On a Theorem of Elie Cartan

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Abstract. We give a new sufficient condition in order that the curvature determines the metric: generically, if two Riemannian manifolds have the same "surjective" (1,3)-curvature tensor fields, then their metrics split into product ones, having the corresponding factors homothetic. The same result holds for some specific pairs of manifolds with indefinite metrics.

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1 Introduction

An important and difficult topic in Global Riemannian Geometry (as a particular case of the so called *equivalence problem*) is the determination of the metric, by imposing (sufficient) restrictions on some geometric invariants, especially curvature-related. The ideas trace back to Riemann ([12]), but the first explicit result belongs to E. Cartan ([4], p.238; [5], p. 157): omitting the details, it asserts that

the (0,4) or the (1,3)- curvature tensor fields, together with the parallel transport, locally determine the Riemannian metric, up to an isometry.

The theorem of Cartan was extended from local to global by Ambrose ([1]).

Nomizu and Yano proved a similar result ([10], [11]): in dimension greater than one, the (1,3)-curvature tensor field, together with its covariant derivatives of any order, determine the Riemanian metric, up to a homothety. This suggests that (the redundant) "(1,3)-curvature + holonomy" determines the metric, up to a homothety.

One may ask if one needs all the holonomy information or it suffices only a part of it. K. Teleman showed ([13]) that the (1,3)-curvature tensor field of an irreducible Riemannian manifold determines the metric, up to a homothety. (Here the "irreducibility" is considered with respect to a subgroup on the holonomy group, spanned only by the curvature operators, without operators provided by the covariant derivatives of the curvature).

Kulkarni ([7], [8]) and Yau ([15]) proved that in dimension greater than 3, the (0,4)-curvature tensor field (or, equivalently, the sectional curvature), completely determines the Riemannian metric, on the set of non-isotropic points. In dimensions 2 and 3, this assertion is not true.

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For more details, see the monograph *A Panoramic View of Riemannian Geometry* ([2]), where Marcel Berger analyses other progress made for extending and generalizing the "Cartan's philosophy". The dominant belief that emerges from this survey is that the problem of the metric determination by the (0,4)-curvature tensor field is basically solved, through the works of Kulkarni and Yau, previously quoted; by contrast, the problem of the metric determination by the (1,3)-curvature tensor field is open, the existing partial results treating only some particular cases. (Given a Riemannian manifold, the information carried by the (0,4)-curvature tensor is equivalent with the information carried by the (1,3)-curvature tensor. It may seem quite strange that the curvature determination of the metric differs so much, in function of the type of the curvature tensor.)

Consider $(M^{(i)}, g^{(i)})$ Riemannian manifolds and ρ^i positive real numbers, with $i = \overline{1,k}$. Then $(M^{(1)} \times ... \times M^{(k)}, g^{(1)} \times ... \times g^{(k)})$ and $(M^{(1)} \times ... \times M^{(k)}, \rho^1 g^{(1)} \times ... \times \rho^k g^{(k)})$ have the same curvature (1,3)-tensor field.So, it is clear that only the (1,3)-curvature tensor does not determine the metric (up to a homothety). Then, *what are the weakest sufficient (additional) conditions in order that the (1,3)-curvature tensor determines the metric, up to a homothety* ?

The purpose of this paper is to prove a local and a global result:

Theorem 1.1. *(the local version) Let M be a connected differentiable manifold and $g, \tilde{g}$ two Riemannian metrics on M. Suppose the (1,3)-curvature tensor fields R and $\tilde{R}$ are surjective and equal. Then:*

(i) locally, there exists a natural number k and a splitting $M^{(1)} \times ... \times M^{(k)}$ of M; there exist k Riemannian metrics $g^{(i)} on M^{(i)}$, $i = \overline{1,k}$; there exist k positive real valued functions ρ^1, ..., ρ^k such that

$$g = g^{(1)} \times ... \times g^{(k)} \quad , \quad \tilde{g} = \rho^1 \, g^{(1)} \times ... \times \rho^k \, g^{(k)}.$$

Moreover, if $n = 2$ or $n = 3$, then $k = 1$.

(ii) if for some j we have $dim M^{(j)} > 3$, or $dim M^{(j)} = 3$ and $Ric^{(j)}$ is non-degenerate, then ρ^j is constant.

(iii) if for some j we have $dim M^{(j)} = 2$, then the function $ln \rho^j$ is harmonic.

Theorem 1.2. *(the global version) Let M be a connected real analytic manifold and $g, \tilde{g}$ two complete Riemannian metrics on M. Suppose the (1,3)-curvature tensor fields R and $\tilde{R}$ are surjective and equal. Then:*

(i) there exists a natural number k and a splitting $M^{(1)} \times ... \times M^{(k)}$ of M; there exist k Riemannian metrics $g^{(i)} on M^{(i)}$, $i = \overline{1,k}$; there exist k positive real valued functions ρ^1, ..., ρ^k such that

$$g = g^{(1)} \times ... \times g^{(k)} \quad , \quad \tilde{g} = \rho^1 \, g^{(1)} \times ... \times \rho^k \, g^{(k)};$$

(ii) if for some j we have $dim M^{(j)} > 3$, or $dim M^{(j)} = 3$ and $Ric^{(j)}$ is non-degenerate, then $\rho^j = 1$.

(iii) if for all $i = \overline{1,k}$ we have $dim M^{(i)} > 3$, or $dim M^{(i)} = 3$ and $Ric^{(i)}$ is non-degenerate, then $g = \tilde{g}$.

2 Proof of Theorem 1.1

(i) Consider on M the local coordinates $(x^1,...,x^n)$ and an orthogonal system of congruences such that the line elements of g and $\tilde{g}$ write $ds^2 = (ds^1)^2 + ... + (ds^n)^2$ and $d\tilde{s}^2 = \theta_1^2(ds^1)^2 + ... + \theta_n^2(ds^n)^2$, respectively. Here, $\theta_1^2, \ ... \ , \theta_n^2$ are the eigenvalues of $\tilde{g}$ with respect to g; we suppose that all of them are positive. We use the local transformations from [14]:

$$ds^a = \lambda_i^a dx^i \ , \ d\tilde{s}^a = \tilde{\lambda}_i^a dx^i \ , \ dx^i = \mu_a^i ds^a \ , \ dx^i = \tilde{\mu}_a^i d\tilde{s}^a \ , \ \tilde{\lambda}_i^a = \theta_a \lambda_i^a \ , \ \mu_a^i = \theta_a \tilde{\mu}_a^i.$$

Denote by γ_{bcd}^a and $\tilde{\gamma}_{bcd}^a$ the curvature coefficients with respect to the system of congruences. We have

$$\gamma_{bcd}^a = R_{jkl}^i \lambda_i^a \mu_b^j \mu_c^k \mu_d^l.$$

Due to the hypothesis $\tilde{R}_{jkl}^i = R_{jkl}^i$, we obtain ([14])

$$\theta_a^{-1}\theta_b\theta_c\theta_d\tilde{\mu}_{bcd}^a = \gamma_{bcd}^a.$$

By permuting the indices a and b, then using the skew-symmetry of γ_{bcd}^a in a and b, we derive

$$(2) \qquad (\theta_a^2 - \theta_b^2)\gamma_{bcd}^a = 0 \quad , \quad (\theta_a^2 - \theta_b^2)\tilde{\gamma}_{bcd}^a = 0$$

for any indices $a, b, c, d = \overline{1,n}$. Due to the surjectivity of R, if $n = 2$ or $n = 3$, it follows that all the eigenvalues θ_i coincide.

Consider $n > 3$. In a fixed point, let $a = \overline{1,n}$; then, there exist b, c, d such that $\gamma_{bcd}^a \neq 0$. It follows that there exists a positive integer $k \geq 1$ such that, locally, modulo an eventual renumbering, we have $\theta_1 = ... = \theta_{i_1}$, ..., $\theta_{i_{k-1}+1} = ... = \theta_n$. If $k = 1$, then the metrics g and $\tilde{g}$ are locally conformal. Without restraining the generality, we suppose in what follows that $k = 2$. (The general case is very similar, with only additional significance-less details). So, we have

$$(3) \qquad \theta_1 = ... = \theta_m = C_1 \quad , \quad \theta_{m+1} = ... = \theta_n = C_2 \quad , \quad C_1 \neq C_2.$$

We denote by $a, b, c, ... = \overline{1,n}$; $i, j, k, ... = \overline{1,m}$; $\alpha, \beta, \delta, ... = \overline{m+1,n}$. From (2), (3) and the first Bianchi identity, it results $\gamma_{\alpha ab}^i = 0$, $\gamma_{a\alpha b}^i = 0$. We use the second Bianchi identity and obtain

$$\gamma_{jkl}^i \gamma_{\alpha\beta}^j = 0 \quad , \quad \gamma_{\beta\delta c}^\alpha \gamma_{ij}^\beta = 0,$$

where

$$\gamma_{bc}^a := \frac{\partial \lambda_d^a}{\partial x^e} - |_{de}^{f}| \ \lambda_f^a)\mu_b^d \mu_c^e.$$

A short calculation gives $\gamma_{\alpha\beta}^i = 0$ and $\lambda_{ij}^\alpha = 0$. The Pfaffian systems $\lambda_a^i dx^a = 0$ and $\lambda_a^\alpha dx^a = 0$ are then completely integrable. So, $\lambda_\alpha^i = 0$ and $\lambda_i^\alpha = 0$; thus $g_{\alpha i} = \tilde{g}_{\alpha i} = 0$. On another hand, $\frac{\partial g_{ij}}{\partial x^\alpha} = 0$ and $\frac{\partial g_{\alpha\beta}}{\partial x^i} = 0$. Similar relations are obtained for the coefficients of $\tilde{g}$.

In conclusion, locally, we get $\tilde{g}_{ij} = C_1^2 g_{ij}$ and $\tilde{g}_{\alpha\beta} = C_2^2 g_{\alpha\beta}$, where C_1 and C_2 depend only on $x^1, ..., x^m$ and respectiely on $x^{m+1}, ..., x^n$.

(ii) is a direct consequence of the

Lemma 2.1. *([9]) Let M be a connex n-differentiable manifold ($n > 3$) and g, $\tilde{g}$ two conformal Riemannian metrics on M. Suppose the (1,3)-curvature tensor fields R and $\tilde{R}$ are surjective and equal. Then g and $\tilde{g}$ are homothetic.*

In dimension 3, the surjectivity of the (1,3)-curvature tensor does not assure anymore the homothety of the metrics. We need the stronger hypothesis of Ricci non-degeneracy (which implies the surjectivity of the (1,3)-curvature tensor):

Lemma 2.2. *Let M be a connected 3-dimensional manifold and g, $\tilde{g}$ two conformal Riemannian metrics on M. Suppose $R = \tilde{R}$ and Ric is non-degenerated. Then g and $\tilde{g}$ are homothetic.*

Proof. Suppose $\tilde{g} = e^{2f}g$, with $f \in \mathcal{F}(\mathcal{M})$.

In dimension 3, the (1,3)-curvature tensor has the form

$$R(Z,W)Y = Ric(Y,W)Z - Ric(Y,Z)W + g(Y,W)ricZ-$$

$$-g(Y,Z)ricW - \frac{s}{2}g(Y,W)Z + \frac{s}{2}g(Y,Z)W$$

for any vector fields Y, Z, W. We denoted by ric the associated Ricci tensor of type (1,1) and by s the scalar curvature. From the equality $R = \tilde{R}$, we derive $Ric(U,X)Y = Ric(U,Y)X$, where $U = gradf$. By contracting and by using the non-degeneracy of Ric, we obtain $U = 0$. The conformal factor e^{2f} is a constant, hence the metrics are homothetic. □

(iii) We use the

Lemma 2.3. *Let M be a connected 2-dimensional manifold, f a real valued differentiable function on M and $\tilde{g} = e^{2f}g$. Then $R = \tilde{R}$ if and only if f is harmonic.*

Proof. From [3], pag.58 we know that $\tilde{R} = e^{2f}\{R - g \bigotimes (\nabla df - df \bigotimes df + \frac{1}{2} \mid\mid df \mid\mid^2 g)\}$, where the curvature tensors are of type (0,4) and $\bigotimes$ is the Kulkarni-Nomizu product. From the equality of the (1,3)-curvature tensors, we derive

$$\{(\nabla_X df)Z - df(X)df(Z) + \frac{1}{2} \mid\mid df \mid\mid^2 g(X,Z)\}Y - \{(\nabla_Y df)Z - df(Y)df(Z)+$$

$$+\frac{1}{2} \mid\mid df \mid\mid^2 g(Y,Z)\}X + g(X,Z)\{\nabla_V gradf - df(Y)gradf + \frac{1}{2} \mid\mid df \mid\mid^2 Y\}-$$

$$-g(Y,Z)\{\nabla_X gradf - df(X)gradf + \frac{1}{2} \mid\mid df \mid\mid^2 X\} = 0$$

for any vector fields X, Y, Z. We contract Y and obtain $\Delta f = 0$, where the Laplacian is determined by using g.

For the converse statement, suppose $\Delta f = 0$ and calculate the two curvature tensors in isothermal coordinates. □

3 Proof of Theorem 1.2

By Theorem 1.1,(i) the Riemannian manifold M splits locally as $M^{(1)} \times ... \times M^{(k)}$. The real analiticity allows to prolong this relation globally. As (M, g) is complete, it follows that each factor $(M^{(i)}, g^{(i)})$ is complete. Then (i) is proved.

Consider now an index j as in the hypothesis of (ii); then, by Theorem 1.1, (ii), one obtains that g is homothetic with $\tilde{g}$ on $M^{(j)}$.

We need now the following lemma:

Lemma 3.1. *([6],vol.I, p.242). If N is a complete Riemannian manifold which is not locally Euclidean, then every homothetic transformation of N is an isometry.*

Since $(M^{(j)}, g^{(j)})$ is not locally Euclidean (due to the surjectivity of the curvature tensor field $R^{(j)}$), it follows that g is isometric with $\tilde{g}$ on $M^{(j)}$. The assertion (ii) is thus proved.

The conclusion (iii) is a direct consequence of (ii), when applied for every index $i = \overline{1, k}$.

4 Comments

(i) Obviously, a curvature tensor R is surjective iff, for every point $p \in M$, the tensor R_p is surjective. As examples of Riemannian manifolds with surjective curvature, we have:

- every surface with nowhere vanishing Gauss curvature;
- every non-flat constant sectional curvature manifold;
- products of manifolds with surjective curvature;
- non-flat invariant submanifolds (in particular, totally geodesic ones) of Riemannian manifolds with surjective curvature.

(ii) For the assertions (ii) and (iii) of the Theorem 1.1, the "surjective" curvature hypothesis is only sufficient. It is not necessary, as may be observed from simple product metrics (for example, on $\mathbf{R} \times S^1$, consider $dt^2 + d\sigma^2$ and $dt^2 + f^2\, d\sigma^2$, with f a non-negative real valued function on S^1).

(iii) The hypothesis concerning the dimension cannot be much improved in Theorem 1.1,(ii), (iii). Indeed, on 2-dimensional manifolds, the surjective curvature property do not imply that conformal metrics are necessarly homothetic. In what concerns the dimension 3, Lemma 1 cannot be applied; we don't know if a counterexample may be constructed, as in [15], or if Theorem 1.1 (ii), (iii) may be proved, by a different argument.

(iv) We may "explain" Theorem 1.1 by the following heuristic and speculative remarks: due to the de Rham decomposition theorem ([6]), the Riemannian manifolds (M, g) and $(M, \tilde{g})$ were known to split into irreductible factors, with respect to the holonomy groups.As $R = \tilde{R}$, the holonomy groups are "close" (even not identical), which determines "some" (local) decompositions of M to coincide (the decomposition in Theorem 1.1 is not necessarly the same as de Rham decomposition). The fact that, on each factor, the metrics g and $\tilde{g}$ are conformal is, perhaps, a reminiscence of the surfaces behaviour (via an "isothermal-like transformation"), induced by the fact that their curvature is the same.

The assertion (ii) of Theorem 1.1 has an intuitive "justification": on each factor of the (local) decomposition of M, the metrics g and $\tilde{g}$ are conformal; the surjectivity of the curvature tensors determines a kind of reduction (similar to Liouville's theorem [5],p.170), and conformal transformations reduce to homotheties.

(v) In [11] is proved the following result: *Let* (M, g) *be a Riemannian manifold with* $dim M \geq 3$. *If a conformal change* $g \to \rho^2 g$ *preserves the (1,3)-curvature* R *and* ∇R, *then* $R = 0$ *or the function* ρ *is constant.* The dichotomy "curvature flatness" versus "homothety" anticipates the assertion of Theorem 1.1; however, it is not completely clear how may be filled the gap between curvature flatness and the non-surjectivity of the (1,3)-curvature tensor field.

(vi) The next step on this topic is to investigate a possible generalization of Theorems 1.1 in the semi-Riemannian setting. For example, the respective result remains true if we replace the hypothesis "Riemannian manifolds" by "semi-Riemannian manifolds, of the same index and with the metrics (locally) simultaneously diagonalizable". The additional assumptions are necessary, as may be seen by considering the canonical metrics on the 2-dimensional sphere and on two pseudo-spheres, with different time axis (they have the same curvature tensor, but cannot be homothetic).

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