

Classification of constant angle surfaces in a warped product

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Abstract. Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \rightarrow \mathbb{R}$ a strictly positive function and denote by \mathbb{E}^2 the Euclidean plane. We classify all surfaces in the warped product manifold $I \times_f \mathbb{E}^2$ for which the unit normal makes a constant angle with the direction tangent to I . Interesting results may be obtained for constant angle surfaces in the hyperbolic space \mathbb{H}^3 .

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1 Introduction

In the last few years, the study of the geometry of surfaces in 3-dimensional spaces, in particular of product type $M^2 \times \mathbb{R}$ was developed by a large number of mathematicians. In particular, in [8], [9] and [10] the authors have studied constant angle surfaces in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$, namely those surfaces for which the unit normal makes a constant angle with the tangent direction to \mathbb{R} . In [12] a classification of surfaces in the 3-dimensional Heisenberg group making a constant angle with the fibers of the Hopf-fibration was obtained. In all these spaces, the angle which is required to be constant is one of the fundamental invariants appearing in the existence and uniqueness theorem for isometric immersions, cfr. [7]. In another recent paper [6] it is proven that if the ambient space is the Euclidean 3-space, the study of surfaces making a constant angle with a fixed direction has some important applications to physics, namely in special equilibrium configurations of nematic and smectic C liquid crystals. In [15] constant angle surfaces in 3-dimensional Minkowski space were studied. Geometric properties (e.g. minimality) are obtained investigating the so called *contact angle* in different ambient spaces such as spheres or the Heisenberg group (see [16], [17]).

On the other hand, several authors have studied surfaces in warped products $I \times_f M$, where I is an interval, M is a surface and f a function on I . For instance in [13] totally umbilical surface are considered. In a sequence of papers [1], [2], [3], [4] the authors study constant mean curvature surfaces. In the present paper, constant

angle surfaces in warped products of an open interval with the Euclidean plane, are classified. Special examples, such as flat or minimal surfaces in this family are given. Finally, we discuss constant angle surfaces in the hyperbolic space \mathbb{H}^3 .

2 Preliminaries

The following notion of warped product or, more generally, warped bundle was introduced by Bishop and O'Neill in [5] in order to construct a large variety of manifolds of negative curvature. Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively. Let $f > 0$ be a smooth positive function on B and denote by $B \times_f F$ the product manifold. The warped product of B and F with warping function f is the Riemannian manifold

$$B \times_f F = (B \times F, g_B + f^2 g_F).$$

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a smooth strictly positive function on an open interval I and consider the warped product of I and the Euclidean plane \mathbb{E}^2 with warping function f

$$(\widetilde{M}, \widetilde{g}) = I \times_f \mathbb{E}^2 = (I \times \mathbb{R}^2, dt^2 + f(t)^2(dx^2 + dy^2))$$

where t is the coordinate on I and x and y are coordinates on \mathbb{E}^2 .

Denote by $\widetilde{\nabla}$ the Levi-Civita connection of $(\widetilde{M}, \widetilde{g})$. Denote by U , V and W lifts of vector fields tangent to \mathbb{E}^2 . One has

$$(2.1.a) \quad \widetilde{\nabla}_U V = D_U V - \frac{f'}{f} \widetilde{g}(U, V) \partial_t$$

$$(2.1.b) \quad \widetilde{\nabla}_U \partial_t = \widetilde{\nabla}_{\partial_t} U = \frac{f'}{f} U$$

$$(2.1.c) \quad \widetilde{\nabla}_{\partial_t} \partial_t = 0$$

where D is the covariant derivative on \mathbb{E}^2 , see for example [19]. Remark that we have identified U and V with their projections onto \mathbb{E}^2 . From these equations, it follows immediately that the curvature tensor \widetilde{R} , defined as $\widetilde{R}(U, V) = [\widetilde{\nabla}_U, \widetilde{\nabla}_V] - \widetilde{\nabla}_{[U, V]}$ is given by

$$(2.2.a) \quad \widetilde{R}(U, \partial_t)V = \frac{f''}{f} \widetilde{g}(U, V) \partial_t$$

$$(2.2.b) \quad \widetilde{R}(U, V)\partial_t = 0$$

$$(2.2.c) \quad \widetilde{R}(U, \partial_t)\partial_t = -\frac{f''}{f} U$$

$$(2.2.d) \quad \widetilde{R}(U, V)W = -\frac{(f')^2}{f^2} (\widetilde{g}(V, W)U - \widetilde{g}(U, W)V).$$

Let $\iota : M \rightarrow \widetilde{M}$ be an immersion of a surface M in \widetilde{M} and let g be the pulled back metric of \widetilde{g} on M . We will not write down ι , unless it is absolutely necessary to avoid confusion. Let ξ be a unit normal vector field on M and denote by A the associated shape operator. The formulas of Gauss and Weingarten state respectively that

$$(G) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (W) \quad \tilde{\nabla}_X \xi = -AX$$

for every X and Y tangent to M . Here, ∇ is the Levi-Civita connection of M and h is the second fundamental form. We have $\tilde{g}(h(X, Y), \xi) = g(X, AY)$ for all X and Y tangent to M . One can decompose ∂_t as

$$(2.3) \quad \partial_t = T + \cos \theta \xi,$$

where $\theta \in [0, \pi)$ is the angle between ∂_t and the normal ξ and T is the projection of ∂_t on the tangent plane of M . We have $\cos \theta = \tilde{g}(\partial_t, \xi)$ and, since ∂_t has unit length, $|T| = \sin \theta$.

If one denotes by R the curvature tensor on M , then it follows from (2.2), (G), (W) and (2.3) that the equations of Gauss and Codazzi can be written respectively as

$$(EG) \quad \begin{aligned} R(X, Y)Z = & g(AY, Z)AX - g(AX, Z)AY \\ & - ((\log f)' \circ \iota)^2 (g(Y, Z)X - g(X, Z)Y) \\ & - ((\log f)'' \circ \iota) \left(g(Y, T)g(Z, T)X - g(X, T)g(Z, T)Y \right. \\ & \left. - g(Y, T)g(X, Z)T + g(X, T)g(Y, Z)T \right) \end{aligned}$$

$$(EC) \quad (\nabla_X A)Y - (\nabla_Y A)X = \cos \theta ((\log f)'' \circ \iota) (g(Y, T)X - g(X, T)Y)$$

for X, Y and Z tangent to M .

Proposition 2.1. *Let X be tangent to M , then*

$$(2.4) \quad \nabla_X T = \cos \theta AX + ((\log f)' \circ \iota) (X - g(X, T)T),$$

$$(2.5) \quad X(\cos \theta) = -g(X, AT) - \cos \theta ((\log f)' \circ \iota) g(X, T).$$

Proof. If X is tangent to M , then $\tilde{g}(X, \partial_t) = g(X, T)$. One can express $\tilde{\nabla}_X \partial_t$ in two ways:

$$\tilde{\nabla}_X \partial_t = ((\log f)' \circ \iota) (X - g(X, T)\partial_t), \text{ by use of (2.1.b) and (2.1.c),}$$

$$\tilde{\nabla}_X \partial_t = \nabla_X T + h(X, T) + X(\cos \theta)\xi - \cos \theta AX, \text{ by use of (G), (W) and (2.3).}$$

Comparing the tangent and the normal parts, one gets the conclusion. \square

From (2.5) we obtain immediately the following.

Proposition 2.2. *If θ is a constant angle, then T is a principal direction and the corresponding eigenvalue of the shape operator is $-\cos \theta ((\log f)' \circ \iota)$.*

From now on, we will assume that θ is constant. In this case we say that $\iota : M \rightarrow \tilde{M}$ is a constant angle surface.

We may assume that $\theta \in [0, \pi/2]$. If $\theta = 0$, then $\iota(M) \subseteq \{t_0\} \times \mathbb{E}^2$. If $\theta \neq 0$, then $T \neq 0$ and we can consider $e_1 = T/|T| = T/\sin \theta$. Let e_2 be a unit tangent vector, orthogonal to e_1 . Then e_2 is also a principal direction, thus there exists a function $\lambda \in C^\infty(M)$ such that $Ae_2 = \lambda e_2$. Combining with (2.4), this yields the following.

Proposition 2.3. *Let M be a constant angle surface in \tilde{M} , with $\theta \neq 0$. Then there exists an orthonormal frame field $\{e_1, e_2\}$ on M such that the shape operator with*

respect to this frame takes the form

$$(2.6) \quad A = \begin{pmatrix} -\cos \theta ((\log f)' \circ \iota) & 0 \\ 0 & \lambda \end{pmatrix}$$

for some $\lambda \in C^\infty(M)$ and the Levi-Civita connection is given by

$$(2.7) \quad \begin{aligned} \nabla_{e_1} e_1 &= 0, & \nabla_{e_2} e_1 &= \frac{1}{\sin \theta} (\lambda \cos \theta + ((\log f)' \circ \iota)) e_2, \\ \nabla_{e_1} e_2 &= 0, & \nabla_{e_2} e_2 &= -\frac{1}{\sin \theta} (\lambda \cos \theta + ((\log f)' \circ \iota)) e_1. \end{aligned}$$

3 The classification theorem

In this section we classify the constant angle surfaces in $(\widetilde{M}, \widetilde{g}) = I \times_f \mathbb{E}^2$.

The first impulse to solve the problem would be to parametrize the surface as a graph $(t(x, y), x, y)$ in \mathbb{E}^3 . It follows that t is a solution of a classical first order PDE $t_x^2 + t_y^2 = \tan \theta f^2(t(x, y))$ if $\theta \neq 0$ known as *eikonal equation*. This technique was used by Cermelli and Di Scala in [6]. The methods of the classical theory of first order PDE can be used to study constant angle surfaces, yet explicit embedding equations are not obtained. Consequently, we will discuss another approach in order to classify our surfaces.

We only have to consider the case $\theta \neq 0$. In that case, we consider the orthonormal frame field $\{e_1, e_2\}$ as above. Then from (2.7) we obtain that $[e_1, e_2]$ is proportional to e_2 . Therefore we can choose coordinates (u, v) such that $\partial_u = e_1$ and $\partial_v = \beta e_2$ for some function β . Then it is clear that g takes the form

$$(3.1) \quad g = du^2 + \beta^2(u, v) dv^2.$$

The Levi-Civita connection is determined by

$$(3.2) \quad \nabla_{\partial_u} \partial_u = 0, \quad \nabla_{\partial_u} \partial_v = \nabla_{\partial_v} \partial_u = \frac{\beta_u}{\beta} \partial_v, \quad \nabla_{\partial_v} \partial_v = -\beta \beta_u \partial_u + \frac{\beta_v}{\beta} \partial_v$$

and β satisfies

$$(3.3) \quad \beta_u = \frac{\beta}{\sin \theta} (\lambda \cos \theta + ((\log f)' \circ \iota)).$$

If we put

$$\iota(u, v) = (t(u, v), x(u, v), y(u, v))$$

then

$$t_u = \widetilde{g}(\iota_u, \partial_t) = \widetilde{g}(e_1, \partial_t) = \widetilde{g}(T/\sin \theta, T + \cos \theta \xi) = \sin \theta$$

and

$$t_v = \widetilde{g}(\iota_v, \partial_t) = \widetilde{g}(\beta e_2, \partial_t) = \widetilde{g}(\beta e_2, T + \cos \theta \xi) = 0$$

such that, after a translation in the u coordinate,

$$(3.4) \quad t(u, v) = u \sin \theta.$$

Theorem 3.1. *An isometric immersion $\iota : M \rightarrow I \times_f \mathbb{E}^2$ defines a surface with constant angle $\theta \in [0, \pi/2]$ if and only if, up to rigid motions of $I \times_f \mathbb{E}^2$, one of the following holds locally.*

(i) *There exist local coordinates (u, v) on M , with respect to which the immersion ι is given by*

$$(3.5) \quad \iota(u, v) = \left(u \sin \theta, \cot \theta \left(\int^{u \sin \theta} \frac{d\tau}{f(\tau)} \right) \cos v - \int^v \alpha(\tau) \sin \tau d\tau, \right. \\ \left. \cot \theta \left(\int^{u \sin \theta} \frac{d\tau}{f(\tau)} \right) \sin v + \int^v \alpha(\tau) \cos \tau d\tau \right)$$

for some smooth function α .

(ii) *$\iota(M)$ is an open part of the cylinder*

$$(3.6) \quad x - \cot \theta \int^t \frac{d\tau}{f(\tau)} = 0.$$

This surface is totally umbilical with mean curvature $H = -\cos \theta f'(u \sin \theta)/f(u \sin \theta)$.

(iii) *$\iota(M)$ is an open part of the surface $t = t_0$ for some real number t_0 , and $\theta = 0$.*

Proof. Let us first check that the surfaces described in the theorem are constant angle surfaces.

For case (i), a basis for the tangent plane to the surface is given by

$$\iota_u = \left(\sin \theta, \frac{\cos \theta \cos v}{f(u \sin \theta)}, \frac{\cos \theta \sin v}{f(u \sin \theta)} \right) \\ \iota_v = \left(\cot \theta \left(\int^{u \sin \theta} \frac{d\tau}{f(\tau)} \right) + \alpha(v) \right) (0, -\sin v, \cos v).$$

Notice that if $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are vectors in $T_{(t,x,y)}(I \times_f \mathbb{E}^2)$, then the vector defined by

$$a \times_f b = (f^2(t)(a_2 b_3 - a_3 b_2), a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

is orthogonal to both a and b . Hence

$$\xi = \frac{\iota_u \times_f \iota_v}{|\iota_u \times_f \iota_v|} = \left(\cos \theta, -\frac{\sin \theta \cos v}{f(u \sin \theta)}, -\frac{\sin \theta \sin v}{f(u \sin \theta)} \right)$$

is a unit normal on the surface. We immediately deduce that $\tilde{g}(\xi, \partial_t) = \cos \theta$.

For case (ii), one can use the parametrization

$$\iota(u, v) = \left(u, \cot \theta \int^u \frac{d\tau}{f(\tau)}, v \right).$$

Then $\xi = (\cos \theta, -\sin \theta/f(u), 0)$ is a unit normal and $\tilde{g}(\xi, \partial_t) = \cos \theta$.

Case (iii) is obvious.

Conversely, let $\iota : M \rightarrow I \times_f \mathbb{E}^2$ be a constant angle surface with constant angle θ . As mentioned before, we may assume that $\theta \in [0, \pi/2]$. If $\theta = 0$ then $\iota(M)$ is of type (iii) described in the theorem. If $\theta = \pi/2$, the vector field ∂_t is everywhere tangent to $\iota(M)$. This implies that $\iota(M)$ is an open part of a cylinder with rulings in the direction of ∂_t or, equivalently that there exist local coordinates (u, v) on M such that $\iota(u, v) = (u, \gamma_1(v), \gamma_2(v))$ for some smooth functions γ_1 and γ_2 . If ι parametrizes a plane, this is case (ii) of the theorem with $\theta = \pi/2$. If ι does not describe a plane, this is case (i) of the theorem with $\theta = \pi/2$.

From now on, assume that $\theta \in (0, \pi/2)$. If we choose local coordinates on M as above, we can write $\iota(u, v) = (u \sin \theta, x(u, v), y(u, v))$. Using (3.1) we obtain

$$(3.7.a) \quad f^2(u \sin \theta) (x_u^2 + y_u^2) = \cos^2 \theta$$

$$(3.7.b) \quad x_u x_v + y_u y_v = 0$$

$$(3.7.c) \quad f^2(u \sin \theta) (x_v^2 + y_v^2) = \beta^2.$$

Define

$$(3.8) \quad \sigma(u) = \log f(u \sin \theta) = ((\log f) \circ \iota)(u, v).$$

Then a straightforward computation, using (2.1), (3.7) and (3.8) yields

$$(3.9.a) \quad \tilde{\nabla}_{\iota_u} \iota_u = \iota_{uu} + 2\sigma' \iota_u - \left(\sin \theta + \frac{1}{\sin \theta} \right) \sigma' \partial_t$$

$$(3.9.b) \quad \tilde{\nabla}_{\iota_u} \iota_v = \iota_{uv} + \sigma' \iota_v$$

$$(3.9.c) \quad \tilde{\nabla}_{\iota_v} \iota_v = \iota_{vv} - \frac{1}{\sin \theta} \beta^2 \sigma' \partial_t.$$

On the other hand, we can express these covariant derivatives by using the formula of Gauss **(G)**. By using (2.3), (2.6), (3.2) and (3.8) we obtain

$$(3.10.a) \quad \tilde{\nabla}_{\iota_u} \iota_u = \sigma' \iota_u - \frac{1}{\sin \theta} \sigma' \partial_t,$$

$$(3.10.b) \quad \tilde{\nabla}_{\iota_u} \iota_v = \frac{\beta_u}{\beta} \iota_v,$$

$$(3.10.c) \quad \tilde{\nabla}_{\iota_v} \iota_v = -(\beta \beta_u + \tan \theta \lambda \beta^2) \iota_u + \frac{\beta_v}{\beta} \iota_v + \frac{1}{\cos \theta} \lambda \beta^2 \partial_t.$$

We will now compare successively (3.9) to (3.10).

From (3.9.a) and (3.10.a) we obtain

$$\iota_{uu} + \sigma' \iota_u - \sin \theta \sigma' \partial_t = 0.$$

This equation is satisfied for the t -component. For the x - and the y -component we obtain respectively $x_{uu} + \sigma' x_u = 0$ and $y_{uu} + \sigma' y_u = 0$, such that $x_u(u, v) = e^{-\sigma(u)} c_1(v)$ and $y_u(u, v) = e^{-\sigma(u)} c_2(v)$ for some functions c_1 and c_2 . From (3.7.a) we obtain $c_1^2(v) + c_2^2(v) = \cos^2 \theta$. If we put $p_1(v) = c_1(v)/\cos \theta$ and $p_2(v) = c_2(v)/\cos \theta$, then

$$(3.11) \quad \iota_u(u, v) = \left(\sin \theta, \cos \theta e^{-\sigma(u)} p_1(v), \cos \theta e^{-\sigma(u)} p_2(v) \right), \quad p_1^2(v) + p_2^2(v) = 1.$$

From (3.9.b) and (3.10.b), we obtain

$$\iota_{uv} + \left(\sigma' - \frac{\beta_u}{\beta} \right) \iota_v = 0.$$

This equation is again satisfied for the t -component. Integrating, we obtain

$$(3.12) \quad \iota_v(u, v) = e^{-\sigma(u)} \beta(u, v) (0, q_1(v), q_2(v)), \quad q_1^2(v) + q_2^2(v) = 1.$$

Remark that the compatibility condition for (3.11) and (3.12) is

$$(3.13) \quad (p'_1, p'_2) = \frac{1}{\cos \theta} (\beta_u - \sigma' \beta) (q_1, q_2).$$

Finally, from (3.9.c) and (3.10.c), we obtain

$$(3.14) \quad \iota_{vv} + (\beta \beta_u + \tan \theta \lambda \beta^2) \iota_u - \frac{\beta_v}{\beta} \iota_v - \beta^2 \left(\frac{\sigma'}{\sin \theta} + \frac{\lambda}{\cos \theta} \right) \partial_t = 0.$$

If we substitute (3.11) and (3.12) into (3.14), the resulting equations for the x - and the y -component yield

$$(3.15) \quad (q'_1, q'_2) = -(\beta_u \cos \theta + \lambda \beta \sin \theta) (p_1, p_2).$$

At this point we can distinct two cases: $(p_1(v), p_2(v))$ is constant or not.

Case 1: $(p_1(v), p_2(v))$ is constant.

Then from (3.13) we obtain that $\beta_u = \sigma' \beta$, and hence $\beta(u, v) = \psi(v) f(u \sin \theta)$. After a change in the v -coordinate, we can assume that $\psi(v) = 1$, such that $\beta(u, v) = f(u \sin \theta)$. From (3.3) we then obtain that $\lambda = -\cos \theta f'(u \sin \theta) / f(u \sin \theta)$. From Proposition 2.3 it follows that M is totally umbilical.

From (3.15) then follows that (q_1, q_2) is constant. Integrating (3.11) and using (3.12) gives us

$$(3.16) \quad \iota(u, v) = \left(u \sin \theta, p_1 \cos \theta \left(\int^u e^{-\sigma(\mu)} d\mu \right) + q_1 v + a_1, \right. \\ \left. p_2 \cos \theta \left(\int^u e^{-\sigma(\mu)} d\mu \right) + q_2 v + a_2 \right)$$

for some constants a_1 and a_2 , which can be taken zero after a translation in x and y . Moreover, since $\tilde{g}(\iota_u, \iota_v) = 0$, we have $p_1 q_1 + p_2 q_2 = 0$. Hence, after a rotation around the t -axis, which is an isometry of $I \times_f \mathbb{E}^2$, we may assume that $(p_1, p_2) = (1, 0)$ and $(q_1, q_2) = (0, 1)$. Hence we obtain after a substitution $\tau = \mu \sin \theta$

$$\iota(u, v) = \left(u \sin \theta, \cot \theta \int^{u \sin \theta} \frac{d\tau}{f(\tau)}, v \right)$$

which corresponds to case (ii) of the theorem.

Case 2: $(p_1(v), p_2(v))$ is not constant. Then from (3.11) we can assume that, after a change of the v -coordinate,

$$(3.17) \quad (p_1(v), p_2(v)) = (\cos v, \sin v).$$

Then (3.13) implies that

$$(3.18) \quad \beta_u - \sigma' \beta = \pm \cos \theta$$

and by changing the sign of u , we can assume the right hand side to be $\cos \theta$. Integrating (3.18) gives

$$(3.19) \quad \beta(u, v) e^{-\sigma(u)} - \cos \theta \int^u e^{-\sigma(\mu)} d\mu = \alpha(v)$$

for some function $\alpha(v)$. Furthermore (3.13) shows that

$$(q_1(v), q_2(v)) = (-\sin v, \cos v).$$

Hence (3.11) and (3.12) reduce to

$$(3.20) \quad \iota_u(u, v) = \left(\sin \theta, \cos \theta e^{-\sigma(u)} \cos v, \cos \theta e^{-\sigma(u)} \sin v \right)$$

$$(3.21) \quad \iota_v(u, v) = e^{-\sigma(u)} \beta(u, v) (0, -\sin v, \cos v).$$

Integrating (3.20) gives

$$(3.22) \quad \iota(u, v) = \left(u \sin \theta, \cos \theta \left(\int^u e^{-\sigma(\mu)} d\mu \right) \cos v + \gamma_1(v), \right. \\ \left. \cos \theta \left(\int^u e^{-\sigma(\mu)} d\mu \right) \sin v + \gamma_2(v) \right)$$

for some smooth functions γ_1 and γ_2 . If we take the derivative with respect to v in (3.22) and compare it to (3.21) we get, using (3.19)

$$(\gamma_1'(v), \gamma_2'(v)) = \alpha(v) (-\sin v, \cos v).$$

After integration, we obtain case (i) of the theorem. \square

Remark 3.2. In case (i), the function λ is given by

$$(3.23) \quad \lambda(u, v) = \frac{1}{\beta(u, v)} \sin \theta - \frac{f'(u \sin \theta)}{f(u \sin \theta)} \cos \theta.$$

This follows from (3.3) and (3.15).

Remark 3.3. Notice that if we take the Euclidean metric on \mathbf{R}^3 , i.e. the warping function is 1, we retrieve the statements of Theorem 7 in [18].

4 Rotational surfaces of constant angle

In this section, we classify constant angle surfaces in $I \times_f \mathbb{E}^2$, which are invariant under rotations with respect to the t -axis.

Let us first remark that any rotation

$$R_\phi : I \times_f \mathbb{E}^2 \rightarrow I \times_f \mathbb{E}^2 : (t, x, y) \mapsto (t, x \cos \phi - y \sin \phi, x \sin \phi + y \cos \phi)$$

is an isometry. Let γ be a curve in the plane containing the t - and the x -axis. Assume that $\gamma(u) = (a(u), b(u), 0)$ is an arc length parametrization, i.e., that

$$(4.1) \quad (a'(u))^2 + f^2(a(u))(b'(u))^2 = 1.$$

We want to investigate under which conditions the surface

$$\iota(u, v) = (a(u), b(u) \cos v, b(u) \sin v)$$

is a constant angle surface in $I \times_f \mathbb{E}^2$.

The unit normal vector field is given by

$$\xi(u, v) = \left(b'(u)f(a(u)), -\frac{a'(u) \cos v}{f(a(u))}, -\frac{a'(u) \sin v}{f(a(u))} \right).$$

Hence, the surface determines a constant angle surface with constant angle θ if and only if

$$(4.2) \quad b'(u)f(a(u)) = \cos \theta.$$

Combining (4.1) and (4.2) yields

$$(4.3) \quad (a'(u))^2 = \sin^2 \theta.$$

There are now two cases to consider.

The case $\sin \theta = 0$ is obvious and it corresponds to case (iii) of the Theorem 3.1. So assume $\sin \theta \neq 0$. Then we see from (4.3) that $a(u) = \pm u \sin \theta + c$ for some real constant c . After a change of the arc length parameter u of γ , we may assume that

$$(4.4) \quad a(u) = u \sin \theta.$$

If $\theta = \pi/2$, then $b = b_0$ is constant and we obtain the circular cylinder $\iota(u, v) = (u, b_0 \cos v, b_0 \sin v)$. In the sequel we will take $\theta \in (0, \pi/2)$.

It then follows from (4.2) that

$$b(u) = \int^u \frac{\cos \theta}{f(\mu \sin \theta)} d\mu = \cot \theta \int^{u \sin \theta} \frac{d\tau}{f(\tau)}.$$

We conclude that the rotational surface immersion becomes

$$(4.5) \quad \iota(u, v) = \left(u \sin \theta, \left(\cot \theta \int^{u \sin \theta} \frac{d\tau}{f(\tau)} \right) \cos v, \left(\cot \theta \int^{u \sin \theta} \frac{d\tau}{f(\tau)} \right) \sin v \right)$$

which corresponds, up to a translation in the x -direction, to a special situation of case (i) of Theorem 3.1, namely when $\alpha(v) = 0$.

5 Examples

5.1 Flat constant angle surfaces.

A surface of type (iii) of Theorem 3.1 is a trivial example of a flat surface with constant angle $\theta = 0$. In order to give an example of flat constant angle surface with $\theta \neq 0$ consider a surface of type (ii) in Theorem 3.1. Using **(EG)** and (2.6), we obtain

$$K = \det A - ((\log f)' \circ \iota)^2 - ((\log f)'' \circ \iota) \sin^2 \theta = - \left(\frac{f''}{f} \circ \iota \right) \sin^2 \theta.$$

The surface is flat if $f(t) = a(t + b)$, with $a \neq 0$. The metric \tilde{g} on the ambient space is called a *cone metric*.

5.2 Minimal constant angle surfaces.

Consider first a constant angle surface of type (iii) of Theorem 3.1. Then ∂_t is a unit normal and it follows from (2.1.b) that the surface is totally umbilical with shape operator $A = f'(t_0)/f(t_0) \text{id}$. Hence, such a surface is minimal if and only if $f'(t_0) = 0$, case in which it is also totally geodesic.

Now assume that the constant angle surface is of type (ii) of Theorem 3.1. Then it is minimal only if it is totally geodesic. Since $H = -\cos \theta f'(u \sin \theta)/f(u \sin \theta)$, either $\theta = \pi/2$, i.e. the surface is a warped product of an interval and a straight line in \mathbb{E}^2 , or $f' = 0$, i.e. the ambient space is a direct product and M is a plane.

Finally, if we assume that the constant angle surface is of type (i) of Theorem 3.1, then from (2.6) and (3.23) it follows that $H = 0$ if and only if

$$(5.1) \quad 2 \cos \theta \beta \sigma' = \sin^2 \theta.$$

Hence β depends only on u . Differentiating (5.1) using (3.18) yields

$$(5.2) \quad \left(\frac{1}{\sigma'} \right)' = \frac{1 + \cos^2 \theta}{\sin^2 \theta}.$$

Integrating (5.2) shows that f has to take the form

$$f(t) = b(t + c)^{\frac{\sin^2 \theta}{1 + \cos^2 \theta}}.$$

Without loss of generality, we can assume $c = 0$. We put $m = \sin^2 \theta / (1 + \cos^2 \theta)$, such that $f(t) = b t^m$, $m \in (0, 1)$. From (3.3) and (3.23) we then obtain that $\lambda = m \cot \theta / u$ and $\beta = u \cos \theta / (1 - m)$. Then it follows that in (3.19) we have to take $\alpha = 0$. Then from the classification Theorem 3.1 we obtain that

$$\iota(u, v) = \left(u \sin \theta, \frac{\cot \theta}{b(1 - m)} (u \sin \theta)^{1-m} \cos v, \frac{\cot \theta}{b(1 - m)} (u \sin \theta)^{1-m} \sin v \right).$$

This represents a constant angle minimal surface, with $\theta = \arccos \sqrt{(1 - m)/(1 + m)}$. Moreover, the surface is a rotation surface.

5.3 Constant angle surfaces with a harmonic height function.

Consider the *height function*

$$\tilde{h} : I \times_f \mathbb{E}^2 \rightarrow \mathbb{R} : (t, x, y) \mapsto t.$$

If $\iota : M \rightarrow I \times_f \mathbb{E}^2$ is an isometric immersion of a surface, then we denote by h the restriction of \tilde{h} to M , i.e. $h = \tilde{g}(\iota, \partial_t)$. Remark that

$$g(X, \text{grad } h) = X(h) = X(\tilde{g}(\iota, \partial_t)) = \tilde{g}(X, \partial_t) = g(X, T)$$

for all X tangent to M and hence

$$\text{grad } h = T.$$

Thus, by using (2.4) we obtain

$$(5.3) \quad \Delta h = \text{div } T = \text{trace}(\nabla T) = 2 \cos \theta H + ((\log f)' \circ \iota) (1 + \cos^2 \theta).$$

Remark that this formula yields the following. See also Lemma 3.1 and Corollary 3.2 in [20].

Proposition 5.1. *There are no compact minimal surfaces in $I \times_f \mathbb{E}^2$ if f is monotonic.*

Proof. Assume that $(\log f)' \geq 0$ (resp. ≤ 0) and that M is a compact, minimal surface in $I \times_f \mathbb{E}^2$. By integrating (5.3) and taking into account that $H = 0$, one obtains

$$0 = \int_M \Delta h \, dM = \int_M ((\log f)' \circ \iota) (1 + \cos^2 \theta) \, dM \geq 0 \text{ (resp. } \leq 0).$$

It follows that $(\log f)' \circ \iota = 0$, that is f is constant on M and the proposition follows immediately. \square

We now consider non-minimal constant angle surfaces with harmonic height function. If $\sin \theta = 0$, then h is constant. If $\cos \theta = 0$, then (5.3) implies that $f' = 0$ on M such that around M the ambient space is Euclidean and M is a part of a cylinder in the t -direction. If the surface is of type (ii) in Theorem 3.1, with $\theta \in (0, \pi/2)$, then it follows from (5.3) that f is constant on M , such that M is part of a plane, hence minimal. If the surface is of type (i) in Theorem 3.1, with $\theta \in (0, \pi/2)$, then h is harmonic if and only if

$$(5.4) \quad \sin \theta \cos \theta \lambda + \sigma' = 0.$$

From (5.4) and (3.23) it follows that

$$\beta = -\cos \theta \frac{1}{\sigma'}.$$

These equations yield that β only depends on u and that $\lambda\beta = 1/\sin \theta$. From (3.3) we easily obtain that β is constant. Therefore λ is constant and from (5.4) we obtain that $f(t) = ae^{bt}$. From (2.2) we conclude that the warped product has constant negative

sectional curvature. If we assume that it is -1 , then we can put $a = b = 1$. It also follows that $\alpha(v) = 0$ in (3.19) and the surface is given by

$$\iota(u, v) = (u \sin \theta, -\cot \theta e^{-u \sin \theta} \cos v, -\cot \theta e^{-u \sin \theta} \sin v).$$

Since β is constant, M is flat. Moreover, the surface is a rotation surface with constant mean curvature $H = -(1 + \cos^2 \theta)/(2 \cos \theta)$. So the surface is a flat constant mean curvature rotation surface in the hyperbolic space.

Remark 5.2. As we have already seen, the ambient space \mathbb{R}^3 endowed with the metric $\tilde{g} = dt^2 + e^{2t}(dx^2 + dy^2)$ has constant sectional curvature -1 . This model for the hyperbolic space is also used in [1] and [14]. Rotation surfaces in this space, as defined above, are the rotation surfaces in the sense of [11]. By changing the t -coordinate one can obtain the upper half space model for the hyperbolic 3-space. More precisely, by considering $z = e^{-t}$ one gets that (\tilde{M}, \tilde{g}) is isometric to (\mathbb{H}_+^3, g_{-1}) , where

$$\mathbb{H}_+^3 = \{(x, y, z) \in \mathbb{R}^3, z > 0\}, \quad g_{-1} = \frac{1}{z^2} (dx^2 + dy^2 + dz^2).$$

In this model, the constant angle surface M obtained above, is given implicitly by $x^2 + y^2 = z^2 \cot^2 \theta$, $\theta \in (0, \frac{\pi}{2})$, which represents a half-cone, i.e. an equidistant surface from a geodesic line in (\mathbb{H}_+^3, g_{-1}) and can be regarded as the analogue of a round cylinder in hyperbolic geometry (see also [21]). In the hyperboloid model for the hyperbolic space, this surface is the product of a circle and a hyperbola.

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