$$

Initially $N \geq 0$, and at each stage of the algorithm the value of $N$ in each new non-zero term has increased by 1. Thus, after applying the algorithm $k(s+1)+1$ times, we have $N > k(s+1)$ for each non-zero term. But $\tilde{J}(j) \leq |\tilde{J}| = k$ and $\tilde{I}_q(i_q) \leq |\tilde{I}_q| = k$, giving $N \leq k(s+1)$. It follows that, after applying the algorithm $k(s+1)+1$ times, all the terms must be zero. □

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