Generalized convex functionals

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Abstract. This paper studies functionals defined by multiple integrals associated to differential forms on the jet bundle of first order corresponding to some Riemannian manifolds; the domain of these functionals consists in submanifold maps satisfying certain conditions of integrability. Our idea is to give geometric properties to the domain of a functional allowing us to properly define convexity. The method we use consists in creating an extended Riemannian submanifold of the first order jet bundle, in connection to this domain and carrying back its geometric properties. This process allows us to consider and use the *geodesic deformations*. Furthermore, fixing a *pairing map*, allows us to define generalized convex (preinvex and invex) functionals.

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1 Introduction

The research in convexity of functions is an old theory turning permanently into new directions: one of these directions consists in studying generalized convex (namely invex, preinvex, quasipreinvex, pseudopreinvex, quasiinvex, pseudoinvex...) functions [12]-[15], [17] and another one focuses on studying convex functions in Riemannian setting [7]-[9], [16], [22]. The most recent works in convexity combine these two ideas, using geodesics in order to define Riemannian preinvex and invex functions [1], [3]-[4], [18].

This paper extends various Riemannian convexities from functions to functionals. The theory developed here is entirely original. The necessity of creating a consistent analyze of Riemannian convex functionals was suggested by their utility in optimal control or variational problems [10], [11], [23]-[25], [26], [28]-[34].

The similitude between this paper and the above-quoted books, and also its novelty in the study of the convex functionals, consist in the introduction and use of geodesic deformations and pairing maps. These two objects become geometric parameters for convexity, since altering one or both of them can create, destroy or preserve convexity.

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To define the Riemannian convexity of functionals, we also join some ideas from Differential Geometry [2], [5], [6], [11], Geometric Dynamics [20], [21], Riemannian Convexity and Optimization [26], [16], [19].

Section 1 contains some bibliographical notes. Section 2 defines and studies the geodesic deformations and their impact on functionals in Riemannian setting. Section 3 and 4 use pairing maps in order to define and study Riemannian preinvex functionals. Sections 5 turns to η-convex and invex functionals. The most important result of this theory asserts that being an invex functional is equivalent with having equality between the set of critical points and the set of global minimum points. Section 6 gives some examples of geometric invex functionals. Section 7 points out the main outcomes of this theory.

2 Geodesic deformations and convex functionals in Riemannian setting

Let (M,g) be a complete n-dimensional Riemannian manifold and (N,h) be a compact m-dimensional Riemannian manifold. We denote by $J^1(N,M)$ the first order jet bundle and by $G = h + g + h^{-1} \otimes g$ its induced metric (see [25]). Let E be a set of submanifold maps from N to $J^1(N,M)$ and $E(I)$ be the set of those submanifold maps that are integral maps for a family I of differential 1-forms on $J^1(N,M)$. If θ is a differential m-form on $J^1(N,M)$ we can associate the functional (multiple integral)

$$J_\theta : E(I) \to R,\ J_\theta[\Phi] = \int_N \Phi^*\theta, \tag{2.1}$$

where $\Phi^*\theta$ denotes the pull-back of θ on N.

For a fixed submanifold map $\Phi \in E(I)$ we associate a deformation map $\varphi : N \times (-\delta, \delta) \to J^1(N,M)$, satisfying $\varphi(\cdot, 0) = \Phi$. We denote by

$$X_\Phi \in \mathcal{X}_\Phi(J^1(N,M)),\ X(\Phi(t)) = \varphi_* \frac{\partial}{\partial \epsilon}|_{\epsilon=0}$$

the infinitesimal deformation induced by φ, that is, X_Φ is the vector field along $\Phi(N)$ satisfying $X_{\Phi(t)} = \frac{\partial \varphi}{\partial \epsilon}(t,0)$, $\forall t \in N$. Since all the maps from $E(I)$ preserve the boundary of N, it follows that X_Φ also satisfies the condition $X_\Phi(t) = 0$, $\forall t \in \partial N$. From now on, $T_\Phi E(I)$ will denote the set of all infinitesimal deformations of Φ as above and

$$TE(I) = \cup_{\Phi \in E(I)} T_\Phi E(I).$$

It seems natural now to consider also the set

$$\mathcal{X}(E(I)) = \{X \in \mathcal{X}(J^1(N,M)) |\ X_\Phi = X|_{\Phi(N)} \in T_\Phi E(I),\ \forall \Phi \in E(I)\}. \tag{2.2}$$

Our basic example is the multitime variational problem: (N,h) a compact m-dimensional Riemannian manifold with local coordinates $(t^1, ..., t^m)$; (M,g) an n-dimensional manifold with local coordinates $(x^1, ..., x^n)$; the induced local coordinates $(t^\gamma, x^i, x^i_\gamma)$ on $J^1(N,M)$; $L : J^1(N,M) \to R$ a Lagrangian and θ the Cartan form associated to L, that is $\theta = Ldt + \frac{\partial L}{\partial x^i_\gamma}\omega^i \wedge dt_\gamma$, where $I = \{\omega^i \in \Lambda^1(J^1(N,M)) |\ \omega^i =$

$dx^i - x^i_\sigma dt^\sigma, i = \overline{1,m}\}$ and $dt_\gamma = i_{\frac{\partial}{\partial t^\gamma}} dt$. If we introduce the set $E(I) = \{\Phi : N \to J^1(N,M) |\ \Phi(t) = (t, x(t), \frac{\partial x}{\partial t}(t))\}$, then its tangent space is

$$T_\Phi E(I) = \{X = (X^\gamma, X^i, X^i_\gamma) |\ X^\gamma \circ \Phi = 0,\ D_\gamma(X^i \circ \Phi) = X^i_\gamma \circ \Phi, \\ X^i(\Phi(t)) = 0,\ \forall t \in \partial N\} \tag{2.3}$$

and $J_\theta : E(I) \to R$ is defined by $J_\theta[\Phi] = \int_N \Phi^* \theta = \int_N L \circ \Phi dt = J_L[\Phi]$. From now on, when dealing with such a variational problem, we replace J_θ by J_L, emphasizing the fact that the m-form θ is related to the Lagrangian L.

We return to the general problem and, after associating to $E(I)$ the previous structures on $J^1(N,M)$, we look for analyzing their geometric properties and carrying these properties back to the set $E(I)$. We remark that $\mathcal{X}(E(I))$ is not an $\mathcal{F}(J^1(N,M))$-module. Therefore, instead $\mathcal{X}(E(I))$, we consider the set $\tilde{\mathcal{X}}(E(I))$ representing the $\mathcal{F}(J^1(N,M))$-module generated by $\mathcal{X}(E(I))$ and we also denote by $\tilde{T}_\Phi E(I)$ the $\mathcal{F}(J^1(N,M))$-module generated by $T_\Phi E(I)$.

Lemma 2.1. *The set $\tilde{\mathcal{X}}(E(I))$ is an involutive distribution on $J^1(N,M)$ and, for each $\Phi \in E(I)$, there is an integral submanifold $A_\Phi \subset J^1(N,M)$ such that $\mathcal{X}_\Psi(A_\Phi) = \tilde{T}_\Psi E(I)$, $\forall \Psi \in E(I)$ a submanifold map resulting after a deformation of Φ in $E(I)$.*

Remark 2.2. *If $E(I)$ is connected, then $\Psi(N) \subset A_\Phi$, $\forall \Phi, \Psi \in E(I)$. Otherwise, for each connected component of $E(I)$ we can associate a submanifold, as we did above, and we consider A the submanifold for which the previous submanifolds are the connected components. The submanifold A is called the image of $E(I)$ on $J^1(N,M)$.*

Definition 2.1. A deformation map $\varphi : N \times [0,1] \to J^1(N,M)$ is called *geodesic deformation* if $\varphi(t,\cdot)$ is a geodesic in (A,G), for each $t \in N$.

Definition 2.2. A subset $F \subset E(I)$ is called *totally convex* if, for all pairs of submanifold maps $\Phi, \Psi \in F$ and all geodesic deformation $\varphi : N \times [0,1] \to J^1(N,M)$, $\varphi(\cdot,0) = \Phi$, $\varphi(\cdot,1) = \Psi$, we have

$$\varphi(\cdot,\epsilon) \in F,\ \forall \epsilon \in [0,1]. \tag{2.4}$$

Definition 2.3. Let $F \subset E(I)$ be a totally convex subset of submanifold maps and let θ be a differential m-form on $J^1(N,M)$. The functional

$$J_\theta : F \to \mathbb{R},\ J_\theta[\Phi] = \int_N \Phi^* \theta$$

is called *Riemannian convex* if

$$J_\theta[\varphi(\cdot,\epsilon)] \leq (1-\epsilon) J_\theta[\Phi] + \epsilon J_\theta[\Psi], \tag{2.5}$$

for all Φ and Ψ in F, for all the geodesic deformations $\varphi : N \times [0,1] \to J^1(N,M)$ connecting Φ and Ψ and for all $\epsilon \in [0,1]$.

The functional J_θ is called *Riemannian strictly convex* if

$$J_\theta[\varphi(\cdot,\epsilon)] < (1-\epsilon) J_\theta[\Phi] + \epsilon J_\theta[\Psi], \tag{2.6}$$

for all Φ, Ψ, φ as above, $\Phi \neq \Psi$ and $\epsilon \in (0,1)$.

Definition 2.4. Let $J_\theta : E(I) \to \mathbb{R}$ be the functional associated to the differential m-form θ. The map

$$dJ_\theta(\Phi) : T_\Phi E(I) \to \mathbb{R},\ dJ_\theta(\Phi)[X_\Phi] = \frac{d}{d\epsilon} J[\varphi(\cdot, \epsilon)]|_{\epsilon=0}, \tag{2.7}$$

where $\varphi : N \times (-\delta, \delta) \to J^1(N, M)$ is a deformation of Φ in $E(I)$ such that $X_{\Phi(t)} = \frac{\partial \varphi}{\partial \epsilon}(t, 0)$, is called *the differential of the functional* J_θ *at* Φ.

Definition 2.5. A map $\Phi \in E(I)$ is called *critical point* of the functional J_θ if $dJ_\theta(\Phi)[X_\Phi] = 0,\ \forall X_\Phi \in T_\Phi E(I)$.

Theorem 2.3. *The functional* $J_\theta : F \to \mathbb{R}$ *is convex iff*

$$J_\theta[\Psi] - J_\theta[\Phi] \geq dJ_\theta(\Phi)[X],\ \forall \Phi, \Psi \in F, \tag{2.8}$$

where $X \in T_\Phi E(I)$ *is the infinitesimal deformation associated to a geodesic deformation between* Φ *and* Ψ.

Moreover, the functional $J_\theta : F \to \mathbb{R}$ *is strictly convex iff*

$$J_\theta[\Psi] - J_\theta[\Phi] > dJ_\theta(\Phi)[X],\ \forall \Phi \neq \Psi \in F. \tag{2.9}$$

Corollary 2.4. *If* L *is a* C^1 *Lagrangian in a multitime variational problem, then* J_L *is convex iff*

$$\int_N L \circ \Psi dt - \int_N L \circ \Phi dt \geq \int_N X(L) \circ \Phi dt,\ \forall \Phi, \Psi \in F, \tag{2.10}$$

where $X \in T_\Phi E(I)$ *is associated again to a geodesic deformation between* Φ *and* Ψ.

3 Riemannian η-preconvex functionals

Definition 3.1. Let $F \subseteq E(I)$ be a nonvoid subset. A vector map

$$\eta : F \times F \to TE(I),\ \eta(\Psi, \Phi) \in T_\Phi E(I) \tag{3.1}$$

is called *pairing map on* F.

Example If $\Phi \in E(I)$ and $V_\Phi = \{\Psi \in E(I)|\ \Psi(t) \in V_{\Phi(t)},\ \forall t \in N\}$, where $V_{\Phi(t)}$ is a neighborhood of $\Phi(t)$ such that $exp_{\Phi(t)} : T_{\Phi(t)} J^1(N, M) \to V_{\Phi(t)}$ is a diffeomorphism, then we consider the map

$$\eta^{(\Phi)} : V_\Phi \to T_\Phi E(I),\ \eta^{(\Phi)}(\Psi)(t) = exp^{-1}_{\Phi(t)}(\Psi(t)). \tag{3.2}$$

Furthermore, we denote by η_0 a pairing map satisfying

$$\eta_0(\Psi, \Phi) = \eta^{(\Phi)}(\Psi),\ \forall \Psi \in V_\Phi. \tag{3.3}$$

Remark 3.1. *For a multitime variational problem and a pairing map* $\eta : F \times F \to TE(I)$ *we write*

$$\begin{aligned} \eta(\Psi, \Phi)(t) = \big(&0, \eta^i(t, x^i(t), y^i(t), x^i_\gamma(t), y^i_\gamma(t)), \\ &D_\alpha[\eta^i(t, x^j(t), y^j(t), x^j_\sigma(t), y^j_\sigma(t))]\big), \end{aligned} \tag{3.4}$$

where D_α *denotes the total derivative with respect to* t^α.

If $\eta : F \times F \to TE(I)$ is a pairing map and $\Phi, \Psi \in F$, we consider $\gamma_{\Psi\Phi\eta} : N \times (-\delta, \delta) \to J^1(N, M)$, $[0, 1] \subset (-\delta, \delta)$, a geodesic deformation satisfying

$$\gamma_{\Psi\Phi\eta}(t, 0) = \Phi(t),\ \forall t \in N \text{ and } \frac{\partial \gamma_{\Psi\Phi\eta}}{\partial \epsilon}(t, 0) = \eta(\Psi, \Phi)(t),\ \forall t \in N. \tag{3.5}$$

Definition 3.2. Let $F \subseteq E(I)$ be a nonvoid subset and $\eta : F \times F \to TE(I)$ be a pairing map on F. The subset F is called *totally η-convex* if

$$\gamma_{\Psi\Phi\eta}(\cdot, \epsilon) \in F,\ \forall \Psi, \Phi \in F,\ \forall \epsilon \in [0, 1]. \tag{3.6}$$

We consider $F \subset E(I)$, $\eta : F \times F \to TE(I)$ a pairing map such that F is totally η-convex, $\theta \in \Lambda^m(J^1(N, M))$ and J_θ the functional defined by multiple integral associated to θ. From now on, $\gamma_{\Psi\Phi\eta}$ denotes a geodesic deformation generated by $\Psi, \Phi \in F$ and the pairing map η as above.

Definition 3.3. The functional $J_\theta : F \to \mathbb{R}$ is called *Riemannian η-preconvex* on F if

$$J_\theta[\gamma_{\Psi\Phi\eta}(\cdot, \epsilon)] \le (1 - \epsilon) J_\theta[\Phi] + \epsilon J_\theta[\Psi],\ \forall \Phi, \Psi \in F,\ \forall \epsilon \in [0, 1]. \tag{3.7}$$

The functional $J_\theta : F \to \mathbb{R}$ is called *Riemannian strictly η-preconvex* on F if

$$J_\theta[\gamma_{\Psi\Phi\eta}(\cdot, \epsilon)] < (1 - \epsilon) J_\theta[\Phi] + \epsilon J_\theta[\Psi],\ \forall \Phi, \Psi \in F,\ \Phi \ne \Psi,\ \forall \epsilon \in (0, 1). \tag{3.8}$$

Definition 3.4. A functional $J_\theta : F \to \mathbb{R}$ is called *Riemannian (strictly) preinvex* on F if there exists a pairing map η such that F is totally η-convex and J_θ is Riemannian (strictly) η-preconvex on F.

The next Theorem ensures us that the usual Riemannian convexity of functionals is a particular case of Riemannian η-preconvexity, when considering η to be the pairing map induced by the inverse of the exponential map (see the example).

Theorem 3.2. *If $F \subseteq E(I)$ is a totally convex subset, $J_\theta : F \to \mathbb{R}$ is a Riemannian (strictly) convex functional on F and η_0 is the pairing map induced by the inverse of the exponential map, then J_θ is Riemannian (strictly) η_0-preconvex.*

Proof. We recall that F is called totally convex if

$$\varphi_{\Phi\Psi}(\cdot, \epsilon) \in F,\ \forall \Phi, \Psi \in F,\ \forall \epsilon \in [0, 1], \tag{3.9}$$

where $\varphi_{\Phi\Psi} : N \times [0, 1] \to J^1(N, M)$ is a geodesic deformation between Φ and Ψ, that is, for each $t \in N$, $\varphi(t, \cdot)$ is a geodesic between $\Phi(t)$ and $\Psi(t)$. Moreover, J_θ is a convex functional if

$$J_\theta[\varphi_{\Phi\Psi}(\cdot, \epsilon)] \le (1 - \epsilon) J_\theta[\Phi] + \epsilon J_\theta[\Psi],\ \forall \Phi, \Psi \in F,\ \forall \epsilon \in [0, 1]. \tag{3.10}$$

For Φ, Ψ and $\varphi_{\Phi\Psi}$ as above, we have

$$\eta_0(\Psi, \Phi)(t) = exp^{-1}_{\Phi(t)}(\Psi(t)) = \frac{\partial \varphi_{\Phi\Psi}}{\partial \epsilon}(t, 0),\ \forall t \in N.$$

If $\gamma = \gamma_{\Psi\Phi\eta_0}$, we have $\gamma(t,0) = \Phi(t) = \varphi_{\Phi\Psi}(t,0)$ and

$$\frac{\partial\gamma}{\partial\epsilon}(t,0) = \eta_0(\Psi,\Phi)(t) = \frac{\partial\varphi_{\Phi\Psi}}{\partial\epsilon}(t,0),\ \forall t \in N.$$

It follows that $\gamma(t,\epsilon) = \varphi_{\Phi\Psi}(t,\epsilon),\ \forall t \in N,\ \forall\epsilon \in [0,1]$ and, consequently, F is a totally η_0-convex set and J_θ is a Riemannian η_0-preconvex functional. □

The following results analyze the behavior of the η-preconvex functionals when changing coordinates.

Theorem 3.3. *Let $F \subset E(I)$ be a totally η-convex subset and $J_\theta : F \to \mathbb{R}$ be a Riemannian η-preconvex functional. If $\mathcal{I}$ is the $\mathcal{F}(J^1(N,M))$-module generated by I and if $f : J^1(N,M) \to J^1(N,M)$ is a diffeomorphism preserving $\mathcal{I}$, that is $f^*(\mathcal{I}) = \mathcal{I}$, then the set $f(F) = \{f\circ\Phi \mid \Phi \in F\}$ is an $\bar\eta$-totally convex subset and $J_{(f^{-1})^*\theta}$ is a Riemannian $\bar\eta$-preconvex functional on $f(F)$, where*

$$\bar\eta(f\circ\Psi, f\circ\Phi)(t) = f_*(\eta(\Psi,\Phi)(t)),\ \forall\Psi,\Phi \in F,\ \forall t \in N.$$

Proof. If $\Phi \in F$, let $\bar\Phi = f\circ\Phi \in \varphi(F)$. Since $f^*\omega \in \mathcal{I}, \forall\omega \in I$, it follows $\Phi^*(f^*\omega) = \bar\Phi^*\omega = 0, \forall\omega \in I$, which proves that $\bar\Phi \in E(I)$. Therefore $f(F) \subset E(I)$.

Let $\gamma = \gamma_{\Psi\Phi\eta} : N\times(-\delta,\delta) \to J^1(N,M)$ be the geodesic deformation associated to $\Phi,\Psi \in F$ and η, and let $\bar\gamma = \gamma_{\bar\Psi\bar\Phi\bar\eta} : N\times(-\delta,\delta) \to J^1(N,M)$ be the geodesic deformation associated to $\bar\Phi$, $\bar\Psi$ and $\bar\eta$. We know that, for each point $t \in N$,

$$\bar\gamma(t,0) = \bar\Phi(t) = f\circ\Phi(t) = (f\circ\gamma)(t,0)$$

and

$$\frac{\partial\bar\gamma}{\partial\epsilon}(t,0) = \bar\eta(\bar\Psi,\bar\Phi)(t) = \frac{\partial(f\circ\gamma)}{\partial\epsilon}(t,0),$$

and it follows that $\bar\gamma = f\circ\gamma$.

Since F is totally η-convex, we have $\gamma(\cdot,\epsilon) \in F,\ \forall\epsilon \in [0,1]$. Therefore $\bar\gamma(\cdot,\epsilon) \in f(F),\ \forall\epsilon$ and $f(F)$ is totally $\bar\eta$-convex.

Moreover, we have

$$\begin{aligned} J_{(f^{-1})^*\theta}[\bar\gamma(\cdot,\epsilon)] &= \int_N \bar\gamma_\epsilon^*((f^{-1})^*\theta) = \int_N (f^{-1}\circ\bar\gamma_\epsilon)^*\theta \\ &= J_\theta[f^{-1}\circ\bar\gamma(\cdot,\epsilon)] \le (1-\epsilon)J_\theta[\Phi] + \epsilon J_\theta[\Psi] \\ &= (1-\epsilon)J_{(f^{-1})^*\theta}[\bar\Phi] + \epsilon J_{(f^{-1})^*\theta}[\bar\Psi]. \end{aligned}$$

Therefore $J_{(f^{-1})^*\theta}$ is a Riemannian $\bar\eta$-preconvex functional. □

Corollary 3.4. *Let J_L be a Riemannian η-preconvex action associated to a multitime variational problem. If $f : J^1(N,M) \to J^1(N,M)$ is a diffeomorphism preserving $\mathcal{I}$ and $\bar\eta$ is the induced pairing map as in the Theorem 3.3, then $J_{L\circ f^{-1}}$ is a Riemannian $\bar\eta$-preconvex functional.*

Proof. Since $f^*(\mathcal{I}) = \mathcal{I}$, we previously proved that $f(E(I)) \subset E(I)$ is a totally $\bar\eta$-convex subset. Let θ be the Cartan form associated to L and $\bar\Phi \in f(E(I))$, that is $\bar\Phi = f\circ\Phi$, with $\Phi \in E(I)$. Then $\bar\Phi^*((f^{-1})^*\theta) = \Phi^*\theta = L\circ\Phi dt = (L\circ f^{-1})\circ\bar\Phi dt$, therefore $J_{(f^{-1})^*\theta} = J_{L\circ f^{-1}}$ and, by applying the Theorem 3.3, we find that $J_{L\circ f^{-1}}$ is an $\bar\eta$-preconvex functional. □

Corollary 3.5. *If $J_L : E(I) \to \mathbb{R}$ is a functional as above, then the property of J_L of being Riemannian preinvex is invariant with respect to any change of coordinates on M.*

Proof. We consider $(x^1, .., x^n)$ and $(\tilde{x}^1, ..., \tilde{x}^n)$, two systems of coordinates on M, and $(t^1, ..., t^m)$ some local coordinates on N. They generate two sets of coordinates (t, x^i, x^i_γ) and $(t, \tilde{x}^i, \tilde{x}^i_\gamma)$ on $J^1(N, M)$. If $f : J^1(N, M) \to J^1(N, M)$ is the diffeomorphism associated to this change of coordinates, then $f^*(\mathcal{I}) = \mathcal{I}$. Indeed,

$$f^*\omega^i = f^*(d\tilde{x}^i - \tilde{x}^i_\sigma dt^\sigma) = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j - \frac{\partial \tilde{x}^i}{\partial x^j} x^j_\sigma dt^\sigma = \frac{\partial \tilde{x}^i}{\partial x^j} \omega^j \in \mathcal{I}.$$

It follows $f^*(I) \subset \mathcal{I}$. Since $f : J^1(N, M) \to J^1(N, M)$ is a diffeomorphism, $f^*(\mathcal{I}) = \mathcal{I}$. If $L(t, x^i, x^i_\gamma) = L \circ f^{-1}(t, \tilde{x}^i, \tilde{x}^i_\gamma)$, then, by applying the previous Corollary, we obtain: if J_L is Riemannian preconvex with respect to a pairing map η, then there exists a pairing map $\bar{\eta}$ such that $J_{L \circ f^{-1}}$ is Riemannian $\bar{\eta}$-preconvex. $\square$

4 Properties of Riemannian η-preconvex functionals

Theorem 4.1. *If $J_\theta : F \to \mathbb{R}$ is a Riemannian η-preconvex functional on F, then every local minimum point for J_θ is also a global minimum point.*

Proof. Let $\Phi \in F$ be a local minimum point for J_θ. There is a neighborhood G of Φ in F such that $J_\theta[\xi] \geq J_\theta[\Phi]$, $\forall \xi \in G$. We suppose that there is $\Psi \in F - G$ such that $J_\theta[\Psi] < J_\theta[\Phi]$ and we consider $\gamma = \gamma_{\Psi\Phi\eta}$. Due to the η-preconvexity of J_θ, we have

$$J_\theta[\gamma(\cdot, \epsilon)] \leq (1 - \epsilon) J_\theta[\Phi] + \epsilon J_\theta[\Psi] < J_\theta[\Phi], \ \forall \epsilon \in (0, 1].$$

On the other hand, there is some $\delta \in (0, 1]$ such that $\gamma(\cdot, \delta) \in G$ and $J_\theta[\gamma(\cdot, \delta)] \geq J_\theta[\Phi]$. We obtain a contradiction, therefore Φ is a global minimum point for J_θ. $\square$

Theorem 4.2. *If $J_\theta : F \to \mathbb{R}$ is the functional associated to a differential m-form θ, the following properties hold:*

1. *if J_θ is a Riemannian η-preconvex functional and $k > 0$, then kJ_θ is also Riemannian η-preconvex;*

2. *if J_θ and J_Ω are two Riemannian preconvex functionals with respect to the same pairing map η, then $J_\theta + J_\Omega$ is also an η-preconvex functional;*

3. *if $\{J_{\theta_i}\}_{i=\overline{1,k}}$ are Riemannian η-preconvex functionals and $\{k_i\}_{i=\overline{1,k}}$ are positive scalars, then $\sum_{i=1}^k k_i J_{\theta_i}$ is also η-preconvex.*

Theorem 4.3. *If $\{J_{\theta_i}\}_{i \in \Lambda}$ are Riemannian η-preconvex functionals, then*

$$(\sup_{i \in \Lambda} J_{\theta_i}) : F \to \mathbb{R}, \ (\sup_{i \in \Lambda} J_{\theta_i})[\Phi] = \sup_{i \in \Lambda}(J_{\theta_i}[\Phi]),$$

is also Riemannian η-preconvex.

Theorem 4.4. *If J_θ is Riemannian η-preincave on F and $J_\theta[\Phi] > 0$, $\forall \Phi \in F$, then $1/J_\theta$ is a Riemannian preconvex functional with respect to η.*

Proof. Let $\Psi, \Phi \in F$ and $\gamma = \gamma_{\Psi\Phi\eta}$. Then

$$J_\theta[\gamma(\cdot,\epsilon)] \geq (1-\epsilon)J_\theta[\Phi] + \epsilon J_\theta[\Psi],\ \forall \epsilon \in [0,1]$$

and

$$(1/J_\theta)[\gamma(\cdot,\epsilon)] \leq 1/[(1-\epsilon)J_\theta[\Phi] + \epsilon J_\theta[\Psi]],\ \forall \epsilon \in [0,1].$$

On the other side, if $x, y > 0$ and $\epsilon \in [0,1]$, we have

$$\begin{aligned}(1-\epsilon)\frac{1}{x} + \epsilon\frac{1}{y} &= \frac{[(1-\epsilon)y + \epsilon x][(1-\epsilon)x + \epsilon y]}{xy[(1-\epsilon)x + \epsilon y]} \\ &= \frac{(1-\epsilon)\epsilon(x-y)^2 + xy}{xy[(1-\epsilon)x + \epsilon y]} \geq \frac{1}{(1-\epsilon)x + \epsilon y}.\end{aligned}$$

Applying the previous inequality for $x = J_\theta[\Phi]$ and $y = J_\theta[\Psi]$, we obtain

$$(1/J_\theta)[\gamma(\cdot,\epsilon)] \leq (1-\epsilon)(1/J_\theta)[\Phi] + \epsilon(1/J_\theta)[\Psi],\ \forall \epsilon \in [0,1],$$

therefore, $1/J_\theta$ is Riemannian η-preconvex. □

Theorem 4.5. *Let $J_\theta : F \to \mathbb{R}$ be a Riemannian η-preconvex functional and $\varphi : \mathbb{R} \to \mathbb{R}$ a convex increasing function. Then $\varphi \circ J_\theta : F \to \mathbb{R}$ is also η-preconvex on F.*

Proof. If $\Phi, \Psi \in F$, let $\gamma = \gamma_{\Psi\Phi\eta}$. Then $J_\theta[\gamma(\cdot,\epsilon)] \leq (1-\epsilon)J_\theta[\Phi] + \epsilon J_\theta[\Psi]$, which implies $(\varphi \circ J_\theta)[\gamma(\cdot,\epsilon)] \leq \varphi((1-\epsilon)J_\theta[\Phi] + \epsilon J_\theta[\Psi]) \leq (1-\epsilon)\varphi(J_\theta[\Phi]) + \epsilon\varphi(J_\theta[\Psi])$. □

5 η-Convexity of functionals

Theorem 5.1. *If $F \subseteq E(I)$ is an open, totally η-convex subset and $J_\theta : F \to \mathbb{R}$ is a Riemannian η-preconvex functional, then J_θ satisfies*

$$J_\theta[\Psi] - J_\theta[\Phi] \geq dJ_\theta(\Phi)[\eta(\Psi,\Phi)],\ \forall \Phi, \Psi \in F.$$

Moreover, if the functional $J_\theta : F \to \mathbb{R}$ is Riemannian strictly η-preconvex, then

$$J_\theta[\Psi] - J_\theta[\Phi] > dJ_\theta(\Phi)[\eta(\Psi,\Phi)],\ \forall \Phi, \Psi \in F,\ \Psi \neq \Phi.$$

Definition 5.1. Let $F \subseteq E(I)$ be an open subset, $J_\theta : F \to \mathbb{R}$ be the functional associated to a differential m-form θ and $\eta : F \times F \to TE(I)$ be a pairing map on F. The functional J_θ is called *η-convex* at $\Phi \in F$ if

$$J_\theta[\Psi] - J_\theta[\Phi] \geq dJ_\theta(\Phi)[\eta(\Psi,\Phi)],\ \forall \Psi \in F. \tag{5.1}$$

The functional J_θ is called *strictly η-convex* at $\Phi \in F$ if

$$J_\theta[\Psi] - J_\theta[\Phi] > dJ_\theta(\Phi)[\eta(\Psi,\Phi)],\ \forall \Psi \in F, \Psi \neq \Phi. \tag{5.2}$$

Definition 5.2. The functional J_θ is called *invex* if there is a pairing map $\eta : F \times F \to TE(I)$ such that J_θ is η-convex.

Remark 5.2. *If $L : J^1(N,M) \to \mathbb{R}$ is a C^1 Lagrangian associated to a multitime variational problem, then the functional $J_L : F \to R$ is η-convex if*

$$\int_N L(\Psi(t))dt - \int_N L(\Phi(t))dt \geq \int_N \eta(\Psi,\Phi)(t)(L)dt,\ \forall \Psi,\Phi \in F \tag{5.3}$$

which, furthermore, is equivalent to

$$\begin{aligned}&\int_N L(t,y^i(t),y^i_\sigma(t))dt - \int_N L(t,x^i(t),x^i_\sigma(t))dt \geq \int_N \{\eta^k(t,x^i(t),y^i(t),x^i_\sigma(t),y^i_\sigma(t))\\ &\cdot \frac{\partial L}{\partial x^k}(t,x^i(t),x^i_\sigma(t)) + D_\gamma[\eta^k(t,x^i(t),y^i(t),x^i_\sigma(t),y^i_\sigma(t))]\frac{\partial L}{\partial x^k_\gamma}(t,x^i(t),x^i_\sigma(t))\}dt,\\ &\forall x,y : N \to M.\end{aligned} \tag{5.4}$$

We establish next the relation between the invexity and the Riemannian convexity of functionals introduced and studied in the previous sections.

Proposition 5.3. *Let $F \subseteq E(I)$ be an open totally convex subset and $J_\theta : F \to \mathbb{R}$ be the functional associated to a differential m-form θ. The functional J_θ is Riemannian convex iff*

$$J_\theta[\Psi] - J_\theta[\Phi] \geq dJ_\theta(\Phi)[X],\ \forall \Psi,\Phi \in F, \tag{5.5}$$

where X is the infinitesimal deformation associated to a geodesic deformation in $E(I)$ between Φ and Ψ.

Theorem 5.4. *If $F \subseteq E(I)$ is an open totally convex subset, θ is a differential m-form on $J^1(N,M)$ and the functional $J_\theta : F \to \mathbb{R}$ is Riemannian convex, then J_θ is η_0-convex, where η_0 is the pairing map induced by the inverse of the exponential map on $J^1(N,M)$.*

Remark 5.5. *We have proved that the preconvexity implies the convexity, and they are equivalent for η_0. Therefore, it seems natural and more appropriate, from now on, to refer to the classic convexity by using the term of η_0-convexity.*

Same as before, the η-convexity is invariant with respect to some coordinate changes.

Theorem 5.6. *Let $F \subset E(I)$ be an open subset and $J_\theta : F \to \mathbb{R}$ be an η-convex functional. If $f : J^1(N,M) \to J^1(N,M)$ is a diffeomorphism preserving $\mathcal{I}$, then the functional $J_{(f^{-1})^*\theta}$ is $\bar{\eta}$-convex on $f(F)$, where*

$$\bar{\eta}(f\circ\Psi,\varphi\circ\Phi)(t) = f_*(\eta(\Psi,\Phi)(t)),\ \forall \Psi,\Phi \in F,\ \forall t \in N.$$

Proof. By computation, we have

$$\begin{aligned} dJ_{(f^{-1})^*\theta}(f\circ\Phi)[\bar{\eta}(f\circ\Psi,\varphi\circ\Phi)] &= \int_N (f\circ\Phi)^*[\bar{\eta}(f\circ\Psi,\varphi\circ\Phi)((f^{-1})^*\theta)]\\ &= \int_N (f\circ\Phi)^*[(f^{-1})^*(\eta(\Psi,\Phi)(\theta))]\\ &= dJ_\theta(\Phi)[\eta(\Psi,\Phi)].\end{aligned}$$

Moreover, since $J_{(f^{-1})^*\theta}[f\circ\Phi] = J_\theta[\Phi]$, $\forall \Phi \in F$ and J_θ is η-convex, it follows that $J_{(f^{-1})^*\theta}$ is $\bar{\eta}$-convex. □

Corollary 5.7. *Let J_L be an η-convex action associated to a multitime variational problem. If $f : J^1(N, M) \to J^1(N, M)$ is a diffeomorphism preserving $\mathcal{I}$ and $\bar{\eta}$ is the induced pairing map as in the Theorem 5.6, then $J_{L \circ f^{-1}}$ is an $\bar{\eta}$-convex functional.*

Corollary 5.8. *If $J_L : E(I) \to \mathbb{R}$ is a functional as above, then the property of J_L of being invex is invariant with respect to any change of coordinates on M.*

Theorem 5.9. *Let $F \subseteq E(I)$ be an open subset, $\eta : F \times F \to TE(I)$ be a fixed pairing map and J_θ be the functional associated to a differential m-form θ. Then,*

1. *if $J_\theta : F \to \mathbb{R}$ is an η-convex functional, the functional kf, $k > 0$ is also η-convex;*
2. *if $J_\theta, J_\Omega : F \to \mathbb{R}$ are η-convex functionals, then $J_\theta + J_\Omega$ is also η-convex;*
3. *if $J_{\theta_i} : F \to \mathbb{R}$, $i = \overline{1, k}$ are η-convex functionals and $k_i > 0$, $\forall i = \overline{1, k}$, then the functional $\sum_{i=1}^{m} k_i J_{\theta_i}$ is η-convex.*

Theorem 5.10. *Let $F \subseteq E(I)$ be an open subset, $\eta : F \times F \to TE(I)$ be a fixed pairing map and J_θ be the functional associated to a differential m-form θ. If $J_\theta : F \to \mathbb{R}$ is η-convex and $\Psi : \mathbb{R} \to \mathbb{R}$ is an increasing C^1 convex function, then $\Psi \circ J_\theta$ is also η-convex.*

Theorem 5.11. *Let $F \subseteq E(I)$ be an open subset and $J_\theta : F \to \mathbb{R}$ be the functional associated to the differential m-form θ. The functional J_θ is invex iff all the critical points of J_θ are global minimum points.*

Proof. Let $\Phi \in F$ be a critical point for J_θ. Then $dJ_\theta(\Phi)[\eta(\Psi, \Phi)] = 0$, $\forall \Psi \in F$ and, since J_θ is invex, it follows that $J_\theta[\Psi] \geq J_\theta[\Phi]$, $\forall \Psi \in F$, therefore Φ is a global minimum point.

Conversely, we suppose that every critical point is a global minimum. If $\Phi \in F$ is a critical point, then $dJ_\theta(\Phi)[X] = 0$, $\forall X \in T_\Phi E(I)$ and, for $\Psi \in F$ arbitrary, we consider $\eta(\Psi, \Phi)(t) = 0$, $\forall t \in N$. If Φ is not a critical point, there is a vector field X such that $J_{X(\theta)}[\Phi] \neq 0$ and we consider

$$\eta(\Psi, \Phi)(t) = \frac{[J_\theta[\Psi] - J_\theta[\Phi]]X(\Phi(t))}{J_{X(\theta)}[\Phi]}.$$

The vector map η is a pairing map and satisfies the condition

$$J_\theta[\Psi] - J_\theta[\Phi] - dJ_\theta(\Phi)[\eta(\Psi, \Phi)] \geq 0,$$

therefore J_θ is η-convex. □

Theorem 5.12. *If $L : J^1(N, M) \to \mathbb{R}$ is a C^1 Lagrangian and all the points of the set $Crit^A(L) = \{(t, x^i, x^i_\alpha) \in J^1(N, M) |\ X(L)(t, x^i, x^i_\alpha) = 0, \forall X \in \mathcal{X}(E(I))\}$ are minimum points for L, then J_L is invex and all the solutions of the Euler-Lagrange PDEs are optimal solutions.*

Proof. The hypotheses allow us to consider a pairing map $\eta : J^1(N, M) \times J^1(N, M) \to TJ^1(N, M)$ such that L is η-convex and the map $\eta'(\Psi, \Phi)(t) = \eta(\Psi(t), \Phi(t))$ satisfies the condition $\eta'(\Psi, \Phi) \in T_\Phi E(I)$ (is a pairing map for J_L). We have:
$J_L[\Psi] - J_L[\Phi] - dJ_L(\Phi)[\eta'(\Psi, \Phi)] = \int_N L(\Psi(t)) - L(\Phi(t)) - \eta(\Psi(t), \Phi(t))(L) dt \geq 0.$ □

6 Examples

In this section we will analyze some examples of variational problems, establishing their invexity by applying the Theorem 5.12. From now on, if L is the Lagrangian associated to a variational problem, then

$$Crit^A(L) = \{(t, x^i, x^i_\alpha) \in J^1(N, M) |\ X(L)(t, x^i, x^i_\alpha) = 0,\ \forall X \in \mathcal{X}(E(I))\},$$

$Crit(L)$ is the set of critical points of L on $J^1(N, M)$ and $Crit(J_L)$ denotes the solutions of the Euler-Lagrange PDEs (i.e. the critical points of the functional J_L). We have $Crit(L) \subset Crit^A(L)$.

Example We consider the functional

$$J[x(\cdot)] = \int_1^2 [\dot{x}(t)^2 + 2x(t)\dot{x}(t) + x(t)^2]dt,$$

with $x(\cdot)$ satisfying the conditions $x(1) = 1$ and $x(2) = 2$.

The Lagrangian is $L(t, x, \dot{x}) = \dot{x}^2 + 2x\dot{x} + x^2$ and the Euler-Lagrange ODE writes as $\ddot{x}(t) - x(t) = 0$. We have

$$Crit(J) = \{x_0 : [1, 2] \to \mathbb{R} |\ x_0(t) = \frac{2 - e^{-1}}{e^2 - 1} e^t + \frac{e^3 - 2e^2}{e^2 - 1} e^{-t}\}$$

A point $(t, x, \dot{x})$ is in $Crit^A(L)$ if $X(L)(t, x, \dot{x}) = 0$, $\forall X \in \mathcal{X}(E(I))$. Since the elementary vector field $X_0 = x\frac{\partial}{\partial x} + \dot{x}\frac{\partial}{\partial \dot{x}}$ is a vector field from $\mathcal{X}(E(I))$ and

$$X_0(L) = 0 \Leftrightarrow 2(x + \dot{x})^2 = 0 \Leftrightarrow \dot{x} = -x.$$

It follows that $Crit^A(L) = \{(0, \alpha, -\alpha) |\ \alpha \in R\}$ and since all the elements of $Crit^A(L)$ are minimum points for the Lagrangian, it follows that L and J are invex, therefore $x_0(\cdot)$ minimizes the functional.

Example We consider the functional

$$J[x(\cdot)] = \int_0^T [\dot{x}(t)^2 \frac{t}{2} + x(t)^2]dt.$$

Same arguments as above ensure us that $Crit^A(L) = \{(t, 0, 0) |\ t \in \mathbb{R}\}$ and since all these points are minimum points for L it follows that L and J are invex and, therefore, the solution of the Euler-Lagrange PDE associated to this variational problem is also a global minimum point. By computation, this minimum point is

$$x : [0, T] \to \mathbb{R},\ x(t) = c_1 \frac{1}{t} + c_2,$$

where c_1 and c_2 are real constants.

Example We look for minimizing the functional

$$J[x(\cdot)] = \int_0^3 \sqrt{1 + \dot{x}(t)^2}dt,$$

between the points $A(0,2)$ and $B(3,5)$.

We have

$$Crit(J) = \{x_0 : [0,3] \to [2,5] |\ x_0(t) = t+2\}$$

and

$$Crit^A(L) = \{(t,x,0) |\ t \in [0,3],\ x \in [2,5]\}.$$

All the elements of $Crit^A(L)$ are minimum points and it follows that x_0 is an optimal solution for the variational problem.

In the following we prove the invexity of volumetric and kinetic energy, when g is considered to be the Euclidean structure and $M = \mathbb{R}^n$. Let N be a compact m-dimensional Riemannian manifold with $(t^1,...,t^m)$ local coordinates and let E be the set of all submanifolds maps from N to M.

Definition 6.1. The functional $J : E \to \mathbb{R},\ J[x(\cdot)] = \frac{1}{2}\int_N det(x^*g)(t)dt$ is called *the volumetric energy associated to* N, where x^*g denotes the pull-back of the Euclidean metric g on N.

Theorem 6.1. *The volumetric energy functional is invex.*

Proof. We introduce the following differentiable functions on $J^1(N,M)$:

$$\overline{g}_{\alpha\beta}(t^\gamma, x^i, x^i_\gamma) = \delta_{ij}x^i_\alpha x^j_\beta;\ \overline{g}(t^\gamma, x^i, x^i_\gamma) = det(\overline{g}_{\alpha\beta}).$$

The Lagrangian corresponding to the previous functional is

$$L(t^\gamma, x^i, x^i_\gamma) = \frac{1}{2}\overline{g}(t^\gamma, x^i, x^i_\gamma)$$

and

$$X_0(L) = 0 \Leftrightarrow \overline{g}\overline{g}^{\alpha\beta}\delta_{ij}x^i_\alpha x^j_\beta = 0 \Leftrightarrow \overline{g} = 0.$$

It follows that

$$Crit^A L \subset \{(t^\gamma, x^i, x^i_\gamma) |\ \overline{g}(t^\gamma, x^i, x^i_\gamma) = 0\},$$

and, because all these critical points are also minimum points, it follows that the Lagrangian is invex on A and, consequently, the functional J is invex. $\square$

Definition 6.2. If h is a Riemannian structure on N, the functional

$$\overline{J} : E \to \mathbb{R},$$

$$\overline{J}(x(\cdot)) = \frac{1}{2}\int_N Tr^h(\overline{g}_{\alpha\beta}(t, x(t), x^i_\gamma(t))\sqrt{h}dt = \int_N (Tr^h(x^*g)\sqrt{h})(t)dt$$

is called *the kinetic energy functional associated to* (N,h).

Theorem 6.2. *The kinetic energy functional is invex.*

Proof. The Lagrangian associated to this functional is

$$L(t^\gamma, x^i, x^i_\gamma) = \frac{1}{2} h^{\alpha\beta}(t)\delta_{ij} x^i_\alpha x^j_\beta$$

and

$$X_0(L) = 0 \Leftrightarrow h^{\alpha\beta}\delta_{ij} x^i_\alpha x^j_\beta = 0 \Leftrightarrow G(T,T) = 0 \Leftrightarrow T = 0,$$

where $T = x^i_\alpha \frac{\partial}{\partial x^i_\alpha}$. Consequently,

$$Crit^A L = \{(t, x, 0) |\ t \in N,\ x \in M\}.$$

Since all the elements of the previous set are also minimum points it follows that L is an invex Lagrangian on A and $\overline{J}$ is an invex functional. □

7 Conclusions

(1) The pairing maps used for defining the preinvexity and the invexity are local generalizations for the inverse of the exponential map on the jet bundle.

(2) The Riemannian η_0-preconvexity or the η_0-convexity (Section 3-5) associated to the elementary pairing map $\eta_0 = exp^{-1}$ are equivalent with the classic Riemannian convexity (Section 2).

(3) The η-convexity, unlike the preconvexity, is a differential concept and not a Riemannian one.

(4) Our results prove a strong correlation between the convex (invex, preinvex) nature of an action associated to a Lagrangian and the convex nature of the Lagrangian itself restricted to a submanifold of the first order jet bundle.

(5) An invex variational problem has the advantage to precisely identify the optimal solutions: all the solutions of the Euler-Lagrange PDEs are solutions for the variational problem.

(6) We can define convexity (invexity, preinvexity) of functionals associated to differential forms outside the variational setting. For that we need an arbitrary Riemannian manifold instead of a jet bundle and the set of the submanifold maps between a differential manifold and a Riemannian one. If so, by customization, we can regain from this theory the Riemannian convexity of functions if taking $N = \{t_0\}$, (M, g) a Riemannian manifold and, for E the set of all the maps from N to M and f a differentiable function on M, by considering

$$J_f : E \to R,\ J_f[\Phi] = f \circ \Phi.$$

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