The Aleksandrov-Fenchel type inequalities for volume differences

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. In this paper we establish the Aleksandrov-Fenchel type inequality for volume differences function of convex bodies and the Aleksandrov-Fenchel inequality for Quermassintegral differences of mixed projection bodies, respectively. As applications, we give positive solutions of two open problems.

M.S.C. 2000: 52A40.
Key words: Quermassintegral difference function; convex body; projection body; the Brunn-Minkowski inequality; the Aleksandrov-Fenchel inequality.

1 Introduction

The well-known classical Aleksandrov-Fenchel inequality can be stated as follows.

The Aleksandrov-Fenchel inequality *Let $K_1, \ldots, K_n$ be compact convex sets in $\mathbb{R}^n$ and $0 \leq r \leq n$. Then*

$$V(K_1, \ldots, K_n)^r \geq \prod_{j=1}^{r} V(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n). \tag{1.1}$$

The quantities $V(K_1, \ldots, K_n)$ and $V(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n)$ are mixed volumes. Proofs of inequality (1.1), established by A. D. Aleksandrov in 1937, can be found in [14]. The equality conditions are unknown even today. An analog of the Aleksandrov-Fenchel inequality for mixed discriminants (see [14]) was used by G. P. Egorychev in 1981 to solve the van der Waerden conjecture concerning the permanent of a doubly stochastic matrix. See [16, Chaper 6] for a wealth of information and references.

In 1975, Lutwak established the dual Aleksandrov-Fenchel inequality as follows (see [10]).

Balkan Journal of Geometry and Its Applications, Vol.15, No.1, 2010, pp. 163-172.

The dual Aleksandrov-Fenchel inequality. *If $K_1,\ldots,K_n$ are star bodies and $0\le r\le n$. Then*

$$\tilde{V}(K_1,\ldots,K_n)^r \le \prod_{j=1}^{r}\tilde{V}(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_n), \tag{1.2}$$

with equality if and only if $K_1,\ldots,K_n$ are dilates of each other.

The quantities $\tilde{V}(K_1,\ldots,K_n)$ and $\tilde{V}(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_n)$ are dual mixed volumes.

In 1993, Lutwak established the Aleksandrov-Fenchel inequality for mixed projection bodies as follows (see [11]).

The Aleksandrov-Fenchel inequality for mixed projection bodies. *If $K_1,\ldots,K_{n-1}$ are convex bodies, then for $0\le r\le n-1$*

$$V(\Pi(K_1,\ldots,K_{n-1}))^r \ge \prod_{j=1}^{r}V(\Pi(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1})), \tag{1.3}$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are homothetic.

The quantities $V(\Pi(K_1,\ldots,K_{n-1}))$ and $V(\Pi(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}))$ denote volumes of mixed projection bodies.

In 2004, Leng and Zhao et al established the polar form of Aleksandrov-Fenchel inequality (1.3) as follows (see [9]).

The Aleksandrov-Fenchel inequality for polars mixed projection bodies. *If $K_1,\ldots,K_{n-1}\in\mathcal{K}^n$, then for $0\le r\le n-1$*

$$V(\Pi^*(K_1,\ldots,K_{n-1}))^r \le \prod_{j=1}^{r}V(\Pi^*(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1})). \tag{1.4}$$

The equality condition in (1.4) are, in general, unknown.

The quantities $V(\Pi^*(K_1,\ldots,K_{n-1}))$ and $V(\Pi^*(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}))$ denote volumes of polars mixed projection bodies.

On the other hand, in 2004, Leng [8] established Minkowski inequality and Brunn-Minkowski inequality for the volume differences, respectively as follows

Theorem 1.1. *Suppose that K and D are compact domains, L is a convex body, and $D\subset K, D'\subset L, D'$ is a homothetic copy of D. Then*

$$(V_1(K,L)-V_1(D,D'))^n \ge Dv(K,D)^{n-1}Dv(L,D'), \tag{1.5}$$

with equality if and only if K and L are homothetic and $(V(K),V(D))=\mu(V(L),V(D'))$, where μ is a constant. Moreover, $Dv(K,D)$ denotes the volume difference function of compact domains K and D (see [8]).

Theorem 1.2. *If K, L, and D be convex bodies in $\mathbb{R}^n$, $D \subset K$, D' is a homothetic copy of D. Then*

$$Dw_i(K+L, D+D')^{1/(n-i)} \geq Dw_i(K,D)^{1/(n-i)} + Dw_i(L,D')^{1/(n-i)}, \tag{1.6}$$

with equality for $0 \leq i < n-1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant. Moreover, $Dw_i(K,D)$ denotes the i-Quermassintegral difference function of convex bodies K and D (see section 2).

According to the classical theory in convex bodies geometry, on getting (1.5) and (1.6), a natural conjecture is whether The Aleksandrov-Fenchel inequality for the volume differences exists? More precisely, an open problem was posed as follows (see [20], also see [18]).

Open Problem 1. *Let $K_i (i = 1, 2, \ldots, n)$, $0 \leq r \leq n$, and $D_i (i = 1, 2, \ldots, n)$ be convex bodies in $\mathbb{R}^n$, $D_i \subset K_i$ and $D_i (i = 1, 2, \ldots, n)$ be homothetic copies of each other, respectively. Does the following inequality hold ?*

$$D_v((K_1, \ldots, K_n), (D_1, \ldots, D_n))^r$$

$$\geq \prod_{j=1}^{r} D_v((\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n), (\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_n)).$$

Where, $V(K_1, \ldots, K_n) - V(D_1, \ldots, D_n)$ is written as $D_v((K_1, \ldots, K_n), (D_1, \ldots, D_n))$, denoting the volume difference function of mixed bodies $(K_1, \ldots, K_n)$ and $(D_1, \ldots, D_n)$.

Recently, Zhao and Cheung [20] extended (1.5) and (1.6) from general volume difference to p-Quermassintegral difference and established the Minkowski inequality and Brunn-Minkowski inequality for p-Quermassintegral difference function. In particular, in [20], they also extended (1.5) and (1.6) from general convex bodies to mixed projection bodies and get the Minkowski inequality and Brunn-Minkowski inequality for Quermassintegral difference function of mixed projection bodies as follows.

Theorem 1.3. ([20]) *Let K, L, and D be convex bodies in $\mathbb{R}^n$, $D \subset K, D'$ be a homothetic copy of D, then for $0 \leq j < n-2$,*

$$Dw_i(\Pi_j(K+L), \Pi_j(D+D'))^{1/(n-i)(n-j-1)}$$

$$\geq Dw_i(\Pi_j K, \Pi_j D)^{1/(n-i)(n-j-1)} + Dw_i(\Pi_j L, \Pi_j D')^{1/(n-i)(n-j-1)}, \tag{1.7}$$

with equality for $0 \leq i < n-1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

Theorem 1.4. ([20]) *Let K, L, and D be convex bodies in $\mathbb{R}^n$, $D \subset K$, D' be a homothetic copy of D, and $0 \leq j < n-1$, then*

$$Dw_i(\Pi_j(K,L), \Pi_j(D,D'))^{n-1} \geq Dw_i(\Pi K, \Pi D)^{n-j-1} Dw_i(\Pi L, \Pi D')^j, \tag{1.8}$$

with equality for $0 \leq i < n-1$ if and only if K and L are homothetic.

Similarly, with inequalities (1.7) and (1.8), another natural conjecture is whether the Aleksandrov-Fenchel inequality for volume differences of mixed projection bodies exists? More precisely, another open problem was posed as follows (see [20], see also [18]).

Open Problem 2. *Let* $K_i(i = 1, \ldots, n-1)$, $0 \le r \le n-1$, *and* $D_i(i = 1, \ldots, n-1)$ *be convex bodies in* $\mathbb{R}^n$, $D_i \subset K_i$ *and* $D_i(i = 1, 2, \ldots, n-1)$ *are homothetic copies of each other*, *respectively. Does the following inequality hold ?*

$$D_v(\Pi(K_1, \ldots, K_{n-1}), \Pi(D_1, \ldots, D_{n-1}))^r$$

$$\ge \prod_{j=1}^{r} D_v(\Pi(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}), \Pi(\underbrace{D_j, \ldots, D_j}_{r}, D_{r+1}, \ldots, D_{n-1})),$$

where, $V(\Pi(K_1, \ldots, K_{n-1})) - V(\Pi(D_1, \ldots, D_{n-1}))$ *is written as* $D_v(\Pi(K_1, \ldots, K_{n-1}), \Pi(D_1, \ldots, D_{n-1}))$, *denoting the volume difference function of mixed prosection bodies* $\Pi(K_1, \ldots, K_{n-1})$ *and* $\Pi(D_1, \ldots, D_{n-1})$.

In this paper, we shall get positive solutions of these two open problems. As applications we prove some interrelated results.

Please see the next section for above interrelated notations, definitions and background materials.

2 Definitions and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u.

Let $h(K, \cdot) : S^{n-1} \to \mathbb{R}$, denote the support function of $K \in \mathcal{K}^n$; i.e. $h(K, u) = Max\{u \cdot x : x \in K\}, u \in S^{n-1}$, where $u \cdot x$ denotes the usual inner product of u and x in $\mathbb{R}^n$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

2.1. Mixed volumes. If $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, r)$ and $\lambda_i (i = 1, 2, \ldots, r)$are non-negative real numbers, then of fundamental importance is the fact that the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial in λ_i given by $V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V_{i_1 \ldots i_n}$, where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by above identity. It is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. If $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is usually written as $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the ith projection measure (Quermassintegral) of K and is written as $W_i(K)$.

2.2. Dual mixed volumes. Now we introduce a vector addition on $\mathbb{R}^n$, which we call radial addition, as follows. If $x_1, \ldots, x_r \in \mathbb{R}^n$, then $x_1 \tilde{+} \ldots \tilde{+} x_r$ is defined to be the usual vector sum of $x_1, \ldots, x_r$, provided $x_1, \ldots, x_r$ all lie in a 1-dimensional subspace of $\mathbb{R}^n$, and as the zero vector otherwise.

If $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}$. For $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \ldots \tilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i, $V(\lambda_1 K_1 \tilde{+} \ldots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n}$, where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in above identity to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is written as $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ and is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume is written as $\tilde{V}_i(K, L)$. The dual mixed volume $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$.

2.3. Mixed projection bodies and its polars. If K is a convex body that contains the origin in its interior, we define the polar body K^* of K, by $K^* := \{x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K\}$. If K is a convex body that contains the origin in its interior, then we also associate with K its radial function $\rho(K, \cdot)$ defined on S^{n-1} by $\rho(K, u) = Max\{\lambda \geq 0 : \lambda u \in K\}, u \in \mathbb{R}^n$. We easily get that $\rho(K, u)^{-1} = h(K^*, u)$.

If $K_i (i = 1, 2, \ldots, n-1) \in \mathcal{K}^n$, then the mixed projection body of $K_i (i = 1, 2, \ldots, n-1)$ is denoted by $\Pi(K_1, \ldots, K_{n-1})$, and whose support function is given, for $u \in S^{n-1}$, by $h(\Pi(K_1, \ldots, K_{n-1}), u) = v(K_1^u, \ldots, K_{n-1}^u)$.

We use $\Pi^*(K_1, \ldots, K_{n-1})$ to denote the polar body of $\Pi(K_1, \ldots, K_{n-1})$, and call it polar of mixed projection body of $K_i (i = 1, 2, \ldots, n-1)$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\Pi(K_1, \ldots, K_{n-1})$ will be written as $\Pi_i(K, L)$. If $L = B$, then $\Pi_i(K, B)$ is called the ith projection body of K and is denoted by $\Pi_i K$. We write $\Pi_0 K$ as ΠK. We will simply write $\Pi_i^* K$ and $\Pi^* K$ rather than $(\Pi_i K)^*$ and $(\Pi K)^*$, respectively.

2.4. Quermassintegral difference function. In 2004, *i-Quermassintegral difference function* of convex bodies was defined by Leng [8] as

$$Dw_i(K, D) = W_i(K) - W_i(D), \quad (K, D \in \mathcal{K}^n,\ D \subset K \text{ and } 0 \leq i \leq n-1).$$

In [8], Leng established a Minkowski inequality for volume difference and a Brunn-Minkowski inequality for i-Quermassintegral difference function.

3 Lemmas

Lemma 3.1. ([11]) *If $K_1, \ldots, K_{n-1}$ are convex bodies, then for $0 \leq i \leq n-1$, $0 \leq r \leq n-1$,*

$$W_i(\Pi(K_1, \ldots, K_{n-1}))^r \geq \prod_{j=1}^{r} W_i(\Pi(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})), \tag{3.1}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are homothetic.

Lemma 3.2. **([7])** *If* $a_1, b_1, \dots, l_1 \geq 0, a_2, b_2, \dots, l_2 > 0$ *and* $\alpha + \beta + \cdots + \lambda = 1$, *then*

$$a_1^{\alpha} b_1^{\beta} \cdots l_1^{\lambda} + a_2^{\alpha} b_2^{\beta} \cdots l_2^{\lambda} \leq (a_1 + a_2)^{\alpha} (b_1 + b_2)^{\beta} \cdots (l_1 + l_2)^{\lambda},$$

with equality if and only if $a_1/a_2 = b_1/b_2 = \cdots = l_1/l_2$.

Obviously, a special case of the Lemma 3.2 is the following result.

For $a_i \geq 0, b_i > 0 (i = 1, 2, \dots, n)$, we have

$$\left(\prod_{i=1}^{n} (a_i + b_i) \right)^{1/n} \geq \left(\prod_{i=1}^{n} a_i \right)^{1/n} + \left(\prod_{i=1}^{n} b_i \right)^{1/n}, \tag{3.2}$$

with equality if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

Further, Taking $c_i = a_i + b_i$ in (3.2), we obtain that for $c_i > 0, b_i > 0$ and $c_i > b_i$, then

$$\left(\prod_{i=1}^{n} (c_i - b_i) \right)^{1/n} \leq \left(\prod_{i=1}^{n} c_i \right)^{1/n} - \left(\prod_{i=1}^{n} b_i \right)^{1/n}, \tag{3.3}$$

with equality if and only if $c_1/b_1 = c_2/b_2 = \cdots = c_n/b_n$.

Lemma 3.3. **([22])** *If* $K_1, \dots, K_{n-1}$ *are convex bodies*, $0 \leq i \leq n-1$, $0 < j < n-1$ *and* $0 \leq r \leq n-1$, *then*

$$\tilde{W}_i(\Pi^*(K_1, \dots, K_{n-1}))^r \leq \prod_{j=1}^{r} \tilde{W}_i(\Pi^*(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_{n-1})). \tag{3.4}$$

4 Positive solutions of two open problems

In this section, the following Aleksandrov-Fenchel inequality for volume differences of mixed projection bodies (the open problem 2) stated in the introduction will be established.

Let $K_i (i = 1, \dots, n-1)$ and $D_i (i = 1, \dots, n-1)$ be convex bodies in $\mathbb{R}^n$, $D_i \subset K_i$ and $D_i (i = 1, \dots, n-1)$ be homothetic copies of each other, respectively. Then for $0 \leq r \leq n-1$,

$$D_v(\Pi(K_1, \dots, K_{n-1}), \Pi(D_1, \dots, D_{n-1}))^r$$
$$\geq \prod_{j=1}^{r} D_v(\Pi(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_n), \Pi(\underbrace{D_j, \dots, D_j}_{r}, D_{r+1}, \dots, D_n)).$$

This is just the special case $i = 0$ of the following

Theorem 1. *Let* $K_m (m = 1, \dots, n-1)$ *and* $D_m (m = 1, \dots, n-1)$ *be convex bodies in* $\mathbb{R}^n$, $D_m \subset K_m$ *and* $D_m (m = 1, \dots, n-1)$ *be homothetic copies of each other, respectively. Then for* $0 \leq r \leq n-1, 0 \leq i < n$,

$$D_{w_i}(\Pi(K_1, \dots, K_{n-1}), \Pi(D_1, \dots, D_{n-1}))^r$$

$$\geq \prod_{j=1}^{r} D_{w_i}(\Pi(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}),\Pi(\underbrace{D_j,\ldots,D_j}_{r},D_{r+1},\ldots,D_{n-1})), \quad (4.1)$$

with equality if and only if $K_m(m=1,2,\ldots,n-1)$ *are homothetic to each other.*

Proof. From Lemma 3.1, for $K_1,\ldots,K_{n-1}$ being convex bodies and $0\leq r\leq n$, we have

$$W_i(\Pi(K_1,\ldots,K_{n-1}))^r \geq \prod_{j=1}^{r} W_i(\Pi(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1})), \quad (4.2)$$

with equality if and only if $K_m(m=1,2,\ldots,n-1)$ are homothetic to each other.

On the other hand, in view of $D_m(m=1,\ldots,n-1)$ are homothetic copies of each other, we obtain that

$$W_i(\Pi(D_1,\ldots,D_{n-1}))^r = \prod_{j=1}^{r} W_i(\Pi(\underbrace{D_j,\ldots,D_j}_{r},D_{r+1},\ldots,D_{n-1})).$$

Hence

$$W_i(\Pi(K_1,\ldots,K_{n-1})) - W_i(\Pi(D_1,\ldots,D_{n-1}))$$
$$\geq \left(\prod_{j=1}^{r} W_i(\Pi(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}))\right)^{1/r} - \left(\prod_{j=1}^{r} W_i(\Pi(\underbrace{D_j,\ldots,D_j}_{r},D_{r+1},\ldots,D_{n-1}))\right)^{1/r}.$$

By using the inequality (3.3) in right side of above inequality, we obtain

$$D_{w_i}(\Pi(K_1,\ldots,K_{n-1}),\Pi(D_1,\ldots,D_{n-1}))$$
$$\geq \left(\prod_{j=1}^{r}(W_i(\Pi(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1})) - W_i(\Pi(\underbrace{D_j,\ldots,D_j}_{r},D_{r+1},\ldots,D_{n-1})))\right)^{1/r}$$
$$= \prod_{j=1}^{r} D_{w_i}(\Pi(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}),\Pi(\underbrace{D_j,\ldots,D_j}_{r},D_{r+1},\ldots,D_{n-1}))^{1/r}. \quad (4.3)$$

In view of the equality conditions of inequality (4.2) and inequality (4.3), it follows that the equality holds if and only if $K_m(m=1,2,\ldots,n-1)$ are homothetic of each other. This completes the proof of Theorem 1. □

Remark 4.1. (i) Let $D_m(m=1,2,\ldots,n-1)$ be single points in (4.1), then (4.1) changes to

$$W_i(\Pi(K_1,\ldots,K_{n-1}))^r \geq \prod_{j=1}^{r} W_i(\Pi(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1})), \quad (4.4)$$

with equality if and only if $K_m(m=1,2,\ldots,n-1)$ are homothetic of each other. This is just the well-known Aleksandrov-Fenchel inequality for mixed projection bodies which was given by Lutwak [11].

(ii) In (4.1), taking $r = n-1$, we obtain

$$D_{w_i}(\Pi(K_1,\ldots,K_{n-1}),\Pi(D_1,\ldots,D_{n-1}))^{n-1} \geq \prod_{j=1}^{n-1} D_{w_i}(\Pi K_j, \Pi D_j). \tag{4.5}$$

Let $D_1, D_2, \ldots, D_{n-1}$ be single points and take $K_1 = \cdots = K_{n-j-1} = K, K_{n-j} = \cdots = K_{n-1} = L$ in (4.5), then (4.5) changes to

$$W_i(\Pi_j(K,L)^{n-1} \geq W_i(\Pi K)^{n-j-1} W_i(\Pi L)^j, \tag{4.6}$$

with equality if and only if K and L are homothetic. This is just the well-known Minkowski inequality for mixed projection bodies which was given by Lutwak [11].

(iii) Taking $i = 0$, $K_1 = \cdots = K_{n-2} = K, K_{n-1} = L, D_1 = \cdots = D_{n-2} = D, D_{n-1} = D'$ in (4.5), it becomes

$$(V(\Pi_1(K,L)) - V(\Pi_1(D,D')))^{n-1} \geq D_v(\Pi K, \Pi D))^{n-2} D_v(\Pi L, \Pi D')).$$

This is just a mixed projection form of the following result which was given by Leng [8]

$$(V_1(K,L) - V_1(D,D'))^n \geq D_v(K,D)^{n-1} D_v(L,D'). \tag{4.7}$$

Theorem 2. *Let $K_i (i = 1,2,\ldots,n)$ and $D_i (i = 1,2,\ldots,n)$ be convex bodies in $\mathbb{R}^n$, $D_i \subset K_i$ and $D_i (i = 1,2,\ldots,n)$ be homothetic copies of each other, respectively. Then for $0 \leq r \leq n$,*

$$D_v((K_1,\ldots,K_n),(D_1,\ldots,D_n))^r$$
$$\geq \prod_{j=1}^{r} D_v((\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_n),(\underbrace{D_j,\ldots,D_j}_{r}, D_{r+1},\ldots,D_n)). \tag{4.8}$$

Proof. From the classical Aleksandrov-Fenchel inequality, for $K_1,\ldots,K_n$ being convex bodies and $0 \leq r \leq n$, we have

$$V(K_1,\ldots,K_n)^r \geq \prod_{j=1}^{r} V(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_n). \tag{4.9}$$

In fact, the sufficient and necessary conditions of the equality in the Aleksandrov-Fenchel inequality (4.9) are, in general, unknown. But the equality holds if $K_1,\ldots,K_n$ are homothetic. Hence, in view of $D_i (i = 1,\ldots,n)$ are homothetic copies of each other, we obtain

$$V(D_1,\ldots,D_n)^r = \prod_{j=1}^{r} V(\underbrace{D_j,\ldots,D_j}_{r}, D_{r+1},\ldots,D_n). \tag{4.10}$$

From (4.9) and (4.10), we have

$$V(K_1,\ldots,K_n) - V(D_1,\ldots,D_n)$$
$$\geq \left(\prod_{j=1}^{r} V(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_n)\right)^{1/r} - \left(\prod_{j=1}^{r} V(\underbrace{D_j,\ldots,D_j}_{r}, D_{r+1},\ldots,D_n)\right)^{1/r}$$

In view of inequality (3.3), we obtain

$$D_v((K_1,\dots,K_n),(D_1,\dots,D_n))$$

$$\geq \left(\prod_{j=1}^{r}(V(\underbrace{K_j,\dots,K_j}_{r},K_{r+1},\dots,K_n)-V(\underbrace{D_j,\dots,D_j}_{r},D_{r+1},\dots,D_n))\right)^{1/r}$$

$$=\prod_{j=1}^{r}D_v((\underbrace{K_j,\dots,K_j}_{r},K_{r+1},\dots,K_n),(\underbrace{D_j,\dots,D_j}_{r},D_{r+1},\dots,D_n))^{1/r}.$$

This completes the proof of Theorem 2. *xBox*

Remark 4.2. In (4.8), taking $r=n$, we obtain

$$D_v((K_1,\dots,K_n),(D_1,\dots,D_n))^n \geq \prod_{j=1}^{n}D_v(K_j,D_j). \tag{4.11}$$

Taking $K_1=\cdots=K_{n-1}=K, K_n=L, D_1=\cdots=D_{n-1}=D, D_n=D'$ in (4.11), it becomes

$$(V_1(K,L)-V_1(D,D'))^n \geq Dv(K,D)^{n-1}Dv(L,D').$$

This is just the inequality (4.7).

On the other hand, let D and D' be single points in (4.11), then (4.11) becomes the classical Brunn-Minkowski inequality. For interrelated research about these classical inequalities, one is directed to [1, 2, 3, 4, 5, 6, 12, 13, 15, 17, 19, 21] et al.

5 Conclusions

In the present paper we present the Aleksandrov-Fenchel type inequalities for volume differences and Quermassintegral differences. These new results will be applied in the area of convex geometry.

Acknowledgements. Research is supported by National Natural Sciences Foundation of China (10971205).

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