Minimal tensor product surfaces of two pseudo-Euclidean curves

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Abstract. In this article, surfaces are studied that arise from taking the tensor product of two curves. More precisely, the classification of minimal tensor product surfaces of two arbitrary curves in pseudo-Euclidean spaces is obtained. This main result generalizes several previously known partial results concerning tensor product surfaces and, moreover, corrects some of these.

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1 Introduction

Tensor products of submanifolds are one of the many interesting topics studied in differential geometry of submanifolds. The tensor product of two immersions of a given Riemannian manifold is introduced in [4] as a generalization of the quadratic representation of a submanifold. In [5], the tensor product of two immersions of, in general, different manifolds, is studied. See [14] for an introduction to and an overview of the origin of the study of tensor products of submanifolds.

A tensor product surface is obtained by taking the tensor product of two curves. In several papers, curvature conditions and other characterizations of tensor product surfaces are considered.

Various results are known for tensor product surfaces of two planar curves. For instance, in [11], minimal, totally real, complex, slant and pseudo-umbilical tensor product surfaces of Euclidean planar curves are studied. A classification of minimal, totally real and pseudo-minimal tensor product surfaces of Lorentzian planar curves is proved in [12]. Minimal and pseudo-minimal tensor product surfaces of a Lorentzian planar curve and a Euclidean planar curve are considered in [13]. In [2], minimal tensor product surfaces of two pseudo-Euclidean planar curves are classified.

Also tensor product surfaces of a planar curve and a space curve are well-studied. A classification of minimal, totally real and slant tensor product surfaces of a Euclidean space curve and a Euclidean planar curve is obtained in [1]. In [7], [8] and

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[9], the authors study minimal, totally real and complex tensor product surfaces of a Lorentzian space curve and a Lorentzian planar curve, a Euclidean space curve and a Lorentzian planar curve, and a Lorentzian space curve and a Euclidean planar curve respectively.

Recently, the minimal tensor product surfaces of two arbitrary Euclidean curves are classified in [3], hereby generalizing partially the previous mentioned results. Some errors in the results of [1] are corrected in [3].

In the present article, a classification of minimal tensor product surfaces of two arbitrary curves in pseudo-Euclidean spaces is proved. All the previous mentioned results on minimal tensor product surfaces are covered by this classification theorem. Also, some corrections of the results in [7], [8] and [9] are made.

Curvature properties of surfaces have already been the subject of many research. This work is a contribution to the study of a minimality condition on a surface in an arbitrary pseudo-Euclidean space. For an examination of relations between two curvatures of a surface in a 3-space see for instance [6] and [10].

2 Preliminaries

The mean curvature vector field of a non-degenerate surface M parametrized by $f(s,t)$ is given by

$$H = \frac{1}{2}\left(g^{11}\frac{\partial^2 f}{\partial s^2} + 2g^{12}\frac{\partial^2 f}{\partial s \partial t} + g^{22}\frac{\partial^2 f}{\partial t^2}\right)^{\perp},$$

where $\perp$ denotes the normal part and (g^{ij}) is the inverse matrix of (g_{ij}) with g_{ij} the components of the induced metric g on the surface M, see for example [15]. The surface M is minimal if and only if the mean curvature vector field is identically zero. That is, if and only if $g(H,n)=0$ for every normal n of the surface M. Thus, the next lemma follows directly.

Lemma 2.1 *A surface M parametrized by $f(s,t)$ is minimal if and only if*

$$g\left(g_{22}\frac{\partial^2 f}{\partial s^2} + g_{11}\frac{\partial^2 f}{\partial t^2} - 2g_{12}\frac{\partial^2 f}{\partial s \partial t}, n\right) = 0,$$

for every normal n of the surface.

Denote by $\mathbb{E}^m_\mu$ the m-dimensional pseudo-Euclidean space of index μ with the standard flat metric g_1. Consider the standard basis $\{U_1,\dots,U_m\}$ on $\mathbb{E}^m_\mu$ with spacelike vectors $U_1,\dots,U_{m-\mu}$ and timelike vectors $U_{m-\mu+1},\dots,U_m$. Analougously, denote the metric on $\mathbb{E}^n_\nu$ by g_2 and consider the standard basis $\{V_1,\dots,V_n\}$ on $\mathbb{E}^n_\nu$ with spacelike vectors $V_1,\dots,V_{n-\nu}$ and timelike vectors $V_{n-\nu+1},\dots,V_n$. Denote the metric matrices of $\mathbb{E}^m_\mu$ and $\mathbb{E}^n_\nu$ by G_1 and G_2 respectively. Consider the elements of $\mathbb{E}^m_\mu$ and $\mathbb{E}^n_\nu$ as column vectors. As in [8], identify in the usual way the space $\mathbb{E}^{mn}$ with the space $\mathcal{M}$ of real-valued $m\times n$ matrices. Define the metric g in $\mathcal{M}$ by

$$g(A,B) = \operatorname{trace}(G_1 A G_2\, {}^tB),$$

with $A,B\in\mathcal{M}$, where ${}^tB$ denotes the transpose of B. Then, $(\mathcal{M},g)$ is isometric to the pseudo-Euclidean space $\mathbb{E}^{mn}_\rho$ of index $\rho = \mu(n-\nu)+\nu(m-\mu)$.

The tensor product is defined as

$$\otimes : \mathbb{E}^m_\mu \times \mathbb{E}^n_\nu \to \mathcal{M} : (X,Y) \mapsto X \otimes Y = X\,{}^tY.$$

Concerning the metric g of $\mathcal{M}$, one has the following lemma.

Lemma 2.2 *If $X, W \in \mathbb{E}^m_\mu$ and $Y, Z \in \mathbb{E}^n_\nu$, then*

$$g(X \otimes Y, W \otimes Z) = g_1(X,W) g_2(Y,Z).$$

Proof. Straightforward calculation using the definitions of the metric g and the tensor product. □

A pseudo-orthogonal transformation of a pseudo-Euclidean space $\mathbb{E}^n_\nu$ is a linear map of $\mathbb{E}^n_\nu$ that preserves the standard flat metric of $\mathbb{E}^n_\nu$. The next lemma is used in the proof of the classification theorem.

Lemma 2.3 *Let O_1 and O_2 be pseudo-orthogonal transformations of $\mathbb{E}^m_\mu$ and $\mathbb{E}^n_\nu$ respectively. Then*

$$\mathcal{H} : \mathcal{M} \to \mathcal{M} : A \mapsto O_1 A\,{}^tO_2$$

is a pseudo-orthogonal transformation of $\mathcal{M}$.

Proof. From

$$\begin{aligned} g(\mathcal{H}(A), \mathcal{H}(B)) &= \operatorname{trace}(G_1 O_1 A\,{}^tO_2 G_2 O_2\,{}^tB\,{}^tO_1) \\ &= \operatorname{trace}({}^tO_1 G_1 O_1 A\,{}^tO_2 G_2 O_2\,{}^tB) \\ &= g(A,B), \end{aligned}$$

it is clear that $\mathcal{H}$ is a pseudo-orthogonal transformation of $\mathcal{M}$. □

3 Minimal tensor product surfaces of two pseudo-Euclidean curves

Let $f(s,t) = \alpha(s) \otimes \beta(t) = \alpha(s)\,{}^t\beta(t) = (\alpha_1(s)\beta_1(t), \alpha_1(s)\beta_2(t), \ldots, \alpha_m(s)\beta_n(t))$ be the tensor product surface of two arbitrary pseudo-Euclidean curves

$$\alpha : \mathbb{R} \to \mathbb{E}^m_\mu : s \mapsto \alpha(s) = (\alpha_1(s), \ldots, \alpha_m(s))$$

and

$$\beta : \mathbb{R} \to \mathbb{E}^n_\nu : t \mapsto \beta(t) = (\beta_1(t), \ldots, \beta_n(t)).$$

Assume $f(s,t) = \alpha(s) \otimes \beta(t)$ defines an immersion of $\mathbb{R}^2$ into $\mathcal{M}$. It follows directly that

$$f_s(s,t) = \frac{\partial f}{\partial s}(s,t) = \alpha'(s) \otimes \beta(t), \qquad f_t(s,t) = \frac{\partial f}{\partial t}(s,t) = \alpha(s) \otimes \beta'(t),$$

$$f_{ss}(s,t) = \alpha''(s) \otimes \beta(t), \quad f_{st}(s,t) = \alpha'(s) \otimes \beta'(t) \quad \text{and} \quad f_{tt}(s,t) = \alpha(s) \otimes \beta''(t),$$

where primes denote ordinary differentiation. From here on, the parameters s and t are often omitted for notational reasons. The components of the induced metric on the surface $f(s,t) = \alpha(s) \otimes \beta(t)$ are

$$\begin{aligned} g_{11} &= g(f_s, f_s) = g_1(\alpha', \alpha') g_2(\beta, \beta), \\ g_{12} &= g(f_s, f_t) = g_1(\alpha, \alpha') g_2(\beta, \beta'), \\ g_{22} &= g(f_t, f_t) = g_1(\alpha, \alpha) g_2(\beta', \beta'). \end{aligned}$$

Since $g_{11}g_{22} - g_{12}^2$ must be distinct from zero in order for the surface to be non-degenerate, the position vectors of α and β cannot be null.

Lemma 3.1 *For $i, j = 1, \ldots, m$ and $p, q = 1, \ldots, n$ with $i \neq j$ and $p \neq q$, the vectors*

$$n^1_{ijpq} = (\alpha_j g_1(U_i, U_i) U_i - \alpha_i g_1(U_j, U_j) U_j) \otimes (\beta_q g_2(V_p, V_p) V_p - \beta_p g_2(V_q, V_q) V_q)\,,$$

$$n^2_{ijpq} = \left(\alpha'_j g_1(U_i, U_i) U_i - \alpha'_i g_1(U_j, U_j) U_j\right) \otimes \left(\beta'_q g_2(V_p, V_p) V_p - \beta'_p g_2(V_q, V_q) V_q\right),$$

are normal to the surface $f(s,t) = \alpha(s) \otimes \beta(t)$.

Proof. The result follows directly from Lemma 2.2. □

It is clear that, without altering the tensor product surface, one of the curves can be multiplied by a non-zero constant, provided the other curve is divided by the same constant.

For pseudo-orthogonal transformations O_1 and O_2 of $\mathbb{E}^m_\mu$ and $\mathbb{E}^n_\nu$ respectively, it is clear that $O_1\alpha \otimes O_2\beta = O_1\alpha\, {}^t\beta\, {}^tO_2 = \mathcal{H}(\alpha \otimes \beta)$. Thus, by Lemma 2.3, the curves α and β are determined up to a pseudo-orthogonal transformation.

The minimal tensor product surfaces $f(s,t) = \alpha(s) \otimes \beta(t)$ are classified in the following theorem.

Theorem 3.2 *A non-degenerate tensor product surface $f(s,t) = \alpha(s) \otimes \beta(t)$ of two pseudo-Euclidean curves $\alpha : \mathbb{R} \to \mathbb{E}^m_\mu : s \mapsto \alpha(s)$ and $\beta : \mathbb{R} \to \mathbb{E}^n_\nu : t \mapsto \beta(t)$ is a minimal surface if and only if*

1. *α is either*

   (a) a circle in a definite plane;

   (b) a hyperbola in a non-degenerate plane of index 1,

   and β is either

   (a) a circle in a non-degenerate plane of index 1;

   (b) a hyperbola in a definite plane;

   (c) a hyperbola in a non-degenerate plane of index 1;

   (d) a hyperbola in a degenerate plane,

or

2. *β is an open part of a non-null straight line through the origin not containing the origin and α is a planar curve,*

or vice versa for α and β.

Proof. From Lemma 2.1, it follows that the tensor product surface $f(s,t)=\alpha(s)\otimes\beta(t)$ is minimal if and only if

$$g_1(\alpha,\alpha)g_2(\beta',\beta')g(f_{ss},n)+g_1(\alpha',\alpha')g_2(\beta,\beta)g(f_{tt},n)-2g_1(\alpha,\alpha')g_2(\beta,\beta')g(f_{st},n)=0,$$

for every normal n of the surface. Calculating this condition for the normal vectors defined in Lemma 3.1, one has the equations

$$g_1(\alpha,\alpha')g_2(\beta,\beta')(\alpha_j\alpha'_i-\alpha_i\alpha'_j)(\beta'_p\beta_q-\beta'_q\beta_p)=0, \tag{3.1}$$

$$\begin{aligned} &g_1(\alpha,\alpha)g_2(\beta',\beta')(\alpha'_j\alpha''_i-\alpha''_j\alpha'_i)(\beta'_q\beta_p-\beta'_p\beta_q)\\ &+g_1(\alpha',\alpha')g_2(\beta,\beta)(\alpha'_j\alpha_i-\alpha'_i\alpha_j)(\beta'_q\beta''_p-\beta'_p\beta''_q)=0, \end{aligned} \tag{3.2}$$

with $i,j=1,\dots,m$ and $p,q=1,\dots,n$. Starting from equation (3.1), two cases can be considered.

Case 1 Neither α nor β is (part of) a straight line through the origin

There exist indices $\tilde\imath,\tilde\jmath=1,\dots,m$ and $\tilde p,\tilde q=1,\dots,n$ such that

$$\alpha'_{\tilde\jmath}\alpha_{\tilde\imath}-\alpha'_{\tilde\imath}\alpha_{\tilde\jmath}\neq 0 \quad \text{and} \quad \beta_{\tilde p}\beta'_{\tilde q}-\beta_{\tilde q}\beta'_{\tilde p}\neq 0.$$

From equation (3.1) for these $\tilde\imath,\tilde\jmath,\tilde p,\tilde q$ either $g_1(\alpha,\alpha')=0$ or $g_2(\beta,\beta')=0$. Since the problem is symmetric in α and β, assume without losing generality that $g_1(\alpha,\alpha')=0$. Thus, $g_1(\alpha,\alpha)$ is a non-zero constant. Possibly after multiplying α with a non-zero constant, one has $g_1(\alpha,\alpha)=\varepsilon_\alpha=\pm1$ and α lies in the pseudosphere $\mathbb{S}^{m-1}_\mu=\{x\in\mathbb{E}^m_\mu \mid g_1(x,x)=1\}$ or in the pseudohyperbolic space $\mathbb{H}^{m-1}_{\mu-1}=\{x\in\mathbb{E}^m_\mu \mid g_1(x,x)=-1\}$. Clearly, α and β are non-null since otherwise $g_{11}g_{22}-g_{12}^2=0$. Reparametrize α such that $g_1(\alpha',\alpha')=\varepsilon_{\alpha'}=\pm1$.

Equation (3.2) is rewritten for $\tilde\imath,\tilde\jmath,\tilde p,\tilde q$ as

$$-\varepsilon_\alpha\varepsilon_{\alpha'}\frac{\alpha'_{\tilde\jmath}\alpha''_{\tilde\imath}-\alpha''_{\tilde\jmath}\alpha'_{\tilde\imath}}{\alpha'_{\tilde\jmath}\alpha_{\tilde\imath}-\alpha'_{\tilde\imath}\alpha_{\tilde\jmath}}=\frac{g_2(\beta,\beta)(\beta'_{\tilde q}\beta''_{\tilde p}-\beta'_{\tilde p}\beta''_{\tilde q})}{g_2(\beta',\beta')(\beta'_{\tilde q}\beta_{\tilde p}-\beta'_{\tilde p}\beta_{\tilde q})}.$$

As a consequence,

$$\frac{g_2(\beta,\beta)(\beta'_{\tilde q}\beta''_{\tilde p}-\beta'_{\tilde p}\beta''_{\tilde q})}{g_2(\beta',\beta')(\beta'_{\tilde q}\beta_{\tilde p}-\beta'_{\tilde p}\beta_{\tilde q})}=c,$$

where $c\in\mathbb{R}$. From equation (3.2) with $\tilde p,\tilde q$ and $i,j=1,\dots,m$, one has

$$\alpha'_j\alpha''_i-\alpha''_j\alpha'_i=-\varepsilon_\alpha\varepsilon_{\alpha'}\frac{g_2(\beta,\beta)(\beta'_{\tilde q}\beta''_{\tilde p}-\beta'_{\tilde p}\beta''_{\tilde q})}{g_2(\beta',\beta')(\beta'_{\tilde q}\beta_{\tilde p}-\beta'_{\tilde p}\beta_{\tilde q})}(\alpha'_j\alpha_i-\alpha'_i\alpha_j).$$

Thus, $\alpha'_j\alpha''_i-\alpha''_j\alpha'_i=-\varepsilon_\alpha\varepsilon_{\alpha'}c(\alpha'_j\alpha_i-\alpha'_i\alpha_j)$ with $i,j=1,\dots,m$. Using this, equation (3.2) becomes

$$g_2(\beta,\beta)(\beta'_q\beta''_p-\beta'_p\beta''_q)=cg_2(\beta',\beta')(\beta'_q\beta_p-\beta'_p\beta_q),$$

with $p,q=1,\dots,n$. Summarizing, the minimality conditions (3.1) and (3.2) reduce to

$$\alpha'_j\alpha''_i-\alpha''_j\alpha'_i = -\varepsilon_\alpha\varepsilon_{\alpha'}c(\alpha'_j\alpha_i-\alpha'_i\alpha_j), \tag{3.3}$$

$$g_2(\beta,\beta)(\beta'_q\beta''_p-\beta'_p\beta''_q) = cg_2(\beta',\beta')(\beta'_q\beta_p-\beta'_p\beta_q), \tag{3.4}$$

for every $i, j = 1, \ldots, m$ and $p, q = 1, \ldots, n$. From equation (3.3), it follows that

$$\alpha'' + \varepsilon_\alpha \varepsilon_{\alpha'} c\alpha = \eta\alpha', \tag{3.5}$$

with η a function of s. Thus, α lies in a plane Π_α through the origin. Therefore, α is either a circle in a plane Π_α for which $g_1|_{\Pi_\alpha}$ is definite or α is a hyperbola in a plane Π_α for which $g_1|_{\Pi_\alpha}$ is non-degenerate of index 1 (see [15] p 112-113).

From the derivative of the assumption $g(\alpha, \alpha') = 0$ and (3.5), clearly $c = 1$.

From equation (3.4), one obtains

$$\beta'' = \frac{g_2(\beta', \beta')}{g_2(\beta, \beta)}\,\beta + \gamma\beta',$$

with γ a function of t. Thus, also β lies in a plane Π_β through the origin. Examine now the four possibilities for the plane Π_β. The expressions used for β are valid possibly after applying an appropriate pseudo-orthogonal transformation.

Case $g_2|_{\Pi_\beta}$ is positive definite

One can assume that $\beta(t) = r(t)\cos t\, V_p + r(t)\sin t\, V_q$ for distinct p and q with $p, q \in \{1, \ldots, n-\nu\}$. Equation (3.4) reduces to the differential equation $rr'' - 3r'^2 - 2r^2 = 0$ with solution $r(t) = \frac{b}{\sqrt{|\cos(2t)|}}$.

Case $g_2|_{\Pi_\beta}$ is negative definite

Thus assume that $\beta(t) = r(t)\cos t\, V_p + r(t)\sin t\, V_q$ for distinct p and q with $p, q \in \{n-\nu+1, \ldots, n\}$. Equation (3.4) reduces to the differential equation $rr''-3r'^2-2r^2 = 0$ with solution $r(t) = \frac{b}{\sqrt{|\cos(2t)|}}$.

Case $g_2|_{\Pi_\beta}$ is non-degenerate of index 1

In this case, one can assume that $\beta(t) = r(t)\cosh t\, V_p + r(t)\sinh t\, V_q$ with $p \in \{1, \ldots, n-\nu\}$ and $q \in \{n-\nu+1, \ldots, n\}$. Equation (3.4) reduces to the differential equation $rr'' - 3r'^2 + 2r^2 = 0$ with solutions

$$r(t) = \frac{b}{\sqrt{\cosh(2t)}} \quad \text{and} \quad r(t) = \frac{b}{\sqrt{|\sinh(2t)|}}.$$

Case $g_2|_{\Pi_\beta}$ is degenerate

Assume that $\beta(t) = \beta_1(t)V_p + \beta_2(t)V_q + \beta_1(t)V_r$ for distinct p and q with $p, q \in \{1, \ldots, n-\nu\}$ and $r \in \{n-\nu+1, \ldots, n\}$. Equation (3.4) for β simplifies to

$$\frac{\beta_1''\beta_2' - \beta_1'\beta_2''}{\beta_2'^2} = \frac{\beta_1\beta_2' - \beta_1'\beta_2}{\beta_2^2},$$

with solution $\beta_2(t) = \frac{b}{\beta_1(t)}$.

Case 2 β is a straight line through the origin

First assume β is non-null. Then, possibly after applying an appropriate pseudo-orthogonal transformation, $\beta(t) = tV_i$ with $i \in \{1, \ldots, n\}$. However, this means that f lies in $\mathbb{E}^m_\mu$ and $f(s,t) = t\alpha = t(\alpha_1, \ldots, \alpha_m)$. Consequently,

$$f_s = t\alpha', \qquad f_t = \alpha, \qquad f_{ss} = t\alpha'', \qquad f_{st} = \alpha', \qquad f_{tt} = 0,$$

and the minimality condition of Lemma 2.1 reduces to $g(\alpha'', n) = 0$. Thus, $\alpha'' \in \operatorname{span}\{\alpha, \alpha'\}$ and α is a planar curve.

If β is a null straight line through the origin, then also the position vector of β is null, which is a contradiction.

All parametrizations referred to in the theorem are obtained. Conversely, it can be shown in a straightforward fashion that the tensor product surfaces of the curves in the statement are minimal. □

To conclude, some remarks on this classification theorem of minimal tensor product surfaces of two arbitrary pseudo-Euclidean curves are made.

Remark 3.3 *If the tensor product surface of two arbitrary pseudo-Euclidean curves is minimal, then the two curves are planar.*

Remark 3.4 *There exist no minimal tensor product surfaces of two null curves. Neither do there exist minimal tensor product surfaces of a null curve and an arbitrary pseudo-Euclidean curve.*

Remark 3.5 *The tensor product surface of a straight line through the origin not containing the origin and an arbitrary pseudo-Euclidean curve α is a cone over the curve α. Hence, the surface is minimal if and only if it is a part of a plane. That is, α is a planar curve.*

Remark 3.6 *For the appropriate choices of m, μ, n and ν, the results of [1], [2], [3], [7], [8], [9], [11], [12] and [13] are reconstructed.*

As mentioned in [3], the sinosoidal spiral solutions for the curve β in [1] is incorrect. Similarly, the logaritmic and hyperbolic spiral solutions in the classification of minimal tensor product surfaces in [8] and [9] are incorrect. In the cases where these curves are found, the normal vectors form no basis of the normal space, leading to the incorrect solutions.

The solutions for which one of the curves is a straight line are missing in [8] and [9] and the solution for which the space curve lies in a degenerate plane is missing in [7].

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