Multitime control strategies for skilled movements

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Abstract. The paper presents a two-time motor control strategies for skilled movements. There are found movements which are optimum with various "costs", given by double integrals and different PDEs constraints (Newton Law as first order PDEs, multitime hyperbolic-parabolic Newton Law, multitime elliptic Newton Law). For simplicity, the movements, the constraints and the costs depend upon two independent variables.

The model-based investigation of human and human-like motions is an important interdisciplinary research topic which involves aspects of biomechanics, physiology, orthopedics, psychology, neurosciences, robotics, sport, computer graphics and applied mathematics. In this context, the detailed study on a joint level of basic locomotion forms such as two-time walking and running is of particular interest due to the high demand on dynamic coordination, actuator efficiency and balance control. Two-time mathematical models can help to better understand the basic underlying mechanisms of these motions and to improve them.

In this paper, we present the mathematical point of view of our research group on dynamic human motions which show how optimization can help to generate very natural two-time looking motions.

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1 Classical investigation of human and human-like motions

Skill acquisition (ability, talent to do something) involves learning to execute movements with the minimum effort to achieve predetermined effects. It is a complex process demanding high levels of sensory perception, integration within the central

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nervous system, and coordination of different muscle groups. There are many different kinds of skill ranging from *fine motor skills*, requiring delicate muscular control (used in activities such as putting and rifle shooting), to *gross motor skills*, requiring coordination of many muscle groups (used in activities such as running). The skills can be classified as: (i) *open skills*, performed in an unpredictable situation (such as a football match, basket match, etc), with outside factors dictating how and when the skill is performed; (ii) *closed skills*, involving movements which can be planned in advance and usually performed in a stable, mainly predictable situation; examples include performing a handstand, serving in tennis, teeing off at golf, and diving from a platform.

The classical model-based investigation of human and human-like motions are based on the movement described by a single-time controlled Newton Law written as a second order ODE

$$\ddot{x}(t) = -b(t)\dot{x}(t) + u(t),\ t \in [0,T] \subset \mathbb{R}_+,\ x(0) = 0, x(T) = D, \tag{1.1}$$

or as a first order ODE system

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -b(t) \end{pmatrix}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),\ \ t \in [0,T] \subset \mathbb{R}_+, \tag{1.2}$$

$$x(0) = 0, x(T) = D, v(0) = 0, v(T) = 0,$$

where $x(t)$ is the *state variable*, $v(t)$ is *velocity variable*, $b(t)$ is a given function, and $u(t)$ is the *control*.

Our aims refer to the introduction of multitime evolution PDEs (Newton Law) and recovery of the single-time equation solution. The original results include: Newton Law as first order PDEs, two-time hyperbolic-parabolic Newton Law approach, two-time elliptic Newton Law approach, each being accompanied by original optimal control problems and bang-bang optimal controls, useful for two-time skilled movements.

2 Newton Law as first order PDEs

In the warped multitime PDE (WaMPDE) approach via first order PDEs approach [1-4], the variations of the state variables are decomposed in several time dimensions. For example, the ODE system (1.1) is transformed into the following first order PDE system

$$\omega(t^2)\frac{\partial}{\partial t^1}\begin{pmatrix} \hat{x} \\ \hat{v} \end{pmatrix}(t^1,t^2) + \frac{\partial}{\partial t^2}\begin{pmatrix} \hat{x} \\ \hat{v} \end{pmatrix}(t^1,t^2)$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -\hat{b}(t^1,t^2) \end{pmatrix}\begin{pmatrix} \hat{x}(t^1,t^2) \\ \hat{v}(t^1,t^2) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{u}(t^1,t^2),\ \ (t^1,t^2) \in \Omega \subset \mathbb{R}^2_+, \tag{2.1}$$

with the boundary conditions

$$\hat{x}(0,0) = 0\,,\ \ \hat{x}(T^1,T^2) = D\,,$$

$$\hat{v}_1(0,0) = \hat{v}_2(0,0) = \hat{v}_1(T^1,T^2) = \hat{v}_2(T^1,T^2) = 0.$$

We shall denote the two-time (t^1, t^2) by $t = (t^\alpha) = (t^1, t^2)$ (a bi-parameter of the evolution).

This achieves a symbolic separation of the (typically slow) rates of frequency modulation (FM) and amplitude modulation (AM) from the (much faster) oscillation rate. The resulting formulation is a multi-time partial-differential equation in warped and unwarped time scales, together with a mapping between multi-time and single-time functions.

Theorem 2.1. *If $\hat{x}(t^1, t^2), \hat{v}(t^1, t^2)$ is a solution of the first order PDE system* (2.1) *and $b(t) = \hat{b}(\phi(t), t)$, $u(t) = \hat{u}(\phi(t), t)$, where $\phi(t) = \int_0^t \omega(\tau) d\tau$, then $x(t) = \hat{x}(\phi(t), t)$, $v(t) = \hat{v}(\phi(t), t)$ solves the first order ODE system* (1.2).

Proof. By computation, we obtain

$$\dot{x}(t) = \frac{\partial \hat{x}}{\partial t^1}(\phi(t), t)\omega(t) + \frac{\partial \hat{x}}{\partial t^2}(\phi(t), t) = \hat{v}(\phi(t), t) = v(t);$$

$$\dot{v}(t) = \frac{\partial \hat{v}}{\partial t^1}(\phi(t), t)\omega(t) + \frac{\partial \hat{v}}{\partial t^2}(\phi(t), t) = -\hat{b}(\phi(t), t)\hat{v}(\phi(t), t) + \hat{u}(\phi(t), t)$$
$$= -b(t)v(t) + u(t).$$ □

Generalization. Let $\hat{h}(t) = (\hat{h}^1(t), \hat{h}^2(t))$, $t \in \Omega$ be a suitable direction at each point. We can extend the ODE system (1.1) to the first order PDE system

$$\hat{h}^\alpha(t)\frac{\partial \hat{x}}{\partial t^\alpha}(t) = \hat{v}(t),\ \hat{h}^\alpha(t)\frac{\partial \hat{v}}{\partial t^\alpha}(t) = -\hat{b}(t)\hat{v}(t) + \hat{u}(t).$$

From a solution $\hat{x}(t^1, t^2), \hat{v}(t^1, t^2)$ of this first order PDE system, we recover the solution of the ODE system (1.2) setting $x(t) = \hat{x}(\phi(t), \psi(t)), v(t) = \hat{v}(\phi(t), \psi(t))$. In fact we use a curve $t^1 = \phi(t), t^2 = \psi(t)$, where $\dot{\phi}(t) = \hat{h}^1(\phi(t), \psi(t)), \dot{\psi}(t) = \hat{h}^2(\phi(t), \psi(t))$. Particularly, for $x(t) = \hat{x}(t, t), v(t) = \hat{v}(t, t)$, the vector h must give a partition of the unity, i.e., $\hat{h}^1(t) + \hat{h}^2(t) = 1$.

2.1 Optimal control problem

Let us consider a two-time optimal control problem with a double integral cost functional

$$\min_u I(u(\cdot)) = \iint_\Omega L(\hat{x}(t), \hat{v}(t), \hat{u}(t)) d\omega, \quad d\omega = dt^1 \wedge dt^2,$$

constrained by the PDE (1.2)+(2.1), i.e.,

$$\omega(t^2)\frac{\partial \hat{x}}{\partial t^1}(t^1, t^2) + \frac{\partial \hat{x}}{\partial t^2}(t^1, t^2) = \hat{v}(t^1, t^2)$$

$$\omega(t^2)\frac{\partial \hat{v}}{\partial t^1}(t^1, t^2) + \frac{\partial \hat{v}}{\partial t^2}(t^1, t^2) = -b(t^1, t^2)\hat{v}(t^1, t^2) + \hat{u}(t^1, t^2),$$

$$\hat{x}(0, 0) = 0\,,\ \ \hat{x}(T^1, T^2) = D\,,$$

$$\hat{v}_1(0, 0) = \hat{v}_2(0, 0) = \hat{v}_1(T^1, T^2) = \hat{v}_2(T^1, T^2) = 0,$$

where $t = (t^\alpha) = (t^1, t^2) \in \Omega \subset \mathbb{R}^2_+$ is the two-time (for multitime optimal control, see also [5-20]).

To solve this problem, we use the Lagrangian

$$L_1 = L(\hat{x}(t), \hat{v}(t), \hat{u}(t)) + p(t)\left(\omega(t^2)\frac{\partial \hat{x}}{\partial t^1}(t) + \frac{\partial \hat{x}}{\partial t^2}(t) - \hat{v}(t)\right)$$
$$+q(t)\left(\omega(t^2)\frac{\partial \hat{v}}{\partial t^1}(t) + \frac{\partial \hat{v}}{\partial t^2}(t) + b(t)\hat{v}(t) - \hat{u}(t)\right).$$

Theorem 2.2. *Suppose that the problem of minimizing the functional $I(u(\cdot))$ constrained by the PDEs (1.2) and the conditions (2.1), with C^1 functions $\omega(t^2)$, $b(t)$, has an interior solution $\hat{u}(t) \in U$ which generates a 2-sheet state variable $\hat{x}(t)$. Then there exists a C^1 costate vector, $(p(t), q(t))$ such that*

(i) adjoint PDEs: $\frac{\partial L}{\partial \hat{x}} - \omega \frac{\partial p}{\partial t^1} - \frac{\partial p}{\partial t^2} = 0,\ \frac{\partial L}{\partial \hat{v}} - \omega \frac{\partial q}{\partial t^1} - \frac{\partial q}{\partial t^2} - p + bq = 0;$

(ii) constraint PDEs: $\frac{\partial L}{\partial p} = 0,\ \frac{\partial L}{\partial q} = 0;$

(iii) critical point condition: $\frac{\partial L_1}{\partial \hat{u}} + q = 0$

are satisfied.

2.2 Bang-bang optimal control

Let us consider an optimal control problem of type *minimal two-time area*. By application of the two-time maximum principle, we obtain necessary conditions for optimality and use them to guess a candidate control policy.

Theorem 2.3. *If we consider $L = -1$, then the optimal control is a bang-bang control.*

Proof. The control Lagrangian becomes

$$L_1 = -1 + p(t)\left(\omega(t^2)\frac{\partial \hat{x}}{\partial t^1}(t) + \frac{\partial \hat{x}}{\partial t^2}(t) - \hat{v}(t)\right)$$
$$+q(t)\left(\omega(t^2)\frac{\partial \hat{v}}{\partial t^1}(t) + \frac{\partial \hat{v}}{\partial t^2}(t) + b(t)\hat{v}(t)\right) - q(t)\hat{u}(t).$$

Let $[-U, U] \subset \mathbb{R}$ be the control set. The adjoint equations are

$$\omega\frac{\partial p}{\partial t^1} + \frac{\partial p}{\partial t^2} = 0,\ \omega\frac{\partial q}{\partial t^1} + \frac{\partial q}{\partial t^2} + p - bq = 0.$$

The maximum of the linear Lagrangian function $u \to L_1$ exists since the control variable belongs to the interval $[-U, U]$; for optimum, the control must be $\hat{u} = U$ or $\hat{u} = -U$ (see linear optimization, simplex method). The optimal control $\hat{u}$ must be the function $\hat{u}(t) = U \operatorname{sgn}(-q(t))$. Consequently the optimal Lagrangian is

$$L_1^* = -1 + p(t)\left(\omega(t^2)\frac{\partial \hat{x}}{\partial t^1}(t) + \frac{\partial \hat{x}}{\partial t^2}(t) - \hat{v}(t)\right)$$
$$+q(t)\left(\omega(t^2)\frac{\partial \hat{v}}{\partial t^1}(t) + \frac{\partial \hat{v}}{\partial t^2}(t) + b(t)\hat{v}(t)\right) + |q(t)|U.$$

The optimal evolution first order PDEs follows automatically. □

3 Multitime hyperbolic-parabolic Newton Law approach

Our second idea is to transform the ODE system (1.1) (*single-time Newton Law*) into a hyperbolic-parabolic PDE system (*two-time hyperbolic-parabolic Newton Law*)

$$\frac{\partial x}{\partial t^\alpha}(t) = v_\alpha(t), \quad \frac{\partial v_\alpha}{\partial t^\beta}(t) = u_{\alpha\beta}(t) - b^\gamma_{\alpha\beta}(t) v_\gamma(t), \ t \in \Omega \subset \mathbb{R}^2_+, \tag{3.1}$$

with $\alpha, \beta, \gamma = 1, 2$ and the boundary conditions

$$\hat{x}(0,0) = 0, \ \hat{x}(T^1, T^2) = D,$$
$$\hat{v}_1(0,0) = \hat{v}_2(0,0) = \hat{v}_1(T^1,T^2) = \hat{v}_2(T^1,T^2) = 0.$$

This law is based on the remark that a change of variable, realized by the decomposition of single-time as sum of two-times, leads the second order differential equation into a system of hyperbolic partial differential equations of second order (and conversely).

Let us use the unknown function

$$y : \mathbb{R}^2 \to \mathbb{R}^3, \quad (t^1, t^2) \xrightarrow{y} (x, v_1, v_2), \quad y = \begin{pmatrix} x \\ v_1 \\ v_2 \end{pmatrix},$$

which transform our PDEs into some more maneuverable. Then, the PDEs (3.1) of the initial problem become

$$\frac{\partial y}{\partial t^\beta}(t) = \frac{\partial}{\partial t^\beta}\begin{pmatrix} x \\ v_\alpha \end{pmatrix}(t) = \begin{pmatrix} \dfrac{\partial x}{\partial t^\beta} \\ \dfrac{\partial v_\alpha}{\partial t^\beta} \end{pmatrix}(t)$$

$$= \begin{pmatrix} v_\beta(t) \\ u_{\alpha\beta}(t) - b^\gamma_{\alpha\beta}(t) v_\gamma(t) \end{pmatrix} = \begin{pmatrix} 0 \\ u_{\alpha\beta}(t) \end{pmatrix} + \begin{pmatrix} v_\beta(t) \\ -b^\gamma_{\alpha\beta}(t) v_\gamma(t) \end{pmatrix}.$$

These can be written explicitly using the variables (x, v_1, v_2). It appears

$$\frac{\partial y}{\partial t^1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -b_{11} & -b_{12} \\ 0 & -b_{21} & -b_{22} \end{pmatrix}\begin{pmatrix} x \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u_{11} \\ u_{12} \end{pmatrix},$$

splits as

$$\frac{\partial x}{\partial t^1}(t^1,t^2) = v_1(t^1,t^2)$$
$$\frac{\partial v_1}{\partial t^1}(t^1,t^2) = -b_{11}(t^1,t^2)v_1(t^1,t^2) - b_{12}(t^1,t^2)v_2(t^1,t^2) + u_{11}(t^1,t^2)$$
$$\frac{\partial v_2}{\partial t^1}(t^1,t^2) = -b_{21}(t^1,t^2)v_1(t^1,t^2) - b_{22}(t^1,t^2)v_2(t^1,t^2) + u_{12}(t^1,t^2)$$

and

$$\frac{\partial y}{\partial t^2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -c_{11} & -c_{12} \\ 0 & -c_{21} & -c_{22} \end{pmatrix} \begin{pmatrix} x \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u_{21} \\ u_{22} \end{pmatrix}$$

written explicitly

$$\frac{\partial x}{\partial t^2}(t^1, t^2) = v_2(t^1, t^2)$$
$$\frac{\partial v_1}{\partial t^2}(t^1, t^2) = -c_{11}(t^1, t^2)v_1(t^1, t^2) - c_{12}(t^1, t^2)v_2(t^1, t^2) + u_{21}(t^1, t^2)$$
$$\frac{\partial v_2}{\partial t^2}(t^1, t^2) = -c_{21}(t^1, t^2)v_1(t^1, t^2) - c_{22}(t^1, t^2)v_2(t^1, t^2) + u_{22}(t^1, t^2).$$

Generally, a pair of linear PDE systems, homogeneous and non-homogeneous,

$$\frac{\partial y}{\partial t^\alpha}(t) = M_\alpha(t)y(t), \ \ \frac{\partial y}{\partial t^\alpha}(t) = M_\alpha(t)y(t) + F_\alpha(t), \ t \in R^m, \ \alpha = 1, 2$$

are simultaneously completely integrable PDEs systems if and only if (see, [6]-[9])

$$\frac{\partial M_\alpha}{\partial t^\beta}(t) + M_\alpha(t)M_\beta(t) = \frac{\partial M_\beta}{\partial t^\alpha}(t) + M_\beta(t)M_\alpha(t)$$

$$M_\alpha(t)F_\beta(t) + \frac{\partial F_\alpha}{\partial t^\beta}(t) = M_\beta(t)F_\alpha(t) + \frac{\partial F_\beta}{\partial t^\alpha}(t).$$

In these conditions, the solution of the Cauchy problem

$$\frac{\partial y}{\partial t^\alpha}(t) = M_\alpha(t)y(t) + F_\alpha(t), \ \ x(t_0) = x_0, \ t \in R^m$$

is given by the *variation of parameters formula*

$$y(t) = \mathcal{X}(t, t_0)y_0 + \int_{\gamma_{t_0 t}} \mathcal{X}(t, s)F_\alpha(s)ds^\alpha,$$

where $\mathcal{X}(t, t_0)$ is the fundamental matrix associated to the homogeneous PDE and $\gamma_{t_0 t}$ is an arbitrary piecewise C^1 curve. If M_α are constant matrices, then $\mathcal{X}(t, t_0) = \exp(M_\alpha(t^\alpha - t_0^\alpha))$.

From a two-time solution of the hyperbolic-parabolic problem, we can recover the solution of the second order ODE using $\hat{x}(t) = \hat{x}(t^1, t^2) = \phi(\frac{t^1+t^2}{\sqrt{2}})$. Indeed

$$\hat{x}_{t^1} = \frac{1}{\sqrt{2}}\dot{\phi}, \ \hat{x}_{t^2} = \frac{1}{\sqrt{2}}\dot{\phi}$$

$$\hat{x}_{t^1 t^1} = \frac{1}{2}\ddot{\phi}, \ \hat{x}_{t^2 t^2} = \frac{1}{2}\ddot{\phi}, \ \hat{x}_{t^1 t^2} = \frac{1}{2}\ddot{\phi}$$

and replacing in the second order PDEs we find the second order ODE in the unknown ϕ.

3.1 Optimal control problem

Let $t = (t^\alpha) = (t^1, t^2) \in \Omega \subset \mathbb{R}^2_+$ be the two-time (the bi-parameter of the evolution), and the coordinates x^i given by $x^1 = x$, $x^2 = v_1$, $x^3 = v_2$. We introduce also two vector fields X^i_α defined by

$$\frac{\partial x}{\partial t^1}(t^1,t^2) = v_1(t^1,t^2) = X^1_1$$
$$\frac{\partial v_1}{\partial t^1}(t^1,t^2) = -b_{11}(t^1,t^2)v_1(t^1,t^2) - b_{12}(t^1,t^2)v_2(t^1,t^2) + u_{11}(t^1,t^2) = X^2_1$$
$$\frac{\partial v_2}{\partial t^1}(t^1,t^2) = -b_{21}(t^1,t^2)v_1(t^1,t^2) - b_{22}(t^1,t^2)v_2(t^1,t^2) + u_{12}(t^1,t^2) = X^3_1;$$

and

$$\frac{\partial x}{\partial t^2}(t^1,t^2) = v_2(t^1,t^2) = X^1_2$$
$$\frac{\partial v_1}{\partial t^2}(t^1,t^2) = -c_{11}(t^1,t^2)v_1(t^1,t^2) - c_{12}(t^1,t^2)v_2(t^1,t^2) + u_{21}(t^1,t^2) = X^2_2$$
$$\frac{\partial v_2}{\partial t^2}(t^1,t^2) = -c_{21}(t^1,t^2)v_1(t^1,t^2) - c_{22}(t^1,t^2)v_2(t^1,t^2) + u_{22}(t^1,t^2) = X^3_2.$$

Now, let us consider the multitime optimal control problem with a double integral cost functional

$$\min_u \; I(u(\cdot)) = \iint_\Omega L(x(t), v(t), u(t))d\omega, \quad d\omega = dt^1 \wedge dt^2,$$

constrained by

$$\frac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t). \tag{3.2}$$

To solve this problem, we can use the multitime maximum principle (for multitime optimal control, see also [5-20]) based on the control Hamiltonian

$$H = -L + p^\alpha_i X^i_\alpha, \;\; \alpha = 1,2; \;\; i = 1,2,3$$

and its anti-trace

$$H^\alpha_\beta(x,p,u) = -\frac{1}{m}\, L\, \delta^\alpha_\beta + p^\alpha_i X^i_\beta,$$

called the *control Hamiltonian tensor field.*

Theorem 3.1. (strong multitime maximum principle) *Suppose that the problem of minimizing the functional $I(u(\cdot))$ constrained by the first order PDEs* (3.2), *with C^1 functions X^i_α, has an interior solution $\hat{u}(t) = (\hat{u}_{\alpha\beta}) \in U$ which generates a 2-sheet state variable $y(t)$. Then there exists a C^1 costate matrix $p(t) = (p^\alpha_i(t))$ such that we have*

$$\frac{\partial p^\alpha_i}{\partial t^\beta}(t) = -\frac{\partial H^\alpha_\beta}{\partial x^i}(x(t), v(t), u(t), p(t)), \qquad (\textit{adjoint PDEs})$$

$$\frac{\partial x^i}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p^\alpha_i}(x(t), v(t), u(t), p(t)), \qquad (\textit{initial PDEs})$$

and

$$\frac{\partial H}{\partial \hat{u}_{\alpha\beta}}(x(t), v(t), u(t), p(t)) = 0. \qquad (\textit{critical point conditions})$$

Also, the function $t \to H(x^*(t), v^*(t), u^*(t), p^*(t))$ *is constant.*

3.2 Bang-bang optimal control

Let us consider an optimal control problem of type *minimal two-time area*. By application of the strong two-time maximum principle, we obtain necessary conditions for optimality and use them to guess a candidate control policy.

Theorem 3.2. *If we consider* $L = -1$*, then the optimal control is a bang-bang control.*

Proof. The control Hamiltonian becomes

$$\begin{aligned} H &= -1 + p_i^\alpha X_\alpha^i \\ &= -1 + p_1^1 v_1 + p_2^1(-b_{11}v_1 - b_{12}v_2 + u_{11}) \\ &+ p_3^1(-b_{21}v_1 - b_{22}v_2 + u_{21}) + p_1^2 v_2 \\ &+ p_2^2(-c_{11}v_1 - c_{12}v_2 + u_{12}) + p_3^2(-c_{21}v_1 - c_{22}v_2 + u_{22}). \end{aligned}$$

Observe that the Hamiltonian H is linear in the control u and has no critical point. Hence, the extremum points of H lie on the boundary of the admissible set for u. So, we have a bang-bang control

$$u_{11} = U \operatorname{sgn} p_2^1,\ u_{21} = U \operatorname{sgn} p_3^1,$$

$$u_{12} = U \operatorname{sgn} p_2^2,\ u_{22} = U \operatorname{sgn} p_3^2,$$

where

$$|u_{\alpha\beta}| \le U = \frac{F_{\max}}{m}$$

(we assume that there is a limit, $F_{\max}$, on the magnitude of applied force).

Since the control Hamiltonian tensor field is

$$H_\beta^\alpha = -1 + p_i^\alpha X_\beta^i$$

the adjoint PDEs are

$$\frac{\partial p_i^\alpha}{\partial t^\beta}(t) = -p_j^\alpha \frac{\partial X_\beta^j}{\partial x^i}(x(t), v(t), u(t), p(t)).$$

Explicitly,

$$\frac{\partial p_1^1}{\partial t^1} = 0,\ \frac{\partial p_2^1}{\partial t^1} = -p_1^1 + b_{11}p_2^1 + b_{21}p_3^1,\ \frac{\partial p_3^1}{\partial t^1} = b_{12}p_2^1 + b_{22}p_3^1,$$

$$\frac{\partial p_1^1}{\partial t^2} = 0,\ \frac{\partial p_2^1}{\partial t^2} = c_{11}p_2^1 + c_{21}p_3^1,\ \frac{\partial p_3^1}{\partial t^2} = -p_1^1 + c_{12}p_2^1 + c_{22}p_3^1.$$

Since $p^1_1 = c^1_1$, we have in fact a nonhomogeneous linear PDE system of the form

$$\frac{\partial}{\partial t^1}\begin{pmatrix} p^1_2 \\ p^1_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}\begin{pmatrix} p^1_2 \\ p^1_3 \end{pmatrix} + c^1_1\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\frac{\partial}{\partial t^2}\begin{pmatrix} p^1_2 \\ p^1_3 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}\begin{pmatrix} p^1_2 \\ p^1_3 \end{pmatrix} + c^1_1\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

With these costate solutions we come back in the relations of the Theorem. From the four possible choices for the controls, suppose

$$(u_{\alpha\beta}) = \begin{pmatrix} U & -U \\ -U & U \end{pmatrix}.$$

In this case the initial PDEs become

$$\frac{\partial x}{\partial t^1}(t^1, t^2) = v_1(t^1, t^2)$$

$$\frac{\partial v_1}{\partial t^1}(t^1, t^2) = -b_{11}v_1(t^1, t^2) - b_{12}v_2(t^1, t^2) + U$$

$$\frac{\partial v_2}{\partial t^1}(t^1, t^2) = -b_{21}v_1(t^1, t^2) - b_{22}v_2(t^1, t^2) - U$$

$$\frac{\partial x}{\partial t^2}(t^1, t^2) = v_2(t^1, t^2)$$

$$\frac{\partial v_1}{\partial t^2}(t^1, t^2) = -c_{11}v_1(t^1, t^2) - c_{12}v_2(t^1, t^2) - U$$

$$\frac{\partial v_2}{\partial t^2}(t^1, t^2) = -c_{21}v_1(t^1, t^2) - c_{22}v_2(t^1, t^2) + U.$$

Suppose that the coefficients $b_{\alpha\beta}$, $c_{\alpha\beta}$ are constants. Then we shall explain how we can find the solutions of the original PDEs. Of course the solutions of the adjoint PDEs can be found in a similar way.

We introduce the matrices

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -b_{11} & -b_{12} \\ 0 & -b_{21} & -b_{22} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -c_{11} & -c_{12} \\ 0 & -c_{21} & -c_{22} \end{pmatrix}$$

(see the homogeneous PDE system) and the matrices

$$F_\alpha = (-1)^{\alpha-1}\begin{pmatrix} 0 \\ U \\ -U \end{pmatrix}$$

(see the non-homogeneous system). Adding the complete integrability conditions, $M_1M_2 = M_2M_1$, $M_1F_2 = M_2F_1$, we find the optimal evolution (solution of the non-homogeneous system)

$$y(t) = (\exp M_\alpha(t^\alpha - t^\alpha_0))\ y_0 + \int_{\gamma_{t_0t}} \exp(-M_\alpha(t^\alpha - t^\alpha_0))\ F_\alpha(s)ds^\alpha,$$

where γ_{t_0t} is an arbitrary piecewise C^1 curve. □

3.3 Using the weak multitime maximum principle

To solve the foregoing problem, we can use also the weak multitime maximum principle (for multitime optimal control, see also [4-19]) based on the control Hamiltonian

$$H = -L + p_i^\alpha X_\alpha^i,\ \alpha = 1,2;\ i = 1,2,3.$$

Theorem 3.3. *Suppose that the problem of minimizing the functional* $I(u(\cdot))$ *constrained by the first order PDEs* (3.2)*, with* C^1 *functions* X_α^i*, has an interior solution* $\hat{u}(t) = (\hat{u}_{\alpha\beta}) \in U$ *which generates a 2-sheet state variable* $y(t)$*. Then there exists a* C^1 *costate matrix* $p(t) = (p_i^\alpha(t))$ *such that we have*

$$\frac{\partial p_i^\alpha}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(x(t), v(t), u(t), p(t)), \qquad (adjoint\ PDEs)$$

$$\frac{\partial x^i}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_i^\alpha}(x(t), v(t), u(t), p(t)), \qquad (initial\ PDEs)$$

and

$$\frac{\partial H}{\partial \hat{u}_{\alpha\beta}}(x(t), v(t), u(t), p(t)) = 0. \qquad (critical\ point\ conditions)$$

The adjoint PDEs are equivalent to

$$\frac{\partial p_1^1}{\partial t^1} + \frac{\partial p_1^2}{\partial t^2} = -\frac{\partial H}{\partial x},\ \frac{\partial p_2^1}{\partial t^1} + \frac{\partial p_2^2}{\partial t^2} = -\frac{\partial H}{\partial v_1},\ \frac{\partial p_3^1}{\partial t^1} + \frac{\partial p_3^2}{\partial t^2} = -\frac{\partial H}{\partial v_2}$$

and for the initial PDEs we have in fact $\dfrac{\partial H}{\partial p_i^\alpha} = X_\alpha^i$.

3.3.1 Bang-bang optimal control

Let us consider an optimal control problem of type *minimal two-time area.* By application of the two-time maximum principle, we obtain necessary conditions for optimality and use them to guess a candidate control policy.

Theorem 3.4. *If we consider* $L = -1$*, then the optimal control is a bang-bang control.*

Proof. The control Hamiltonian becomes

$$\begin{aligned} H &= -1 + p_i^\alpha X_\alpha^i \\ &= -1 + p_1^1 v_1 + p_2^1(-b_{11}v_1 - b_{12}v_2 + u_{11}) \\ &+ p_3^1(-b_{21}v_1 - b_{22}v_2 + u_{21}) + p_1^2 v_2 \\ &+ p_2^2(-c_{11}v_1 - c_{12}v_2 + u_{12}) + p_3^2(-c_{21}v_1 - c_{22}v_2 + u_{22}). \end{aligned}$$

Observe that the Hamiltonian H is linear in the control u and has no critical point. Hence, the extremum points of H lie on the boundary of the admissible set for u. So, we have a bang-bang control

$$u_{11} = U \text{ sgn}\, p_2^1,\ u_{21} = U \text{ sgn}\, p_3^1,$$

$$u_{12} = U \text{ sgn } p_2^2, \; u_{22} = U \text{ sgn } p_3^2,$$

where

$$|u_{\alpha\beta}| \le U = \frac{F_{\max}}{m}$$

(we assume that there is a limit, $F_{\max}$, on the magnitude of applied force).

Suppose that the coefficients $b_{\alpha\beta}$, $c_{\alpha\beta}$ are constants. Writing explicitly the adjoint PDEs, it follows the following system

$$\begin{cases} \dfrac{\partial p_1^1}{\partial t^1} + \dfrac{\partial p_1^2}{\partial t^2} = 0 \\ \dfrac{\partial p_2^1}{\partial t^1} + \dfrac{\partial p_2^2}{\partial t^2} = -p_1^1 + b_{11}p_2^1 + b_{21}p_3^1 + c_{11}p_2^2 + c_{21}p_3^2 \\ \dfrac{\partial p_3^1}{\partial t^1} + \dfrac{\partial p_3^2}{\partial t^2} = -p_1^2 + b_{12}p_2^1 + b_{22}p_3^1 + c_{12}p_2^2 + c_{22}p_3^2. \end{cases} \tag{3.3}$$

The first PDE of (3.3) gives us

$$\frac{\partial p_1^1}{\partial t^1} = -\frac{\partial p_1^2}{\partial t^2}$$

and we have

$$p_1^1(t) = \int_0^{t^1} \varphi(\tau, t^2) d\tau \Rightarrow p_1^2(t) = -\int_0^{t^2} \varphi(t^1, \tau) d\tau.$$

Then we split the second and third PDEs of (3.3) in two subsystems as

$$\begin{cases} \dfrac{\partial p_2^1}{\partial t^1} = -p_1^1 + b_{11}p_2^1 + b_{21}p_3^1 \\ \dfrac{\partial p_3^1}{\partial t^1} = b_{12}p_2^1 + b_{22}p_3^1 \end{cases}$$

and

$$\begin{cases} \dfrac{\partial p_2^2}{\partial t^2} = c_{11}p_2^2 + c_{21}p_3^2 \\ \dfrac{\partial p_3^2}{\partial t^2} = -p_1^2 + c_{12}p_2^2 + c_{22}p_3^2, \end{cases}$$

with additional conditions

$$\begin{cases} \dfrac{\partial p_2^1}{\partial t^1} + \dfrac{\partial p_2^2}{\partial t^2} = 0 \\ \dfrac{\partial p_3^1}{\partial t^1} + \dfrac{\partial p_3^2}{\partial t^2} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial p_2^1}{\partial t^1} = -\dfrac{\partial p_2^2}{\partial t^2} \\ \dfrac{\partial p_3^1}{\partial t^1} = -\dfrac{\partial p_3^2}{\partial t^2} \end{cases}$$

and we have

$$p_2^1(t) = \int_0^{t^1} \varphi(\tau, t^2) d\tau \Rightarrow p_2^2(t) = -\int_0^{t^2} \psi(t^1, \tau) d\tau$$

$$p_3^1(t) = \int_0^{t^1} \xi(\tau, t^2) d\tau \Rightarrow p_3^2(t) = -\int_0^{t^2} \xi(t^1, \tau) d\tau.$$

With these costate values we come back in the relations of the Theorem. From the four possible choices for the controls, suppose

$$(u_{\alpha\beta}) = \begin{pmatrix} U & -U \\ -U & U \end{pmatrix}.$$

In this case the initial PDEs become

$$\begin{aligned}
&\frac{\partial x}{\partial t^1}(t^1,t^2) = v_1(t^1,t^2)\\
&\frac{\partial v_1}{\partial t^1}(t^1,t^2) = -b_{11}v_1(t^1,t^2) - b_{12}v_2(t^1,t^2) + U\\
&\frac{\partial v_2}{\partial t^1}(t^1,t^2) = -b_{21}v_1(t^1,t^2) - b_{22}v_2(t^1,t^2) - U\\
&\frac{\partial x}{\partial t^2}(t^1,t^2) = v_2(t^1,t^2)\\
&\frac{\partial v_1}{\partial t^2}(t^1,t^2) = -c_{11}v_1(t^1,t^2) - c_{12}v_2(t^1,t^2) - U\\
&\frac{\partial v_2}{\partial t^2}(t^1,t^2) = -c_{21}v_1(t^1,t^2) - c_{22}v_2(t^1,t^2) + U.
\end{aligned}$$

The solution of this system was written in the previous explanations. □

4 Multitime elliptic Newton Law approach

Our third idea is to transform the ODE system (1.1) (*single-time Newton Law*) into an elliptic PDE equation (*two-time elliptic Newton Law*)

$$\frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) + b^\alpha(t)\frac{\partial x}{\partial t^\alpha}(t) = u(t),\ t \in \Omega \subset \mathbb{R}^2_+, \tag{4.1}$$

with $\alpha, \beta = 1, 2$ and the boundary conditions

$$x(0,0) = 0,\ x(T^1,T^2) = D. \tag{4.2}$$

This law is based on the remark that the change of variable, realized by a decomposition of single-time as sum of two-times, leads the second order differential equation into an elliptic partial differential equation of second order (and conversely).

From a two-time solution of the foregoing problem, we can recover the solution of the second order ODE using $\hat{x}(t) = \hat{x}(t^1,t^2) = \phi(\frac{t^1+t^2}{\sqrt{2}})$. Indeed

$$\hat{x}_{t^1} = \frac{1}{\sqrt{2}}\dot{\phi},\ \hat{x}_{t^2} = \frac{1}{\sqrt{2}}\dot{\phi}$$

$$\hat{x}_{t^1t^1} = \frac{1}{2}\ddot{\phi},\ \hat{x}_{t^2t^2} = \frac{1}{2}\ddot{\phi}$$

and replacing in the second order elliptic PDE we find the second order ODE in the unknown ϕ.

4.1 Optimal control problem

Let us consider the two-time optimal control problem with a double integral cost functional

$$\min_u \ I(u(\cdot)) = \iint_\Omega L(x(t), v(t), u(t))d\omega, \quad d\omega = dt^1 \wedge dt^2 \tag{4.3}$$

constrained by the PDEs (4.1)-(4.2) (for multitime optimal control, see also [5-20]).

To find the necessary conditions, let us start with the generalized Lagrangian

$$\mathcal{L} = L + p(t)\left(\frac{1}{2}\delta^{\alpha\beta}(t)\frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) + b^\alpha(t)\frac{\partial x}{\partial t^\alpha}(t) - u(t)\right)$$

and follow the following steps [see, [5]). The Lagrange multiplier $p(t)$ is a C^1 function. In the case of second order PDEs, we cannot use a canonical Hamiltonian, as in the case of first order PDEs. For that we work directly with the Lagrangian $\mathcal{L}$.

Theorem 4.1. *Suppose that the problem of maximizing the functional* (4.3) *constrained by* (4.1)-(4.2) *has an interior optimal solution $u^*(t)$, which determines the optimal evolution $x(t)$. Then there exists the costate function $p(t)$ such that*

(i) the initial PDE $$\frac{\partial \mathcal{L}}{\partial p} = 0,$$

(ii) the adjoint or dual equation $$\frac{\partial \mathcal{L}}{\partial x} + \frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 p}{\partial t^\alpha \partial t^\beta} - \frac{\partial (pb^\alpha)}{\partial t^\alpha} = 0,$$

(iii) the critical point condition $$\frac{\partial \mathcal{L}}{\partial u} = p$$

hold.

Proof. (for details, see [5]) Firstly, we find the *infinitesimal deformation of elliptic PDE*. We fix the control $u(t)$ and we variate the state $x(t)$ into $x(t,\epsilon)$. Denoting $\frac{\partial x}{\partial \epsilon}(t,0) = y$, the infinitesimal deformation PDE is

$$\frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 y}{\partial t^\alpha \partial t^\beta}(t) + b^\alpha(t)\frac{\partial y}{\partial t^\alpha}(t) = 0.$$

The *adjoint PDE* is

$$\frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 p}{\partial t^\alpha \partial t^\beta}(t) - \frac{\partial (b^\alpha p)}{\partial t^\alpha}(t) = 0.$$

The adjointness has the sense $pLy - yMp = 0$, where L and M are linear second order partial differential operators.

The variation of the control determines the variation of the state. It follows that the adjoint PDE equation

$$\frac{\partial L}{\partial x} + \frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 p}{\partial t^\alpha \partial t^\beta} - \frac{\partial (b^\alpha p)}{\partial t^\alpha} = 0$$

and the critical point condition

$$\frac{\partial L}{\partial u} - p = 0$$

must be satisfied. □

4.2 Bang-bang optimal control

Let us consider an optimal control problem of type *minimal two-time area*. By application of the two-time maximum principle, we obtain necessary conditions for optimality and use them to guess a candidate control policy.

Theorem 4.2. *If we consider $L = -1$, then the optimal control is a bang-bang control.*

Proof. The control Lagrangian becomes

$$\mathcal{L} = -1 + p(t)\left(\frac{1}{2}\delta^{\alpha\beta}(t)\frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) + b^\alpha(t)\frac{\partial x}{\partial t^\alpha}(t)\right) - p(t)u(t).$$

Let $[-U, U] \subset \mathbb{R}$ be the control set. The adjoint PDE is

$$\frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 p}{\partial t^\alpha \partial t^\beta} - \frac{\partial(b^\alpha p)}{\partial t^\alpha} = 0.$$

The maximum of the linear Lagrangian function $u \to \mathcal{L}$ exists since the control variable belongs to the interval $[-U, U]$; for optimum, the control must be $\hat{u} = U$ or $\hat{u} = -U$ (see linear optimization, simplex method). The optimal control $\hat{u}$ must be the function $\hat{u}(t) = U \operatorname{sgn}(-p(t))$. Consequently, the optimal Lagrangian is

$$\mathcal{L}^* = -1 + p(t)\left(\frac{1}{2}\delta^{\alpha\beta}(t)\frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) + b^\alpha(t)\frac{\partial x}{\partial t^\alpha}(t)\right) + |p(t)|U.$$

The optimal evolution is the solution of the problem

$$\frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) + b^\alpha(t)\frac{\partial x}{\partial t^\alpha}(t) = U,\ t \in \Omega \subset \mathbb{R}^2_+,$$

$$x(0,0) = 0,\ x(T^1, T^2) = D.$$

□

5 Conclusion

Our work is the first which introduce and study the theory of multi-temporal of robots based on multi-temporal variants of Newton Law and appropriate functionals. The basic idea is to find an optimal multi-temporal evolution to relieve a robot by unnecessary efforts. The importance of the subject is imposed by the requirements of Applied Sciences. From a mathematical perspective, is to analyze what is the meaning of multi-dimensionality for the variables of evolution. Although our point of view seems quite strange, we believe that this approach will be followed in the future for further developments of robots theory.

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